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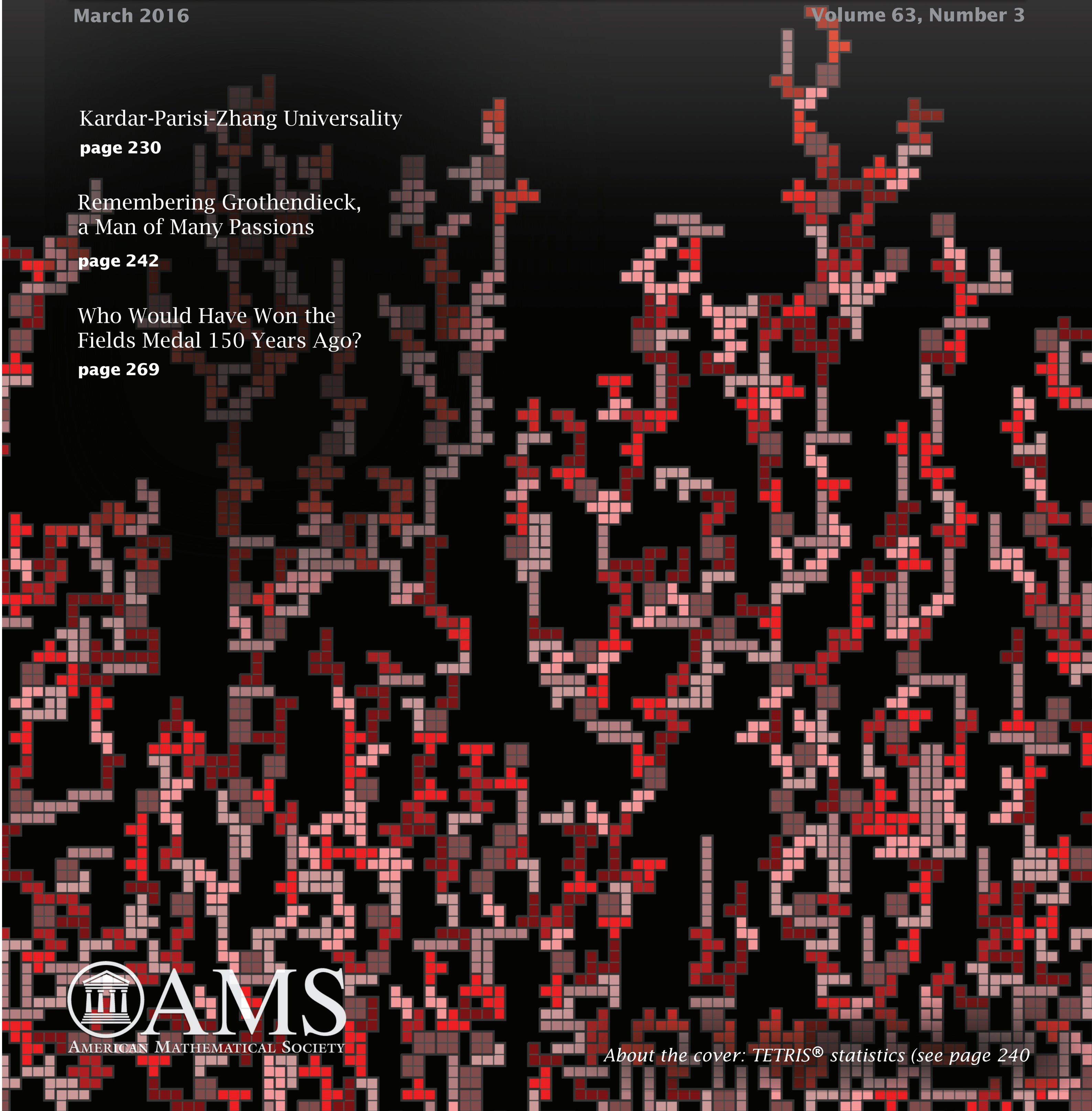

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About the cover: TETRIS® statistics (see page 240

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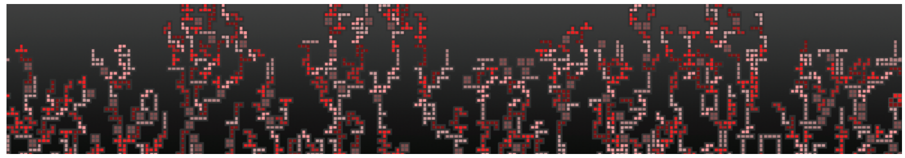
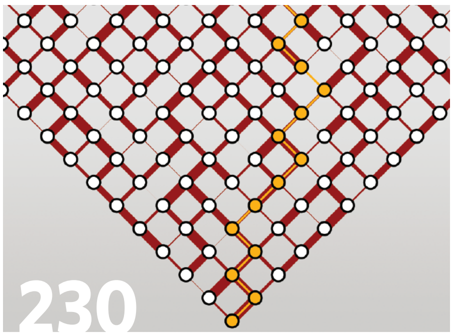
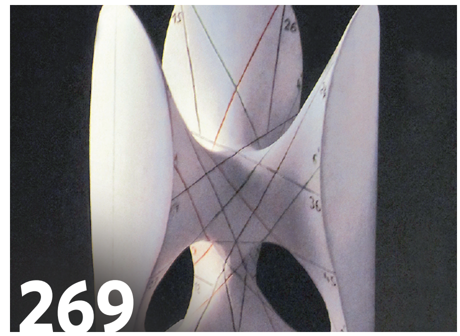


FEATURES





March brings an intriguing examination of universal statistical behaviors by Ivan Corwin; the first of our two-part tribute to Alexandre Grothendieck; some fanciful history by Jeremy Gray; a Doceamus column on why teaching matters; an insider's look at I.M. Gelfand's famous seminar, and much more—right up to the Back Page, which features a new installment of the Notices' original "My TA" comic strip, responded to by the new "My Professor" comic strip in the Graduate Student section. You can post general comments and suggestions for this and future issues on our webpage `www.ams.org/notices`. *—Frank Morgan, Editor-in-Chief*

THE GRADUATE STUDENT SECTION

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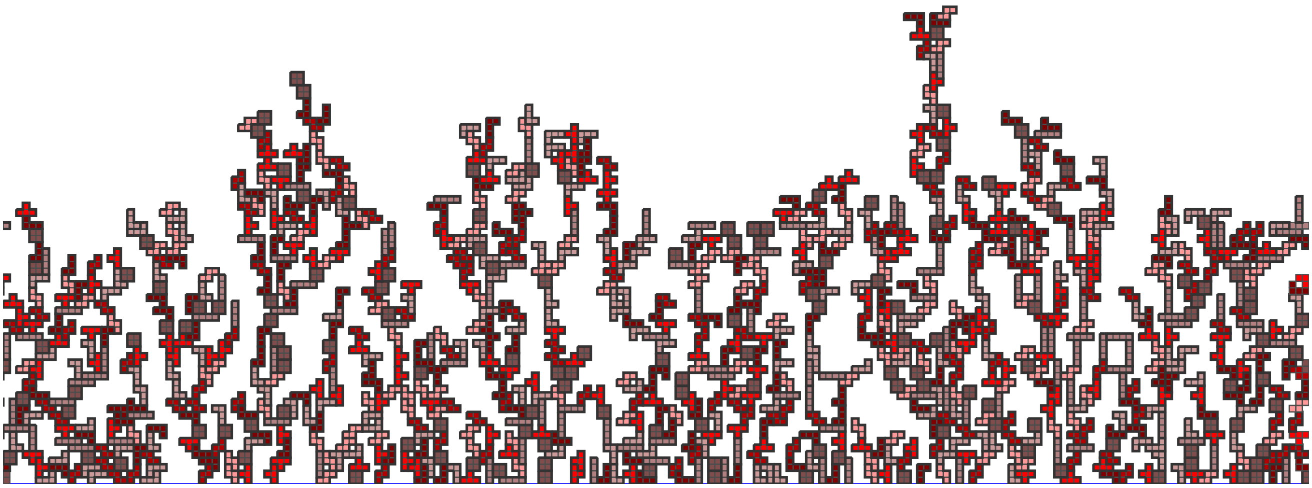
Kardar-Parisi-Zhang Universality



I. Corwin

Universality in Random Systems

Universality in complex random systems is a striking concept which has played a central role in the direction of research within probability, mathematical physics and statistical mechanics. In this article we will describe how a variety of physical systems and mathematical models, including randomly growing interfaces, certain stochastic PDEs, traffic models, paths in random environments, and random matrices all demonstrate the same universal statistical behaviors in their long-time/large-scale limit. These systems are said to lie in the *Kardar-Parisi-Zhang (KPZ) universality class*. Proof of universality within these classes of systems (except for random matrices) has remained mostly elusive. Extensive computer simulations, nonrigorous physical arguments/heuristics, some laboratory experiments, and limited mathematically rigorous results provide important evidence for this belief.

Ivan Corwin is professor of mathematics at Columbia University, and research fellow of the Clay Mathematics Institute, Packard Foundation, as well as previous holder of the Poincaré Chair at the Institut Henri Poincaré and of the Schramm Fellowship at Microsoft Research and MIT.
His email address is `ivan.corwin@gmail.com`.

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The last fifteen years have seen a number of breakthroughs in the discovery and analysis of a handful of special *integrable probability systems* which, due to enhanced algebraic structure, admit many exact computations and ultimately asymptotic analysis revealing the purportedly universal properties of the KPZ class. The structures present in these systems generally originate in representation theory (e.g. symmetric functions), quantum integrable systems (e.g. Bethe ansatz), algebraic combinatorics (e.g. RSK correspondence), and the techniques in their asymptotic analysis generally involve Laplace's method, Fredholm determinants, or Riemann-Hilbert problem asymptotics.

This article will focus on the phenomena associated with the KPZ universality class [4] and highlight how certain integrable examples expand the scope of and refine the notion of universality. We start by providing a brief introduction to the Gaussian universality class and the integrable probabilistic example of random coin flipping and the random deposition model. A small perturbation to the random deposition model leads us to the ballistic deposition model and the KPZ universality class. The ballistic deposition model fails to be integrable; thus to gain an understanding of its long-time behavior and that of the entire KPZ class, we turn to the corner growth model. The rest of the article focuses on various sides of this rich model: its role as a random growth process, its relation to the KPZ stochastic PDE, its interpretation in terms of interacting particle systems, and its relation to optimization problems involving paths in random environments. Along the way, we include some other generalizations of this process whose integrability springs from the same sources. We close the article by reflecting upon some open problems.

A survey of the KPZ universality class and all of the associated phenomena and methods developed or utilized in its study is far too vast to be provided here. This article presents only one of many stories and perspectives regarding this rich area of study. To even provide a representative cross-section of references is beyond this scope. Additionally, though we will discuss integrable examples, we will not describe the algebraic structures and methods of asymptotic analysis behind them (despite their obvious importance and interest). Some recent references which review some of these structures include [2], [4], [8] and references therein. On the more physics oriented side, the collection of reviews and books [1], [3], [5], [6], [7], [8], [9], [10] provides some idea of the scope of the study of the KPZ universality class and the diverse areas upon which it touches.

We start now by providing an overview of the general notion of universality in the context of the simplest and historically first example—fair coin flipping and the Gaussian universality class.

Gaussian Universality Class

Flip a fair coin N times. Each string of outcomes (e.g. head, tail, tail, tail, head) has equal probability 2^{-N}. Call H the (random) number of heads and let $\mathbb{P}$ denote the probability distribution for this sequence of coin flips. Counting shows that

$$\mathbb{P}(H=n) = 2^{-N}\binom{N}{n}.$$

Since each flip is independent, the expected number of heads is $N/2$. Bernoulli (1713) proved that H/N converges to $1/2$ as N goes to infinity. This was the first example of a *law of large numbers*. Of course, this does not mean that if you flip the coin 1,000 times, you will see exactly 500 heads. Indeed, in N coin flips one expects the number of heads to vary randomly around the value $N/2$ in the scale $\sqrt{N}$. Moreover, for all $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\Big(H < \tfrac{1}{2}N + \tfrac{1}{2}\sqrt{N}x\Big) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy.$$

De Moivre (1738), Gauss (1809), Adrain (1809), and Laplace (1812) all participated in the proof of this result. The limiting distribution is known as the Gaussian (or sometimes normal or bell curve) distribution.

A proof of this follows from asymptotics of $n!$, as derived by de Moivre (1721) and named after Stirling (1729). Write

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t}t^n dt = n^{n+1}\int_0^\infty e^{nf(z)}dz$$

where $f(z) = \log z - z$ and the last equality is from the change of variables $t = nz$. The integral is dominated, as n grows, by the maximal value of $f(z)$ on the interval $[0,\infty)$. This occurs at $z=1$, thus expanding $f(z) \approx -1 - \frac{(z-1)^2}{2}$, and plugging this into the integral yields the final expansion

$$n! \approx n^{n+1}e^{-n}\sqrt{2\pi/n}.$$

This general route of writing exact formulas for probabilities in terms of integrals and then performing asymptotics is quite common to the analysis of integrable models in the KPZ universality class, though those formulas and analyses are considerably more involved.

The universality of the Gaussian distribution was not broadly demonstrated until work of Chebyshev, Markov, and Lyapunov around 1900. The *central limit theorem* (CLT) showed that the exact nature of coin flipping is immaterial—any sum of independent identically distributed (iid) random variables with finite mean and variance will demonstrate the same limiting behavior.

Theorem 1. *Let $X_1, X_2, \ldots$ be iid random variables of finite mean m and variance v. Then for all $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} \mathbb{P}\Big(\sum_{i=1}^{N} X_i < mN + v\sqrt{N}x\Big) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy.$$

Proofs of this result use different tools than the exact analysis of coin flipping, and much of probability theory deals with the study of Gaussian processes which arise through various generalizations of the CLT. The Gaussian distribution is ubiquitous, and, as it is the basis for much of classical statistics and thermodynamics, it has had immense societal impact.

Random versus Ballistic Deposition

The *random deposition model* is one of the simplest (and least realistic) models for a randomly growing one-dimensional interface. Unit blocks fall independently and in parallel from the sky above each site of $\mathbb{Z}$ according to exponentially distributed waiting times (see Figure 1). Recall that a random variable X has exponential distribution of rate $\lambda > 0$ (or mean $1/\lambda$) if $\mathbb{P}(X > x) = e^{-\lambda x}$. Such random variables are characterized by the memoryless property—conditioned on the event that $X > x$, $X - x$ still has the exponential distribution of the same rate. Consequently, the random deposition model is *Markov*—its future evolution depends only on the present state (and not on its history).

The random deposition model is quite simple to analyze, since each column grows independently. Let $h(t,x)$ record the height above site x at time t and assume $h(0,x) \equiv 0$. Define random waiting times $w_{x,i}$ to be the time for the i-th block in column x to fall. For any n, the event $h(t,x) < n$ is equivalent to $\sum_{i=1}^{n} w_{x,i} > t$. Since the $w_{x,i}$ are iid, the law of large numbers and central limit theory apply here. Assuming $\lambda = 1$,

$$\lim_{t\to\infty} \frac{h(t,x)}{t} = 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{h(t,x)-t}{t^{1/2}} \Rightarrow N(x)$$

jointly over $x \in \mathbb{Z}$, where $\{N(x)\}_{x\in\mathbb{Z}}$ is a collection of iid standard Gaussian random variables. The top of Figure 2

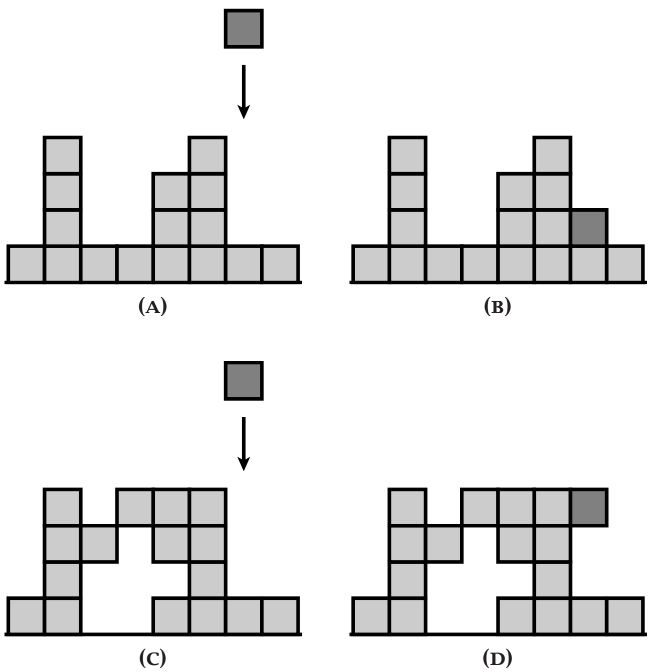

Figure 1. (A) and (B) illustrate the *random deposition model*, and (C) and (D) illustrate the *ballistic deposition model*. In both cases, blocks fall from above each site with independent exponentially distributed waiting times. In the first model, they land at the top of each column, whereas in the second model they stick to the first edge to which they become incident.

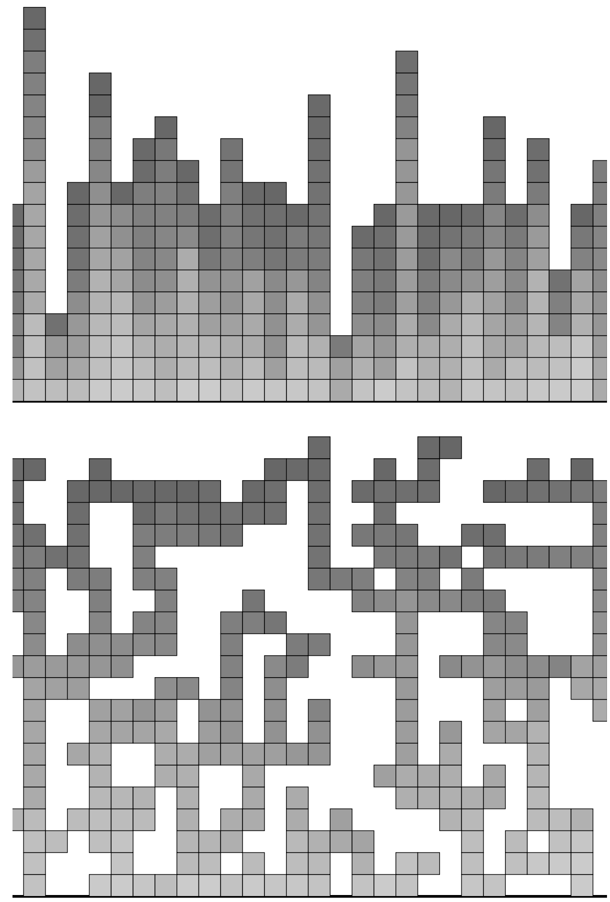

Figure 2. Simulation of random (top) versus ballistic (bottom) deposition models driven by the same process of falling blocks. The ballistic model grows much faster and has a smoother, more spatially correlated top interface.

shows a simulation of the random deposition model. The linear growth speed and lack of spatial correlation are quite evident. The fluctuations of this model are said to be in the Gaussian universality class since they grow like $t^{1/2}$, with Gaussian limit law and trivial transversal correlation length scale t^0. In general, fluctuation and transversal correlation exponents, as well as limiting distributions, constitute the description of a universality class, and all models which match these limiting behaviors are said to lie in the same universality class.

While the Gaussian behavior of this model is resilient against changes in the distribution of the $w_{x,i}$ (owing to the CLT), generic changes in the nature of the growth rules shatter the Gaussian behavior. The *ballistic deposition (or sticky block) model* was introduced by Vold (1959) and, as one expects in real growing interfaces, displays spatial correlation. As before, blocks fall according to iid exponential waiting times; however, now a block will stick to the first edge against which it becomes incident. This mechanism is illustrated in Figure 1. This creates overhangs, and we define the height function $h(t,x)$ as the maximal height above x which is occupied by a box. How does this microscopic change manifest itself over time?

It turns out that sticky blocks radically change the limiting behavior of this growth process. The bottom of Figure 2 records one simulation of the process. Seppäläinen (1999) gave a proof that there is still an overall linear growth rate. Moreover, by considering a lower bound by a width two system, one can see that this velocity exceeds that of the random deposition model. The exact value of this rate, however, remains unknown.

The simulation in Figure 2 (as well as the longer time results displayed in Figure 3) also shows that the scale of fluctuations of $h(t,x)$ is smaller than in random deposition and that the height function remains correlated transversally over a long distance. There are exact conjectures for these fluctuations. They are supposed to grow like $t^{1/3}$ and demonstrate a nontrivial correlation structure in a transversal scale of $t^{2/3}$. Additionally, precise predictions exist for the limiting distributions. Up to certain (presently undetermined) constants c_1, c_2, the sequence of scaled heights $c_2 t^{-1/3}(h(t,0) - c_1 t)$ should converge to the so-called *Gaussian orthogonal ensemble (GOE) Tracy-Widom* distributed random variable. The Tracy-Widom distributions can be thought of as modern-day bell curves, and

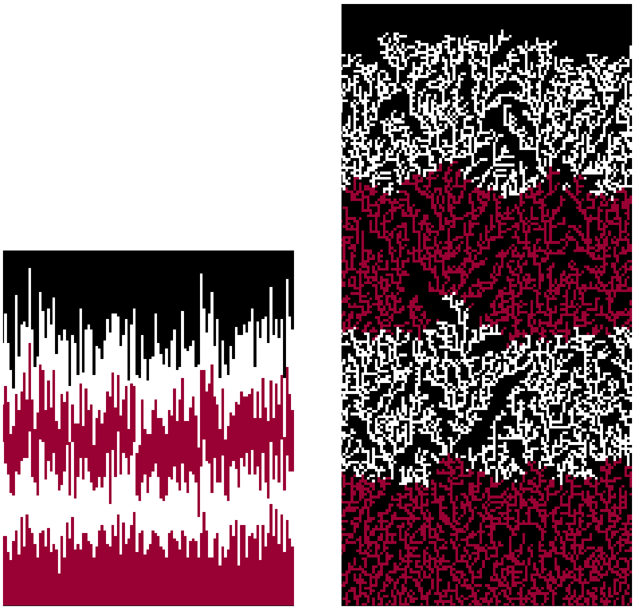

Figure 3. Simulation of random (left) versus ballistic (right) deposition models driven by the same process of falling blocks and run for a long time. The red and white colors represent different epochs of time in the simulation. The size of boxes in both figures are the same.

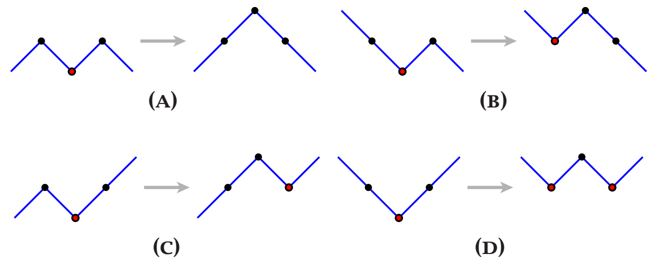

Figure 4. Various possible ways that a local minimum can grow into a local maximum. The red dots represent the local minimum at which growth may occur.

their names GOE or GUE (for *Gaussian unitary ensemble*) come from the random matrix ensembles in which these distributions were first observed by Tracy-Widom (1993, 1994).

Ballistic deposition does not seem to be an integrable probabilistic system, so where do these precise conjectures come from? The exact predictions come from the analysis of a few similar growth processes which just happen to be integrable! Ballistic deposition shares certain features with these models, which are believed to be key for membership in the KPZ class:

- Locality: Height function change depends only on neighboring heights.
- Smoothing: Large valleys are quickly filled in.
- Nonlinear slope dependence: Vertical effective growth rate depends nonlinearly on local slope.
- Space-time independent noise: Growth is driven by noise which quickly decorrelates in space and time and does not display heavy tails.

It should be made clear that a proof of the KPZ class behavior for the ballistic deposition model is far beyond what can be done mathematically (though simulations strongly suggest that the above conjecture is true).

Corner Growth Model

We come to the first example of an integrable probabilistic system in the KPZ universality class, the *corner growth model*. The randomly growing interface is modeled by a height function $h(t,x)$ which is continuous, piecewise linear, and composed of $\sqrt{2}$-length line increments of slope $+1$ or -1, changing value at integer x. The height function evolves according to the Markovian dynamics that each local minimum of h (looking like $\vee$) turns into a local maximum (looking like $\wedge$) according to an exponentially distributed waiting time. This happens independently for each minimum. This change in height function can also be thought of as adding boxes (rotated by $45°$). See Figures 4 and 5 for further illustration of this model.

Wedge initial data means that $h(0,x) = |x|$, while *flat* initial data (as considered for ballistic deposition) means that $h(0,x)$ is given by a periodic sawtooth function which goes between heights 0 and 1. We will focus on wedge initial data. Rost (1980) proved a law of large numbers for the growing interface when time, space, and the height function are scaled by the same large parameter L.

Theorem 2. *For wedge initial data,*

$$\lim_{L\to\infty} \frac{h(Lt,Lx)}{L} = \mathfrak{h}(t,x) := \begin{cases} t\,\frac{1-(x/t)^2}{2} & |x| < t, \\ |x| & |x| \geq t. \end{cases}$$

Figure 6 displays the result of a computer simulation wherein the limiting parabolic shape is evident. The function $\mathfrak{h}$ is the unique viscosity solution to the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t}\mathfrak{h}(t,x) = \frac{1}{2}\Big(1 - \big(\frac{\partial}{\partial x}\mathfrak{h}(t,x)\big)^2\Big).$$

This equation actually governs the evolution of the law of large numbers from arbitrary initial data.

The fluctuations of this model around the law of large numbers are believed to be universal. Figure 6 shows that the interface (blue) fluctuates around its limiting shape (red) on a fairly small scale, with transversal correlation on a larger scale. For $\epsilon > 0$, define the scaled and centered height function

$$h_\epsilon(t,x) := \epsilon^b h(\epsilon^{-z}t, \epsilon^{-1}x) - \frac{\epsilon^{-1}t}{2}$$

where the *dynamic scaling exponent* $z = 3/2$ and the *fluctuation exponent* $b = 1/2$. These exponents are easily remembered, since they correspond with scaling time : space : fluctuations like 3 : 2 : 1. These are the characteristic exponents for the KPZ universality class. Johansson (1999) proved that for fixed t, as $\epsilon \to 0$, the random variable $h_\epsilon(t,0)$ converges to a GUE Tracy-Widom distributed random variable (see Figure 7). Results for the

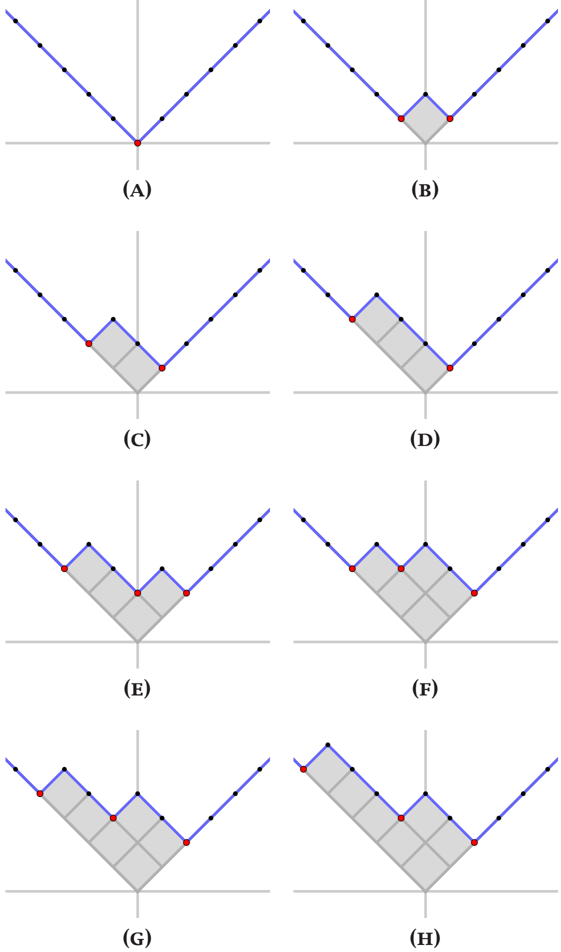

Figure 5. The corner growth model starts with an empty corner, as in (A). There is only one local minimum (the red dot), and after an exponentially distributed waiting time, this turns into a local maximum by filling in the site above it with a block, as in (B). In (B) there are now two possible locations for growth (the two red dots). Each one has an exponentially distributed waiting time. (C) corresponds to the case when the left local minimum grows before the right one. By the memoryless property of exponential random variables, once in state (C) we can think of choosing new exponentially distributed waiting times for the possible growth destinations. Continuing in a similar manner, we arrive at the evolution in (D) through (H).

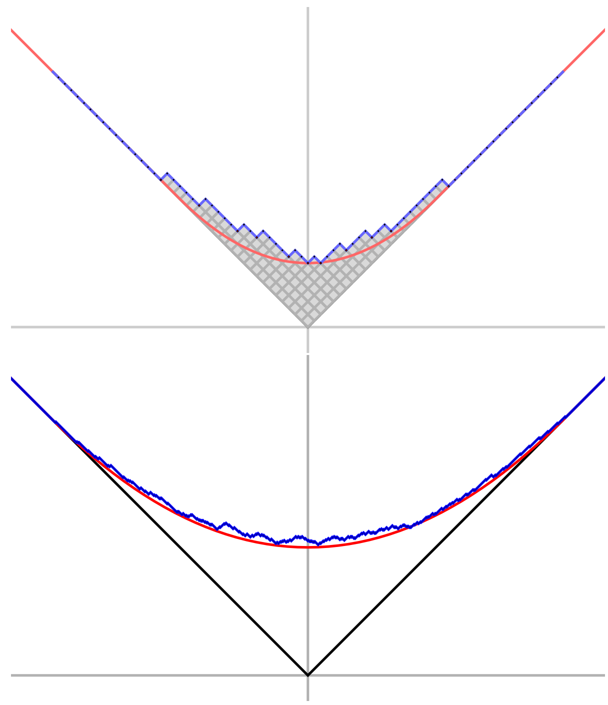

Figure 6. Simulation of the corner growth model. The top shows the model after a medium amount of time, and the bottom shows it after a longer amount of time. The blue interface is the simulation, while the red curve is the limiting parabolic shape. The blue curve has vertical fluctuations of order $t^{1/3}$ and decorrelates spatially on distances of order $t^{2/3}$.

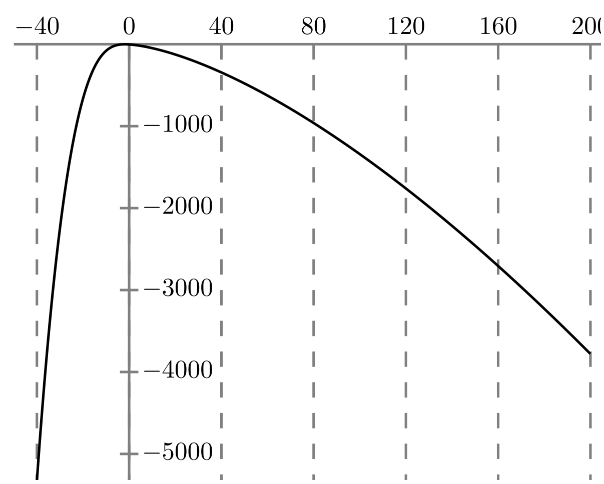

Figure 7. The density (top) and log of the density (bottom) of the GUE Tracy-Widom distribution. Though the density appears to look like a bell curve (or Gaussian), this comparison is misleading. The mean and variance of the distribution are approximately -1.77 and 0.81. The tails of the density (as shown in terms of the log of the density in the bottom plot) decay like $e^{-c_-|x|^3}$ for $x \ll 0$ and like $e^{-c_+x^{3/2}}$ for $x \gg 0$ for certain positive constants c_- and c_+. The Gaussian density decays like e^{-cx^2} in both tails, with the constant c related to the variance.

related model of the longest increasing subsequence in a random permutation were provided around the same time by Baik-Deift-Johansson (1999). For that related model, two years later Prähofer-Spohn (2001) computed the analog to the joint distribution of $h_\epsilon(t, x)$ for fixed t and varying x.

The entire scaled growth process $h_\epsilon(\cdot, \cdot)$ should have a limit as $\epsilon \to 0$ which would necessarily be a fixed point under the 3:2:1 scaling. The existence of this limit (often called the *KPZ fixed point*) remains conjectural. Still, much is known about the properties this limit should enjoy. It should be a stochastic process whose evolution depends on the limit of the initial data under the same scaling. The one-point distribution for general initial data, the multipoint and multitime distribution for wedge initial data, and various aspects of its continuity are all understood. Besides the existence of this limit, what is missing is a useful characterization of the KPZ fixed point. Since the KPZ fixed point is believed to be the universal scaling limit of all models in the KPZ universality class and since corner growth enjoys the same key properties as ballistic deposition, one also is led to conjecture that ballistic deposition scales to the same fixed point and hence enjoys the same scalings and limiting distributions. The reason why the GOE Tracy-Widom distribution came up in our earlier discussion is that we were dealing with flat rather than wedge initial data.

One test of the universality belief is to introduce partial asymmetry into the corner growth model. Now we change local minimum into local maximum at rate p and turn local maximum into local minimum at rate q (all waiting times are independent and exponentially distributed, and $p+q=1$). See Figure 8 for an illustration of this partially asymmetric corner growth model. Tracy-Widom (2007–09) showed that so long as $p > q$, the same law of large numbers and fluctuation limit theorem hold for the partially asymmetric model, provided that t is replaced by $t/(p-q)$. Since $p-q$ represents the growth drift, one simply has to speed up to compensate for this drift being smaller.

Clearly for $p \leq q$ something different must occur than for $p > q$. For $p = q$ the law of large numbers and fluctuations change nature. The scaling of time : space : fluctuations becomes 4 : 2 : 1, and the limiting process under these scalings becomes the stochastic heat equation with additive white noise. This is the *Edwards-Wilkinson* (EW) universality class, which is described by the stochastic heat equation with additive noise. For $p < q$ the process approaches a stationary distribution where the probability of having k boxes added to the empty wedge is proportional to $(p/q)^k$.

So, we have observed that for any positive asymmetry the growth model lies in the KPZ universality class, while for zero asymmetry it lies in the EW universality class. It is natural to wonder whether in critically scaling parameters (i.e. $p-q \to 0$) one might encounter a crossover regime between these two universality classes. Indeed, this is the case, and the crossover is achieved by the KPZ equation, which we now discuss.

KPZ Equation

The *KPZ equation* is written as

$$\frac{\partial h}{\partial t}(t,x) = \nu \frac{\partial^2 h}{\partial x^2}(t,x) + \tfrac{1}{2}\lambda\Big(\frac{\partial h}{\partial x}(t,x)\Big)^2 + \sqrt{D}\xi(t,x),$$

where $\xi(t,x)$ is Gaussian space-time white noise; $\lambda, \nu \in \mathbb{R}$; $D > 0$; and $h(t,x)$ is a continuous function of time $t \in \mathbb{R}_+$ and space $x \in \mathbb{R}$, taking values in $\mathbb{R}$. Due to the white noise, one expects $x \mapsto h(t,x)$ to be only as regular as in Brownian motion. Hence, the nonlinearity does not a priori make any sense (the derivative of Brownian motion has negative Hölder regularity). Bertini-Cancrini (1995) provided the physically relevant notion of solution (called the *Hopf-Cole solution*) and showed how it arises from regularizing the noise, solving the (now well-posed) equation, and then removing the noise and subtracting a divergence.

The equation contains the four key features mentioned earlier: the growth is local, depending on the Laplacian (smoothing), the square of the gradient (nonlinear slope dependent growth), and white noise (space-time uncorrelated noise). Kardar, Parisi, and Zhang introduced their eponymous equation and 3 : 2 : 1 scaling prediction in 1986 in an attempt to understand the scaling behaviors of random interface growth.

How might one see the 3 : 2 : 1 scaling from the KPZ equation? Define $h_\epsilon(t,x) = \epsilon^b h(\epsilon^{-z}t, \epsilon^{-1}x)$; then h_ϵ satisfies the KPZ equation with scaled coefficients $\epsilon^{2-z}\nu$, $\epsilon^{2-z-b}\frac{1}{2}\lambda$ and $\epsilon^{b-\frac{z}{2}+\frac{1}{2}}\sqrt{D}$. It turns out that two-sided Brownian motion is stationary for the KPZ equation; hence any nontrivial scaling must respect the Brownian scaling of the initial data and thus have $b = 1/2$. Plugging this in, the only way to have no coefficient blow up to infinity and not every term shrink to zero (as $\epsilon \to 0$) is to choose $z = 3/2$. This suggests the plausibility of the 3 : 2 : 1 scaling. While this heuristic gives the right scaling, it does not provide for the scaling limit. The limit as $\epsilon \to 0$ of the equation (the inviscid Burgers equation, where only the nonlinearity survives) certainly does not govern the limit of the solutions. It remains something of a mystery as to exactly how to describe this limiting KPZ fixed point. The above heuristic says nothing of the limiting distribution of the solution to the KPZ equation, and there does not presently exist a simple way to see what this should be.

It took just under twenty-five years until Amir-Corwin-Quastel (2010) rigorously proved that the KPZ equation is in the KPZ universality class.

It took just under twenty-five years until Amir-Corwin-Quastel (2010) rigorously proved that the KPZ equation is in the KPZ universality class. That work also computed an exact formula for the probability distribution of the solution to the KPZ equation, marking the first instance of a non-linear stochastic PDE for which this was accomplished. Tracy-Widom's work on the partially asymmetric corner growth model and work of Bertini-Giacomin (1997) which relates that model to the KPZ equation were the two main inputs in this development. See [4] for further details regarding this as well as the simultaneous

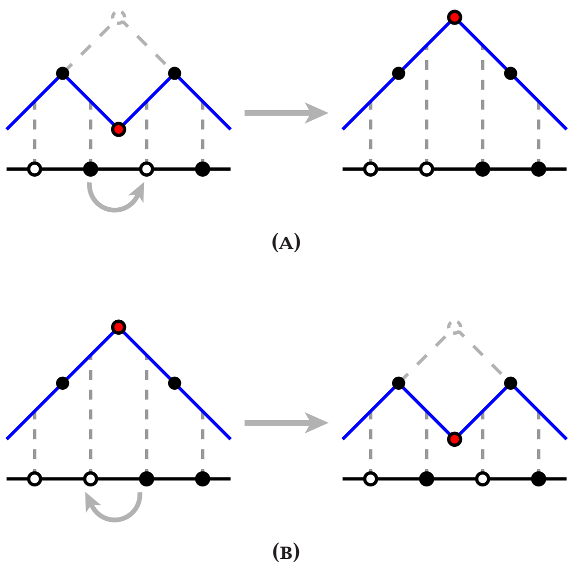

Figure 8. Mapping the partially asymmetric corner growth model to the partially asymmetric simple exclusion process. In (A), the red dot is a local minimum, and it grows into a maximum. In terms of the particle process beneath it, the minimum corresponds to a particle followed by a hole, and the growth corresponds to said particle jumping into the hole to its right. In (B), the opposite is shown. The red dot is a local maximum and shrinks into a minimum. Correspondingly, there is a hole followed by a particle, and the shrinking results in the particle moving into the hole to its left.

exact but nonrigorous steepest descent work of Sasamoto-Spohn (2010) and nonrigorous replica approach work of Calabrese-Le Doussal-Rosso (2010) and Dotsenko (2010).

The proof that the KPZ equation is in the KPZ universality class was part of an ongoing flurry of activity surrounding the KPZ universality class from a number of directions, such as integrable probability [4], experimental physics [10], and stochastic PDEs. For instance, Bertini-Cancrini's Hopf-Cole solution relies upon a trick (the Hopf-Cole transform) which linearizes the KPZ equation. Hairer (2011), who had been developing methods to make sense of classically ill-posed stochastic PDEs, focused on the KPZ equation and developed a direct notion of solution which agreed with the Hopf-Cole one but did not require use of the Hopf-Cole transform trick. Still, this does not say anything about the distribution of solutions or their long-time scaling behaviors. Hairer's KPZ work set the stage for his development of regularity structures in 2013—an approach to construction solutions of certain types of ill-posed stochastic PDEs—work for which he was awarded a Fields Medal.

Interacting Particle Systems

There is a direct mapping (see Figure 8) between the partially asymmetric corner growth model and the *partially asymmetric simple exclusion process* (generally abbreviated ASEP). Associate to every -1 slope line increment a particle on the site of $\mathbb{Z}$ above which the increment sits, and to every $+1$ slope line increment associate an empty site. The height function then maps onto a configuration of particles and holes on $\mathbb{Z}$, with at most one particle per site. When a minimum of the height function becomes a maximum, it corresponds to a particle jumping right by one into an empty site, and likewise when a maximum becomes a minimum, a particle jumps left by one into an empty site. Wedge initial data for corner growth corresponds with having all sites to the left of the origin initially occupied and all to the right empty; this is often called *step initial data* due to the step function in terms of particle density. ASEP was introduced in biology literature in 1968 by MacDonald-Gibbs-Pipkin as a model for RNA's movement during transcription. Soon after it was independently introduced within the probability literature in 1970 by Spitzer.

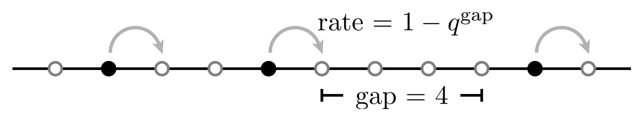

Figure 9. The q-TASEP, whereby each particle jumps one to the right after an exponentially distributed waiting time with rate given by $1 - q^{\mathrm{gap}}$.

The earlier quoted results regarding corner growth immediately imply that the number of particles to cross the origin after a long time t demonstrates KPZ class fluctuation behavior. KPZ universality would have that generic changes to this model should not change the KPZ class fluctuations. Unfortunately, such generic changes destroy the model's integrable structure. There are a few integrable generalizations discovered in the past five years which demonstrate some of the resilience of the KPZ universality class against perturbations.

TASEP (the totally asymmetric version of ASEP) is a very basic model for traffic on a one-lane road in which cars (particles) move forward after exponential rate one waiting times, provided the site is unoccupied. A more realistic model would account for the fact that cars slow down as they approach the one in front. The model of q-TASEP does just that (Figure 9). Particles jump right according to independent exponential waiting times of rate $1 - q^{\mathrm{gap}}$, where gap is the number of empty spaces to the next particle to the right. Here $q \in [0, 1)$ is a different parameter than in the ASEP, though when q goes to zero, these dynamics become those of TASEP.

Another feature one might include in a more realistic traffic model is the cascade effect of braking. The q-pushASEP includes this (Figure 10). Particles still jump right according to q-TASEP rules; however, now particles may also jump left after exponential rate L waiting times. When such a jump occurs, it prompts the next particle to the left to likewise jump left, with a probability given by q^{gap}, where gap is the number of empty spaces between the original particle and its left neighbor. If that jump occurs, it may likewise prompt the next left particle to

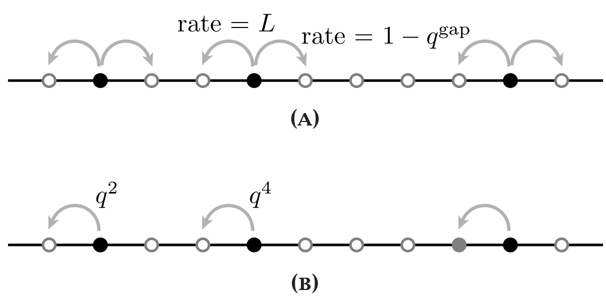

Figure 10. The q-pushASEP. As shown in (A), particles jump right according to the q-TASEP rates and left according to independent exponentially distributed waiting times of rate L. When a left jump occurs, it may trigger a cascade of left jumps. As shown in (B), the rightmost particle has just jumped left by one. The next particle (to its left) instantaneously also jumps left by one with probability given by q^{gap}, where gap is the number of empty sites between the two particles before the left jumps occurred (in this case gap $= 4$). If that next left jump is realized, the cascade continues to the next-left particle according to the same rule; otherwise it stops and no other particles jump left in that instant of time.

jump, and so on. Of course, braking is not the same as jumping backwards; however, if one goes into a moving frame, this left jump is like a deceleration. It turns out that both of these models are solvable via the methods of Macdonald processes as well as stochastic quantum integrable systems, and thusly it has been proved that, just as for ASEP, they demonstrate KPZ class fluctuation behavior (see the review [4]).

Paths in a Random Environment

There is yet another class of probabilistic systems related to the corner growth model. Consider the totally asymmetric version of this model, started from wedge initial data. An alternative way to track the evolving height function is to record the time when a given box is grown. Using the labeling shown in Figure 11, let us call $L(x, y)$ this time, for x, y positive integers. A box (x, y) may grow once its parent blocks $(x-1, y)$ and $(x, y-1)$ have both grown, though even then it must wait for an independent exponential waiting time which we denote by $w_{x,y}$. Thus $L(x, y)$ satisfies the recursion

$$L(x, y) = \max\left(L(x-1, y), L(x, y-1)\right) + w_{x,y}$$

subject to boundary conditions $L(x, 0) \equiv 0$ and $L(0, y) \equiv 0$. Iterating yields

$$L(x, y) = \max_{\pi} \sum_{(i,j)\in\pi} w_{i,j}$$

where the maximum is over all up-right and up-left lattice paths between boxes $(1, 1)$ and (x, y). This model is called *last passage percolation* with exponential weights. Following from the earlier corner growth model results, one readily sees that for any positive real (x, y), for large t, $L(\lfloor xt\rfloor, \lfloor yt\rfloor)$ demonstrates KPZ class fluctuations. A very

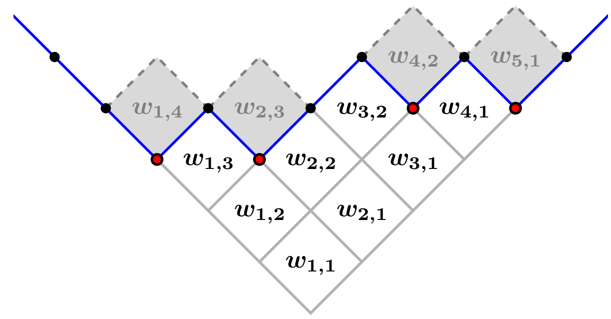

Figure 11. The relation between the corner growth model and last passage percolation with exponential weights. The $w_{i,j}$ are the waiting times between when a box can grow and when it does grow. $L(x, y)$ is the time when box (x, y) grows.

compelling and entirely open problem is to show that this type of behavior persists when the distribution of the $w_{i,j}$ is no longer exponential. The only other solvable case is that of geometric weights. A certain limit of the geometric weights leads to maximizing the number of Poisson points along directed paths. Fixing the total number of points, this becomes equivalent to finding the longest increasing subsequence of a random permutation. The KPZ class behavior for this version of last passage percolation was shown by Baik-Deift-Johansson (1999).

There is another related integrable model which can be thought of as describing the optimal way to cross a large grid with stoplights at intersections. Consider the first quadrant of $\mathbb{Z}^2$ and to every vertex (x, y) assign waiting times to the edges, leaving the vertex rightwards and upwards. With probability $1/2$ the rightward edge has waiting time zero, while the upward edge has waiting time given by an exponential rate 1 random variables; otherwise reverse the situation. The edge waiting time represents the time needed to cross an intersection in the given direction (the walking time between lights has been subtracted). The minimal passage time from $(1, 1)$ to (x, y) is given by

$$P(x, y) = \min_{\pi} \sum_{e\in\pi} w_e,$$

where π goes right or up in each step and ends on the vertical line above (x, y) and w_e is the waiting time for edge $e \in \pi$. From the origin there will always be a path of zero waiting time whose spatial distribution is that of the graph of a simple symmetric random walk. Just following this path one can get very close to the diagonal $x = y$ without waiting. On the other hand, for $x \neq y$, getting to $(\lfloor xt\rfloor, \lfloor yt\rfloor)$ for large t requires some amount of waiting. Barraquand-Corwin (2015) demonstrated that as long as $x \neq y$, $P(\lfloor xt\rfloor, \lfloor yt\rfloor)$ demonstrates KPZ class fluctuations. This should be true when π is restricted to hit exactly (x, y), though that result has not yet been proved. Achieving this optimal passage time requires some level of omnipotence, as you must be able to look forward before choosing your route. As such, it could be considered as a benchmark against which to test various routing algorithms.

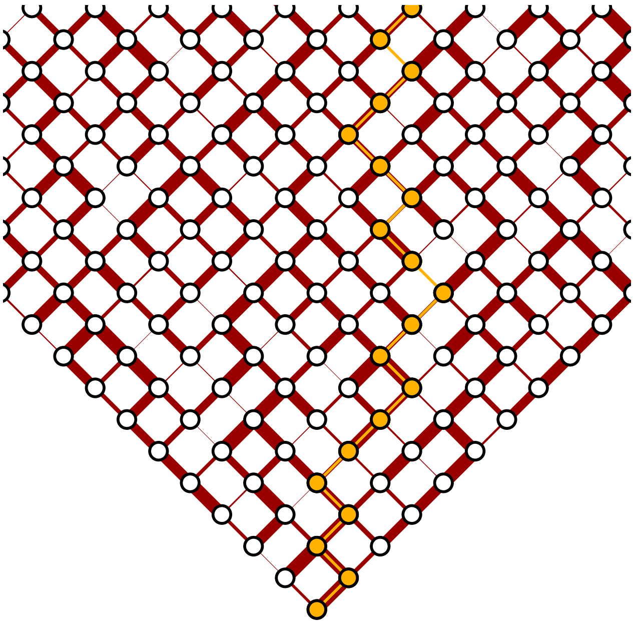

Figure 12. The random walk in a space time random environment. For each pair of up-left and up-right pointing edges leaving a vertex (y,s), the width of the red edges is given by $u_{y,s}$ and $1-u_{y,s}$, where $u_{y,s}$ are independent uniform random variables on the interval $[0,1]$. A walker (the yellow highlighted path) then performs a random walk in this environment, jumping up-left or up-right from a vertex with probability equal to the width of the red edges.

In addition to maximizing or minimizing path problems, the KPZ universality class describing fluctuations of "positive temperature" version of these models in energetic or probabilistic favoritism is assigned to paths based on the sum of space-time random weights along its graph. One such system is called *directed polymers in random environment* and is the detropicalization of LPP where in the definition of $L(x,y)$ one replaces the operations of $(\max,+)$ by $(+,\times)$. Then the resulting (random) quantity is called the *partition function* for the model, and its logarithm (the *free energy*) is conjectured for very general distributions on $w_{i,j}$ to show KPZ class fluctuations. There is one known integrable example of weights for which this has been proved: the inverse-gamma distribution, introduced by Seppäläinen (2009) and proved in the work by Corwin-O'Connell-Seppäläinen-Zygouras (2011) and Borodin-Corwin-Remenik (2012).

The stoplight system discussed above also has a positive temperature lifting, of which we will describe a special case (see Figure 12 for an illustration). For each space-time vertex (y,s) choose a random variable $u_{y,s}$ distributed uniformly on the interval $[0,1]$. Consider a random walk $X(t)$ which starts at $(0,0)$. If the random walk is in position y at time s, then it jumps to position $y-1$ at time $s+1$ with probability $u_{y,s}$ and to position $y+1$ with probability $1-u_{y,s}$. With respect to the same environment of u's, consider N such random walks. The fact that the environment is fixed causes them to follow certain high-probability channels. This type of system is called a *random walk in a space-time random environment*, and the behavior of a single random walker is quite well understood. Let us, instead, consider the maximum of N walkers in the same environment $M(t,N)=\max_{i=1}^{N} X^{(i)}(t)$. For a given environment, it is expected that $M(t,N)$ will localize near a given random environment dependent value. However, as the random environment varies, this localization value does as well in such a way that for $r\in(0,1)$ and large t, $M(t,e^{rt})$ displays KPZ class fluctuations.

Big Problems

It took almost two hundred years from the discovery of the Gaussian distributions to the first proof of their universality (the central limit theorem). So far, KPZ universality has withstood proof for almost three decades and shows no signs of yielding.

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Besides universality, there remain a number of other big problems for which little to no progress has been made. All of the systems and results discussed herein have been $(1+1)$-dimensional, meaning that there is one time dimension and one space dimension. In the context of random growth, it makes perfect sense (and is quite important) to study surface growth $(1+2)$-dimensional. In the isotropic case (where the underly growth mechanism is roughly symmetric with respect to the two spatial dimensions) there are effectively no mathematical results, though numerical simulations suggest that the $1/3$ exponent in the $t^{1/3}$ scaling for corner growth should be replaced by an exponent of roughly .24. In the anisotropic case there have been a few integrable examples discovered which suggest very different (logarithmic scale) fluctuations such as observed by Borodin-Ferrari (2008).

Finally, despite the tremendous success in employing methods of integrable probability to expand and refine the KPZ universality class, there seems to still be quite a lot of room to grow and new integrable structures to employ. Within the physics literature, there are a number of exciting new directions in which the KPZ class has been pushed, including: out-of-equilibrium transform and energy transport with multiple conservation laws, front propagation equations, quantum localization with directed paths, and biostatistics. Equally important is understanding what type of perturbations break out of the KPZ class.

Given all of the rich mathematical predictions, one might hope that experiments have revealed the KPZ class behavior in nature. This is quite a challenge, since determining scaling exponents and limiting fluctuations requires immense numbers of repetition of experiments. However, there have been a few startling experimental confirmations of these behaviors in the context of liquid crystal growth, bacterial colony growth, coffee stains, and fire propagation (see [10] and references therein). Truly, the study of the KPZ universality class demonstrates the unity of mathematics and physics at its best.

Acknowledgments. The author appreciates comments on a draft of this article by A. Borodin, P. Ferrari, and H. Spohn. This text is loosely based on his "Mathematic Park" lecture entitled "Universal phenomena in random systems", delivered at the Institut Henri Poincaré in May 2015. The author is partially supported by NSF grant DMS-1208998, by a Clay Research Fellowship, by the Poincaré chair, and by a Packard Fellowship for Science and Engineering.

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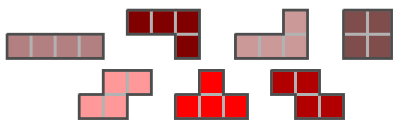
Photo courtesy of Craig Tracy.

Ivan Corwin

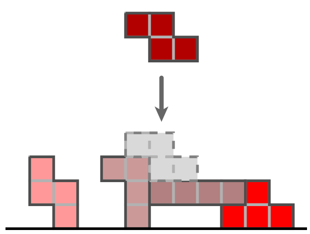
About the Cover

TETRIS® Statistics

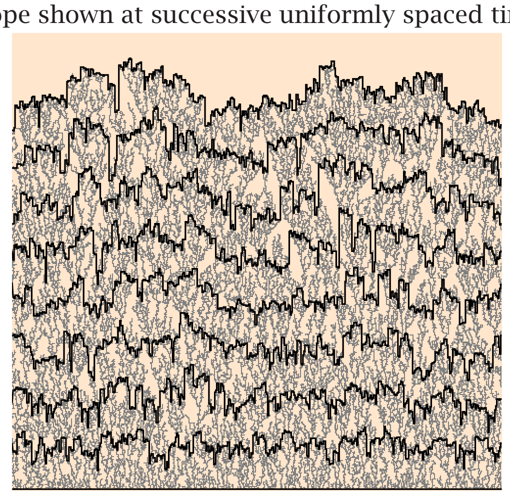
The theme of the cover is a mathematical version of the familiar game TETRIS®. The basic objects are the 7 oriented polyominoes of size 4, which are called tetrominoes.



At each moment a random tetromino, oriented randomly, descends to a random horizontal location and sticks on to the lowest point at which it will not overlap with the current configuration.



As time goes on these pile up, albeit rather loosely. The following figure illustrates a typical run, with the top envelope shown at successive uniformly spaced times:



Mathematical questions arise immediately. For example, *how does the interface at the top of the configuration grow?* Conjecturally, the answer to this question is related to the Kardar-Parisi-Zhang (KPZ) statistics discussed in Ivan Corwin's article in this issue. Corwin writes,

"The first natural conjecture is that the envelope proceeds to first order with a constant velocity, whose value is the same for all runs. To make a precise formulation, assume the boundary to be periodic, say of width W. Drop W tetrominoes in each unit of time, at a constant rate. With this normalization, does the envelope grow linearly with the same velocity for all widths and almost all runs? If so, this would be an analogue of the law of large numbers. In fact, if one drops unit squares instead of tetrominoes then each stack grows independently of other stacks, and linear growth at a unit rate follows exactly from the law of large numbers. I call this the random deposition model. Such a prediction has been verified for at least one of the models referred to in my article. (Finding exactly what that velocity is, however, seems impossible.)

Despite the overall linear growth of the envelope height, it fluctuates significantly, and the size of fluctuations seems to grow over time. What order of magnitude is this growth as a function of time? For example, if we were dropping unit squares the fluctuation would be $O(\sqrt{t})$. But this is where the KPZ universality class predictions come into play. If $\sigma(t)$ is the standard deviation of the envelope from its mean, then it is expected that

$$\sigma(t) \approx \begin{cases} t^{1/3} & t \ll W^{3/2} \\ W^{1/2} & t \gg W^{3/2} \end{cases}$$

with a crossover occurring on the time scale $W^{3/2}$. Here, we assume also that W is very large, and the $\approx$ symbol means up to constants (whose values are not known). This prediction could be checked numerically by running random TETRIS many times and for many different choices of time so as to estimate the curve $\sigma(t)$. A log-log plot would then be best suited then to reveal whether the prediction holds.

There are two very striking features of this prediction. The first is that the fluctuations of the envelope grow like $t^{1/3}$, unlike in the case of the random deposition model where they would grow like $t^{1/2}$. The intuitive reason for this is that in TETRIS attachment is not independent, and there is some evident correlation in the envelope. In TETRIS, if one section gets high, then nearby sections tend to catch up quickly because pieces latch on and spread outwardly. In fact, it is exactly this spatial correlation which leads to the second striking feature of this prediction—the saturation of the standard deviation on the time scale $O(W^{3/2})$. The reason for this is that the KPZ prediction holds that spatial correlations grow over time like $t^{2/3}$. This means that in time $t = W^{3/2}$, the correlations in space will be of order $t^{2/3} = O(W)$. Therefore, the system will feel its finiteness and will approach a steady state after which the height function fluctuations cannot continue to grow.

There is more to the KPZ prediction. In particular, using connections to integrable systems, the exact statistical distributions describing the fluctuations of the height function have been predicted (and proved in some very simple analogous models). Confirming this finer scale distributional behavior is more subtle."

—Bill Casselman
Graphics Editor
notices-covers@ams.org

Alexandre Grothendieck 1928–2014, Part 1

Michael Artin, Allyn Jackson, David Mumford, and John Tate, Coordinating Editors

In the eyes of many, Alexandre Grothendieck was the most original and most powerful mathematician of the twentieth century. He was also a man with many other passions who did all things his own way, without regard to convention.

This is the first part of a two-part obituary; the second part will appear in the April 2016 *Notices*. The obituary begins here with a brief sketch of Grothendieck's life, followed by a description of some of his most outstanding work in mathematics. After that, and continuing into the April issue, comes a set of reminiscences by some of the many mathematicians who knew Grothendieck and were influenced by him.

Biographical Sketch

Alexandre Grothendieck was born on March 28, 1928, in Berlin. His father, a Russian Jew named Alexander Shapiro, was a militant anarchist who devoted his life to the struggle against oppression by the powerful. His mother, Hanka Grothendieck, came from a Lutheran family in Hamburg and was a talented writer. The upheavals of World War II, as well as the idealistic paths chosen by his parents, marked his early childhood with dislocation and deprivation. When he was five years old, his mother left him with a family she knew in Hamburg, where he remained until age eleven. He was then reunited with his parents in France, but before long his father was deported to Auschwitz and perished there.

By the war's end the young Alexandre and his mother were living in Montpellier, where he was able to attend the university. In 1948 he made contact with leading mathematicians in Paris, who recognized both his brilliance and his meager background. A year later, on the advice of Henri Cartan and André Weil, he went to the Université Nancy, where he solved several outstanding problems in the area of topological vector spaces. He earned his doctoral degree in 1953, under the direction of Laurent Schwartz and Jean Dieudonné.

Grothendieck as a child.

Because Grothendieck was stateless at the time, obtaining a regular position in France was difficult. He held visiting positions in Brazil and the United States before returning to France in 1956, where he obtained a position in the Centre National de la Recherche Scientifique (CNRS). In 1958, at the International Congress of Mathematicians in Edinburgh, he gave an invited address that proved to be a prescient outline of many of the mathematical themes that would occupy him in the coming years.

That same year he was approached by a French mathematician businessman, Léon Motchane, who planned to launch a new research institute. This was the start of the Institut des Hautes Études Scientifiques (IHES), now located in Bures sur Yvette, just outside Paris. Grothendieck and

Michael Artin is professor of mathematics at the Massachusetts Institute of Technology. His email address is artin@math.mit.edu.

Allyn Jackson is deputy editor of the Notices. *Her email address is* axj@ams.org.

David Mumford is professor of mathematics at Brown University. His email address is David_Mumford@brown.edu.

John Tate is professor of mathematics at Harvard University and University of Texas, Austin. His email address is tate@math.utexas.edu.

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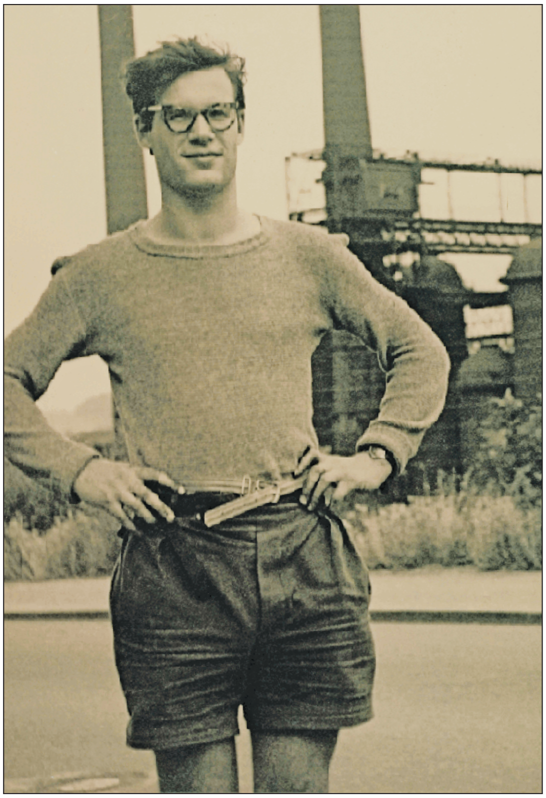

Photo courtesy of Paulo Rihenhoim.

Grothendieck as a student in Nancy, 1950.

Dieudonné were the institute's first two professors. While he was at the IHES, Grothendieck devoted himself completely to mathematics, running a now-legendary seminar and collecting around him a dedicated group of students and colleagues who helped carry out his extraordinary mathematical ideas. Much of the resulting work from this era is contained in two foundational series, known by the acronyms EGA and SGA: *Éléments de Géométrie Algébrique* and *Séminaire de Géométrie Algébrique du Bois Marie*.

In 1970 Grothendieck abruptly resigned from the IHES and changed his life completely. The reasons for this change are complex and difficult to summarize, but it is clear that he was deeply affected by the student unrest that seized France in 1968 and became convinced that he should focus his energy on pressing social issues, such as environmental degradation and the proliferation of weapons. He began to lecture on these subjects and founded an international group called Survivre et Vivre (called simply Survival in English). While this effort was not a political success, Grothendieck did have, at the grassroots level, a significant influence on others sharing his concerns. After his death leaders in the "back to the land" movement wrote tributes to him. He briefly held positions at the Collège de France and the Université de Paris Orsay before leaving Paris in 1973. He then took a position at the Université de Montpellier and lived in the French countryside.

Photo by Paul R. Halmos. Paul R. Halmos Photograph Collection, e_ph_0113_01, Dolph Briscoe Center for American History, University of Texas at Austin.

Grothendieck in Chicago, around 1955.

In 1984 Grothendieck applied to the CNRS for a research position. His application consisted of his now-famous manuscript *Esquisse d'un Programme (Sketch of a Program)*, which contained the seeds for many new mathematical ideas subsequently developed by others. This marked his first public foray into mathematics after his break with the IHES, but not his last. While he never again returned to producing mathematics in a formal, theorem-and-proof style, he went on to write several unpublished manuscripts that had deep influence on the field, in particular *La Longue Marche à Travers la Théorie de Galois (The Long March through Galois Theory)* and *Pursuing Stacks*.

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Leila Schneps, editor, *Alexandre Grothendieck: A Mathematical Portrait*. International Press, 2014.


Photo by Erika Ifang.

Grothendieck in 1988.

With his CNRS position he remained attached to the Université de Montpellier but no longer taught. From 1983 to 1986 he wrote another widely circulated piece, *Récoltes et Semailles (Reaping and Sowing)*, which is in part an analysis of his time as a mandarin of the mathematical world. *Récoltes et Semailles* became notorious for its harsh attacks on his former colleagues and students.

Grothendieck's severance from the mathematical community meant that he received far fewer prizes and awards compared to other mathematicians of his stature. He received the Fields Medal in 1966 while he was still at the IHES and still active in mathematics. Much later, in 1988, he and Pierre Deligne were awarded the Crafoord Prize from the Royal Swedish Academy of Sciences; Grothendieck declined to accept it.

Grothendieck retired in 1988. He devoted himself to his writing, which focused increasingly on spiritual themes. Around this time he had episodes of deep psychological trauma. In 1991 he went to live in complete isolation in Lasserre, a small village in the French Pyrénées, where he continued to write prodigiously. When he died on November 13, 2014, he left behind thousands of pages of writings.

Grothendieck's Mathematical Work

The greatest accomplishments in Grothendieck's mathematical life were in algebraic geometry and took place in a twelve-year period of the most intense concentration from roughly 1956 to 1968. Before this he had done major work in functional analysis in the period 1950–54, and later, at Montpellier, he worked on many ideas, some of which are summarized in his *Esquisse d'un programme* but which remain mostly unpublished. To cover all this work would require many experts, and in this review we will only sketch what we believe to be his four most outstanding contributions to algebraic geometry. What is most stunning is that in each of them he created a major new abstract theory that then led to the solution of a major problem in algebraic geometry as it stood when he started. Thus we omit many important parts of his work, notably the early work on topological vector spaces, then the theories of duality, flat descent, crystalline cohomology, motives, and topoi done at the IHES, and finally his "dessin d'enfants" and much more from the Montpellier period.

K-theory and the Grothendieck-Riemann-Roch Theorem for Morphisms $f : X \to Y$

The first stunning innovation of Grothendieck was his generalization of the Riemann-Roch theorem that he proved in 1956. In 1954 Hirzebruch had generalized the classical Riemann-Roch theorem for curves and surfaces. His theorem calculated the Euler characteristic of any vector bundle $\mathbb{E}$ on a smooth projective variety X over $\mathbb{C}$ in terms of the Chern classes of $\mathbb{E}$ and of the tangent bundle of X. Following his philosophy that theorems will always fall out naturally when the appropriate level of generality is found, Grothendieck did three things:

(a) he replaced the bundle by an arbitrary coherent sheaf,

(b) he replaced the smooth complex variety X by a proper morphism $f : X \to Y$ between smooth quasi-projective varieties over any field, and

(c) he defined a group $K(X)$ and treated the sheaf as a member of this group.

What then is $K(X)$? It is the free abelian group generated by elements $[A]$, one for each coherent sheaf A, with the relation $[B] - [A] - [C] = 0$ for all exact sequences $0 \to A \to B \to C \to 0$. This seemingly simple definition has led to the development of the major field known as K-theory.

An element $[A]$ of $K(X)$ can be viewed as an abstract "Euler characteristic". Thus, using the higher direct images of the sheaf with respect to the morphism f, the classical Euler characteristic $\sum_k (-1)^k \dim H^k(X, \mathbb{E}) \in \mathbb{Z}$ is replaced by $\sum (-1)^k [R^k f_*(\mathcal{F})]$ in $K(Y)$. The amazing power of treating all sheaves and all morphisms and not just vector bundles on a fixed variety is that, by playing with compositions, products and blow-ups, the result for a general morphism f can be reduced to two cases: the injection of a smooth codimension-one subvariety X into Y and the projection from $\mathbb{P}^n$ to a point. The value of using the K group appears because Hilbert's syzyzgy theorem shows immediately that $K(\mathbb{P}^n)$ is generated by the powers of its basic line bundle $\mathcal{O}(1)$.

Formal Schemes, Nilpotents and the Fundamental Group

The problem of describing the fundamental group of a curve in characteristic p had attracted a lot of attention in the 1950s, and this was the next major problem in algebraic geometry on which Grothendieck made huge progress. To make this progress, he required schemes that went beyond varieties in two essential ways: schemes with nilpotents and schemes of mixed characteristic. This application showed clearly that schemes were the correct setting in which to do algebraic geometry.

In algebraic geometry paths cannot be defined algebraically, so the fundamental group is described in terms of finite coverings. It was known that abelian coverings of degree prime to the characteristic behave in the same way as in characteristic zero, and though not the same as in characteristic zero, coverings of degree p were understood. The nonabelian coverings were a complete mystery. Grothendieck proved a stunning theorem, that the Galois coverings of degree prime to the characteristic are the same as those in characteristic zero and that the fundamental group of a curve in characteristic p is a quotient of the group in characteristic zero. The techniques that he developed for the proof seem amazing still today.

Grothendieck first discovered that if two schemes are given by structure sheaves on the same underlying space, differing only in their nilpotent ideals, they have the same fundamental group. To apply this observation, he considered what Weil had called a "specialization" of a characteristic-zero variety to characteristic p. Suppose, for instance, that a family X of curves is given over the p-adic integers $\mathbb{Z}_p$. Then the fibre X_0 obtained by working modulo p will be a curve over the prime field $\mathbb{F}_p$, and the fibre X_η over the p-adic field $\mathbb{Q}_p$ will have characteristic zero. In this situation one can also consider the scheme X_n obtained by working modulo p^{n+1}, a family of curves over the ring $\mathbb{Z}/p^{n+1}\mathbb{Z}$. The schemes X_n form a sequence $X_0 \subset X_1 \subset \cdots$, and they differ only in their nilpotent elements. So if a covering of X_0 is given, one can extend it to every X_n.

This approach was revolutionary, though nothing technically difficult was needed up to this point. Grothendieck's biggest step was to go from a family of coverings of the sequence $\{X_n\}$ to a covering of the scheme X itself. Once this was done, standard methods related the covering of the curve X_0 in characteristic p to a covering of the characteristic-zero curve X_η.

It was while studying this last step that Grothendieck found a key Existence Theorem. To state that theorem, we begin with a scheme X projective (or proper) over a complete local ring R. It might be a curve over the ring of p-adic integers. Let R_n denote the truncation of R modulo a power of the maximal ideal, and let X_n be the corresponding truncation of X. The schemes X_n form a sequence $X_0 \subset X_1 \subset \cdots$ that Grothendieck calls a *formal scheme*. Given a coherent sheaf M on X, one obtains a sequence of coherent sheaves $\cdots \longrightarrow M_1 \longrightarrow M_0$ on the schemes X_n by truncation: $M_n = M \otimes_R R_n$. The Grothendieck Existence Theorem allows one to go the other way. It states that there is an equivalence of categories between coherent sheaves M on X and sequences of sheaves M_n on X_n such that $M_{n-1} = M_n \otimes_{R_n} R_{n-1}$. Grothendieck then stated the covering problem in terms of coherent sheaves and was able to complete his proof.

Grothendieck's Existence Theorem is a cornerstone of modern algebraic geometry, and the categorical properties that are necessary for a theorem of that type are still not understood.

Functors and the Hilbert, Picard, and Moduli Schemes

Prior to Grothendieck's work, both Weil and Zariski had struggled with deciding what should be called the *points* of a variety when it was defined over a nonalgebraically closed ground field k: should these be the maximal ideals in their affine coordinate rings, or should they be the solutions of the defining equations in the algebraic closure $\bar{k}$? And they needed some concept of *generic points*; they were first defined by van der Waerden in his classical series of papers on algebraic geometry, in the same way as Weil and Zariski. This confusion disappeared when Grothendieck took the radical step of defining two sorts of points on a scheme X: on the one hand, all *prime* ideals in the affine coordinate rings of X became the points of the scheme, but on the other, morphisms from any scheme S to X were called *S-valued points* of X. What was traditionally thought to be the underlying point set is the case that $S = \mathrm{Spec}(\bar{k})$. If $X(S)$ is the set of S-valued points of X and $S \to T$ is a morphism, composition defines a map from $X(T)$ to $X(S)$. Thus $S \mapsto X(S)$ is a *functor* from the category of schemes to the category of sets.

Grothendieck introduced the term *representable functor*, a functor that is isomorphic to $\mathrm{Hom}(\cdot, X)$ for some object X. Moreover, he insisted on the systematic use of fibred products, using them to define the concept of relative representability. A morphism of functors $F \longrightarrow G$ is relatively representable if, given a morphism $\mathrm{Hom}(\cdot, X) \longrightarrow G$, i.e., an element of $G(X)$, the fibred product $X \times_G F$ is representable. For example, F is an open subfunctor of G if for every such morphism, the fibred product is represented by an open subset of X.

There had been substantial work at this time defining varieties parametrizing certain structures; that is, their points were in one-to-one correspondence with the set of all such structures. Chow had defined a union of varieties whose points parametrize subvarieties of projective space of given degree and dimension, Weil had defined varieties whose points parametrize divisor classes of degree zero on a curve, and Baily had defined a variety whose points correspond to isomorphism classes of curves of fixed genus over the complex numbers. Grothendieck immediately realized that in each of these constructions, one should look for a suitable representable functor. Instead of Chow's formulation, he considered *subschemes* of a given projective space $\mathbb{P}^n$ with fixed Hilbert polynomial P, made it into a functor by looking at flat families of subschemes, and proved that this functor was represented by a scheme that he named the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$. He described these ideas in a series of Bourbaki talks in 1959–62 and in a seminar at Harvard in 1961.

Once again, recasting old problems in their natural more abstract settings solved old problems. Going back to the first decades of the twentieth century, a central problem in the theory of algebraic surfaces F over the complex numbers had been showing that the *irregularity* that we now call $\dim H^1(\mathcal{O}_F)$ was the dimension of the Picard variety that classifies topologically trivial divisor classes. This had been proven by complex analytic methods by Poincaré, but despite multiple attempts by Enriques and Severi, had not been proven algebraically. Grothendieck's approach was to define a Picard *scheme* whose S-valued points correspond to the set of line bundles[1] on $F \times S$. Taking $S = \mathrm{Spec}\, k[x]/(x^2)$, he saw that $H^1(\mathcal{O}_F)$ was the tangent space to the Picard scheme at the origin. Thus the old problem became: show that the Picard scheme is reduced, i.e. has no nilpotent elements in its structure sheaf. But the Picard scheme is a group, and in characteristic zero algebraic groups have an exponential map, hence no nilpotents. In characteristic p this need not be true, and life is richer.

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[1] *Technical point: the line bundles should be trivialized on* $\{x\} \times S$ *for some rational point* $x \in F$.

In the case of moduli spaces, the functorial approach first solved their local structure using the idea of *prorepresenting* a functor, F: fix an element a of $F(\mathrm{Spec}(k))$ and seek a complete local ring whose Spec defines the subfunctor of F of all nilpotent extensions of a. Criteria for prorepresentability were established by Grothendieck, Lichtenbaum, and Schlessinger, and for moduli in particular this led to the concept of the cotangent complex due to Grothendieck and Illusie.

The global theory of the moduli space, however, went in two directions. One sought quasi-projective moduli schemes and was pursued by Mumford. Grothendieck's idea, however, was to find simple general properties of a functor that characterized those that were representable, solving special cases like moduli as a corollary. But Hironaka found a simple 3-dimensional scheme with an involution whose quotient by this involution fails to be a scheme; hence schemes themselves need to be further generalized if there is to be a nice characterization of the functors they represent. This led to the concept of an *algebraic space*, a more general type of object. A remarkable "approximation" theorem discovered by M. Artin in 1969 led to his characterization of the functors represented by these spaces in 1971, fully vindicating Grothendieck's vision.

Étale Cohomology

Interest in a cohomology theory for varieties in characteristic p was stimulated by André Weil's talk in 1954 at the International Congress of Mathematicians (see also his earlier paper "Numbers of Solutions of Equations in Finite Fields" (*AMS Bulletin*, vol. 55 (1949), 497–508)). In this talk, he compared analytic and algebraic methods in algebraic geometry. The problem of defining cohomology algebraically hadn't attracted much interest before, because the classical topology was available for varieties over the complex numbers. But the culmination of Weil's talk was his explanation that, because rational points on a variety V over a finite field were the fixed points of a Frobenius automorphism, one might be able to count them by the Lefschetz Fixed Point Formula, which asserts that the number of fixed points of an automorphism φ is equal to the alternating sum $\sum_i (-1)^i \mathrm{Trace}_{H^i(V)}(\varphi^*)$ of traces of the maps induced by φ on the cohomology. However, a definition of the cohomology groups was required, and the Zariski topology was useless for this. That a definition should exist with the properties Weil predicted became known as the Weil Conjectures.

There was no problem with cohomology in dimension 1, because $H^1(V, \mathbb{Z}/n)$ can be constructed from the group of n-torsion divisor classes. Therefore the cohomology of curves was understood. In fact, Weil's conjectures were based on the known case of curves, for which the zeta function had been analyzed and for which the analogue of the Riemann Hypothesis had been proved by E. Artin, H. Hasse, and Weil himself.

Grothendieck's idea for defining cohomology was to replace open sets of a topology by unramified coverings of Zariski open sets. There were some hints that this might work. Previously, Serre had defined what he called *local isotriviality*. A bundle B over a variety X is locally isotrivial if for every point p of X there is a finite covering U' of a Zariski open neighborhood U of p such that the pullback of B to U' is trivial. Moreover, Kawada and Tate had shown that one could recover the cohomology groups of a curve in terms of the cohomology of its fundamental group.

M. Artin took up this idea in 1961 when Grothendieck visited Harvard. Using unramified coverings that were not finite, i.e. all étale maps, he succeeded in showing that, over the complex numbers, one did indeed obtain the same cohomology with torsion coefficients as with the classical topology. In retrospect, the étale topology was a natural thing to try, since it is stronger than the Zariski topology and weaker than the classical topology. It wasn't at all obvious at the time, because the étale topology isn't a topology in the usual sense. Open sets are replaced by étale maps, which aren't mapped injectively to the base space. The thought that one could do sheaf theory in such a setting was novel. And one needs to work with torsion coefficients to have a reasonable theory. Cohomology with nontorsion coefficients, which is needed for the Fixed Point Theorem, is defined by an inverse limit as ℓ-adic cohomology.

Then Grothendieck proved a series of theorems, notably the Proper Base Change Theorem, which allows one to control the cohomology of varieties by induction on the dimension, using successive fibrations and beginning with the known case of dimension 1. The Proper Base Change Theorem concerns a proper map $X \longrightarrow S$ and a point s of S. The theorem asserts that the cohomology of the fibre X_s over s, $H^q(X_s, A)$, is isomorphic to the limit of the cohomology $H^q(X', A)$ of pullbacks X' of X to the étale neighborhoods S' of s. To prove the theorem, Grothendieck adapted a method that had been introduced by Serre. Artin, Grothendieck, and Verdier developed the full theory jointly at the IHES in 1963–64.

Grothendieck then defined L-series for cohomology of arbitrary constructible sheaves. This allowed him in 1964 to prove rationality of L-series and to find a functional equation, using the Base Change Theorem and Verdier's duality theorem to reduce to the case of dimension 1. The Riemann Hypothesis for varieties over finite fields was proved by Deligne in 1974.

Michael Atiyah

Grothendieck As I Knew Him

My first encounter with the whirlwind that was Grothendieck occurred at the very first, and very small, Bonn Arbeitstagung in July 1957. I have vivid memories of Grothendieck talking for hours every day, expounding his new K-theory generalization of the Hirzebruch-Riemann-Roch Theorem (HRR). According to Don Zagier, Arbeitstagung records show that he spoke for a total of twelve hours spread over four days. It was

Michael Atiyah is Honorary Professor at the University of Edinburgh and Fellow of Trinity College, Cambridge. His email address is M.Atiyah@ed.ac.uk.

an exhilarating experience: brilliant ideas, delivered with verve and conviction. Fortunately I was young at the time, almost exactly the same age as Grothendieck, and so able to absorb and eventually utilize his great work.

In retrospect we can see that he was the right man at the right time. Serre had laid the new foundations of algebraic geometry, including sheaf cohomology, and Hirzebruch had developed the full cohomological formalism based on the Chern classes, which he and Borel had streamlined. To many it seemed that HRR was the culmination of centuries of algebraic geometry, the pinnacle of the subject. But Grothendieck, standing on the shoulders of Bourbaki, looked to the future, where abstract structural ideas of universality and functoriality would become dominant. His introduction and development of K-theory rested on his mastery of homological algebra and his technical virtuosity, which steamrollered its way through where mere mortals feared to tread. The outcome, the Grothendieck-Riemann-Roch Theorem, was a brilliant functorialization of HRR, which reduced the proof to an exercise left to Borel and Serre!

Grothendieck, standing on the shoulders of Bourbaki, looked to the future…

This great triumph, following his earlier work in functional analysis, established Grothendieck as a mathematician and led to his receiving a Fields Medal (protesting the Soviet regime, he famously did not attend the 1966 Moscow Congress, where the medal was awarded). His new philosophy attracted a host of disciples, who together developed grand new ideas beyond my powers to describe.

For me personally his K-theory, together with more topological ideas germinating in subsequent Arbeitstagungen, led in the end to topological K-theory as developed by Hirzebruch and me, resting on the famous Bott periodicity theorems. Subsequently, through ideas of Quillen and others, algebraic K-theory emerged as a major framework that linked topology, algebraic geometry, and number theory in a deep and beautiful way with great promise and daunting problems for the future. This is part of the Grothendieck legacy.

The first Arbeitstagung also had an educational aspect for me. At the Institute for Advanced Study in Princeton in the fall of 1959, Saturday mornings were devoted to a detailed technical seminar run by Borel, Serre, and Tate, which expounded the algebraic foundations of schemes à la Grothendieck. I was a diligent student and learned enough commutative algebra to deliver a short course of lectures in Oxford, which ended up as my joint textbook with Ian Macdonald. It was not quite a best seller, but read by students worldwide, mainly because of its slim size and affordability. It also gave the mistaken impression that I was an expert on commutative algebra, and I still get emails asking me tricky questions on the subject!

Photo courtesy of the estate of Friedrich Hirzebruch.

Grothendieck (left) and Michael Atiyah, on a boat trip during the Arbeitstagung, around 1961.

I continued to meet Grothendieck frequently in subsequent years in Bonn, Paris, and elsewhere, and we had friendly relations. He liked one of my early papers, which derived Chern classes in a sheaf-theoretic framework, based on what became known as "the Atiyah class". On the other hand, he rather dismissed the Atiyah-Bott fixed point formula, which led to the Hermann Weyl formula for the characters of representations of compact Lie groups, as a routine consequence of his general theories. Technically he was right, but neither he nor anyone else had ever made the connection with the Weyl formula.

These two reactions to my own work are illuminating. He was impressed by my early paper because it was not part of his general theory, but the Atiyah-Bott result, which I consider much more significant, was only part of his big machine and hence not surprising or interesting.

There are two episodes in my memory that deserve to be recorded. The first occurred on one of the famous boat trips on the Rhine, which were central to the Bonn Arbeitstagung. Grothendieck and I were sitting together on a bench on the upper deck, and he had his feet up on the opposite bench. A sailor came up and told him, quite reasonably, to take his feet off the bench. Grothendieck literally dug his heels in and refused. The sailor returned with a senior officer who repeated the request, but Grothendieck again refused. This process then escalated right to the top. The captain came and threatened to return the boat to harbour, and it took all Hirzebruch's diplomatic skills to prevent a major international incident. This story shows how uncompromising Grothendieck could be in his personal life and parallels I think his uncompromising attitude in mathematics. The difference is that in mathematics he was, in the main, successful, but in the real world his uncompromising nature led inevitably to disaster and tragedy.

My second personal recollection is of Grothendieck confiding to me that, when he was forty, he would quit mathematics and become a businessman. He sounded quite serious, though I took it with a grain of salt. In fact he did essentially leave mathematics around that age, and he became an unconventional businessman, operating not in the narrow mercantile world but, as befitted such a visionary man, on the grand scale of world affairs. Unfortunately the talent that had stood him in good stead in the academic world of mathematics was totally inadequate or inappropriate in the broader world. The compromises that make politics the "art of the possible" were anathema to Grothendieck.

He was a tragic figure in the Shakespearian mould, the hero who is undone by his own internal failings. The very characteristics that made Grothendieck a great mathematician, with enormous influence, were also those that unfitted him for the very different role that he chose for himself in later life.

Hyman Bass

Bearing Witness to Grothendieck

Grothendieck had a big, but mostly indirect, impact on my mathematical life. I had only limited personal contact with him, but during the late 1960s I was a fairly close witness to the fundamental transformation of algebraic geometry that he led and inspired. He was a visionary, bigger-than-life figure. Though prodigiously creative, his massive agenda needed distributed effort, and his stable enlisted some of the best young mathematical talent in France—Verdier, Raynaud, Illusie, Demazure,…—with whom I had closer contact. My main intermediary and mentor in that environment was Serre, another universal mathematician but of a totally different style and accessibility. If Serre was a Mozart, Grothendieck was a Wagner. Serre seemed to know the most significant and strategic problems to be addressed across a broad expanse of mathematics, and he had an uncanny sense of exactly where to productively direct the attention of other individual mathematicians, of whatever stature, myself included.

The Grothendieck seminar at IHES, though small in numbers, was intense, almost operatic. On one occasion, Cartier was presenting and struggling with Grothendieck's questioning of the proof of a lemma. At one point, Grothendieck said, "Si tu n'as pas ça, tu n'as rien!" I remember feeling that the events of this period were an important human as well as mathematical story and that it was sad that there was no historian with the technical competence to capture its intellectual and human dimensions in depth.

Grothendieck's influence on my own work began with the exposition, by Borel and Serre, of Grothendieck's proof of his generalized Riemann-Roch Theorem. This seminal paper sowed the birth of both topological (Atiyah and Hirzebruch) and algebraic K-theory. The latter occupied more than two decades of my ensuing work, mostly at the periphery of the Grothendieck revolution.

Hyman Bass is a professor in the School of Education and the Samuel Eilenberg Distinguished University Professor in the Department of Mathematics at the University of Michigan. His email address is hybass@umich.edu.

Pierre Cartier

Some Youth Recollections about Grothendieck

The scientific birth of Grothendieck occurred in October 1948 at age twenty. After getting his *licence* degree (equivalent to a BS) from the University of Montpellier, he obtained a fellowship for doctoral studies in Paris. This year was the beginning of the famous Cartan Seminar. Grothendieck attended it but was not really attracted. He then moved to Nancy to begin his work on functional analysis, leading to his famous thesis.

My scientific birth occurred in October 1950, when I was accepted as a student at the École Normale Supérieure. I was really eager to learn everything, and there I started a lifelong interest in algebraic topology and homological algebra, joined with a lasting friendship with H. Cartan and S. Eilenberg.

During this time, Grothendieck's fame at Nancy developed rapidly, and even in Paris (!!) we took notice of it. I don't remember exactly when he and I met for the first time, probably around 1953, at the occasion of some Bourbaki Seminar.[2] My first acquaintance with his work came through L. Schwartz. When Schwartz left Nancy for Paris, we had another mathematical father (the first one was H. Cartan). He was very famous for his invention of "distributions" and taught functional analysis to an enthusiastic following (J.-L. Lions, B. Malgrange, A. Martineau, F. Bruhat, me). His first seminar in Paris was devoted to Grothendieck's thesis, and I participated actively, taking a special interest in the "theorem of kernels" and the topological version of Künneth's theorem. Two rather unexpected developments came from Grothendieck's thesis. First, in France, there was a fruitful collaboration between H. Cartan, J.-P. Serre, and L. Schwartz using deep analytical methods to put the finishing touch on the cohomology theory of complex-analytic functions. Then, on the other side, Gelfand, in the then-Soviet Union, used topological tensor products and nuclear spaces for applications to probability theory (Minlos's theorem and random distributions) and mathematical physics (quantum field theory). It would be interesting to trace the transition in Grothendieck's work from functional analysis to algebraic geometry. I plan to develop this some day, but this is not the proper place.

The period in which we were very close is approximately from 1955 to 1961, and there Bourbaki plays a major role. I vividly remember one of our first encounters, which took place at the Institut Henri Poincaré. It was in March 1955 at the Bourbaki seminar after a special lecture that Grothendieck gave about convexity inequalities. He told me: "Very soon, both of us will join Bourbaki." I began

Pierre Cartier is an emeritus research professor at the Centre National de la Recherche Scientifique, a visitor at the Institut des Hautes Études Scientifiques, and an associate member of the University of Paris-Diderot. His email address is cartier@ihes.fr.

[2] *At the time, this was the general meeting, three times a year, of all French mathematicians. The French Mathematical Society was then a charming sleeping old lady!*

regularly attending Bourbaki meetings in June 1955. Grothendieck joined soon and participated actively from 1956 to 1960. In June 1955 one of the most interesting pieces to read during our meeting was a first draft of his famous *Tôhoku* paper, where he gives a new birth to homological algebra. One of the major challenges at the time, especially after the appearance of Serre's paper "Faisceaux algébriques cohérents" in 1955, was to devise a theory of sheaf cohomology valid for the most general topological spaces (especially not Hausdorff and not locally compact). What was required was the construction of an injective resolution, but no one knew how to make it for sheaves.[3] In what later became his favorite method, Grothendieck solved the problem from above: looking for the axiomatic properties required of a category to admit injective resolutions, and then checking that the category of sheaves on the most general topological space satisfied these properties.

Let us come back to Bourbaki. There was a turning point, a change of generation. The so-called first part (in six series) was devoted to the foundations, basing everything on set theory and the pervasive notions of isomorphisms and structures. At that time, the publication of this first part was well under way,[4] but what should come after? Among many other projects, it was felt that geometry—both differential geometry, heritage of Elie Cartan,[5] and algebraic geometry, dear to Chevalley, Lang, Samuel, Serre, and Weil—was a cornerstone. We wanted to give a unified presentation of all kinds of manifolds, and there were three competing proposals: ringed spaces (Cartan, Serre), local categories of "charts" (Eilenberg), and a more algebraic version of differential calculus (Weil, Godement, Grothendieck). None was finally accepted, but Grothendieck used them all in his theory of schemes.

Let me add a few personal recollections. All these summer meetings of Bourbaki took place in the Alps, first near Die[6] in Etablissement Thermo-résineux de Salières-les-Bains, a kind of elaborate sauna, then in Pelvoux-le-Poët, in a quiet inn in the mountains. In Die, I remember the late arrival of Grothendieck; having missed an appointment with Serre, who wanted to bring both of them by car, he missed another appointment with us for a night train, then took the wrong train and ended up hitchhiking from Valence to Die! Serre was not especially happy. Another time, he handed me a document to read, where, between the pages, was a letter (in German!) from an unhappy Brazilian girlfriend.

I remember a less exotic event. In the vicinity of Die, deep in the mountains, lived Marcel Légaut, who was an old friend of H. Cartan and A. Weil. Weil's autobiography refers to Légaut as an author of "works of piety," and in the 1970s Grothendieck referred repeatedly to those books. Légaut had left mathematics to raise sheep and became the guru of a kind of phalanstery, long before the wave of hippy communes. With the proper instructions of H. Cartan, Grothendieck and I walked a long way together to visit this guru. On the way, he confessed to me that mathematics was 99 percent labor and 1 percent excitement and that he wanted to leave mathematics to write novels and poetry. Which he did in the end! This was around the time of his mother's death, and it is known that his mother wanted to be a writer.[7] At one of our meetings, he brought his mother, who remained shy.

He was always a rebel.

During a Bourbaki meeting in the summer of 1960, there was a clash between Weil and Grothendieck. It started rather unexpectedly during our reading of a report by Grothendieck about differential calculus. Weil made one of his familiar unpleasant remarks that no one took seriously, except Grothendieck, who immediately left the room and did not come back for a couple of days. Both were uneasy characters, and we didn't understand what was especially at stake. Despite diplomatic efforts of S. Lang and J. Tate, Grothendieck didn't reconsider his self-imposed exile from Bourbaki.

I would need much more space to tell the long tale of the political activities of Grothendieck in the 1970s. He was always a dissident among the dissidents (think of the Vietnam War). Even if your political line was rather close to his own, it was often a painful experience to be on his side, because he wanted to refuse any kind of compromise—and this was the way he always lived his life. He was always a *rebel*.

Pierre Deligne

The first time I attended Grothendieck's seminar, early in 1965, I followed his lecture tenuously. I knew what cohomology groups were but could not understand the expression "*objet de cohomologie*," which kept recurring. After the lecture, I asked him what it meant. Very gently, he explained that if in an abelian category the composite fg is zero, the kernel of f, divided by the image of g, is the cohomology object.

I view his tolerance of what appeared to be crass ignorance and his lack of condescension as typical of him. It encouraged me to not refrain from asking "stupid" questions.

He taught me my trade by asking me to write up, using his notes, the talks XVII and XVIII of SGA4. By "trade" I mean both a feeling for the cohomology of algebraic varieties and how to write. My first draft was returned with comments and injunctions: "never use both sides of a page," "keep ample empty space between lines," as well as

[3] *After the publication of Grothendieck's* Tôhoku *paper, Godement gave an elementary construction of injective sheaves in his well-known textbook.*

[4] *A number of years were still required to finish it, revise it, and produce a so-called "final version".*

[5] *The then-deceased father of Henri Cartan.*

[6] *Site of the family summer house of H. Cartan, where he was the regular organ player in the Huguenot church.*

[7] *She wrote a kind of autobiography entitled* Eine Frau (A Woman), *in German.*

Pierre Deligne is professor emeritus of mathematics at the Institute for Advanced Study in Princeton. His email address is deligne@math.ias.edu.

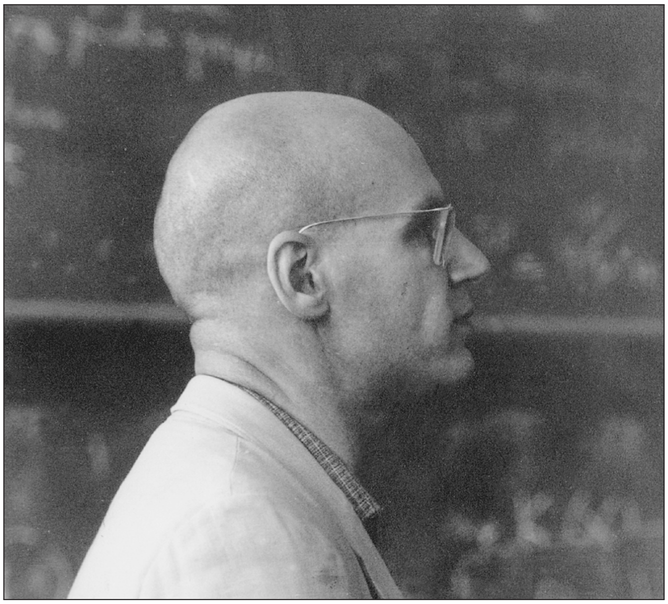


Photo courtesy of the estate of Friedrich Hirzebruch.

Grothendieck in 1965.

"proofs, as well as statements of compatibilities, should be complete." A key rule was: "One is not allowed to make a false assertion." Where sign questions in homological algebra are concerned, this rule is very hard to follow.

To use an image from *Récoltes et Semailles*, at that time Grothendieck and those of us around him were building a house. He was the architect-builder. We were helping as we could and bringing a few stones.

I feel extremely fortunate that he was my Master. What I learned from him, especially the philosophy of motives, has been a guiding thread in the works of mine I like the most, such as the formalism of mixed Hodge structures.

From him and his example, I have also learned not to take glory in the difficulty of a proof: difficulty means we have not understood. The ideal is to be able to paint a landscape in which the proof is obvious. I admire how often he succeeded in reaching this ideal.

In *Récoltes et Semailles* Grothendieck criticized me harshly. I always considered this to be a sign of affection. My task was to decide for myself what in these criticisms was true to be able to profit from them.

I am deeply grateful for his helping me to become a mathematician and for sharing his visions.

Michel Demazure

In 1985, I received a heavy parcel. It was Grothendieck's *Récoltes et Semailles.* On the first page, opposite a photo of the young Shurik, was this dedication, in his well-known and characteristic handwriting: "Pour Michel Demazure—cette réflexion sur un passé et sur un présent, qui nous impliquent l'un et l'autre. Amicalement,—October 1985, Alexandre Grothendieck." As usual with him, every word was carefully weighed, from the dual meaning of "reflection", the balanced "past/present", and the choice of "l'un et l'autre" instead of the obvious "tous les deux" (both).

And the strong "impliquent":[8] yes, I am "implicated" by a common past, between my twentieth and my thirtieth year. I first met him through his two heavy and hard-to-read monographs, "EVT" (*Espaces vectoriels topologiques*) and "PTT" (*Produits tensoriels topologiques*), and then followed his talks, watching with enthusiasm the infancy of the "new" algebraic geometry (new, and obviously the "right" one, to those of us of the younger generation). I spent the academic year 1959–60 in Princeton at the graduate college, and I remember a seminar at the Institute for Advanced Study where I gave a talk following the manuscript of EGA I. My English was very primitive, and I lost the listeners by pronouncing "jay sub jee" instead of "jee sub jay". After I returned to France and completed two years of military service, Grothendieck was my thesis adviser (1962–64), and I assisted him in the production of SGA3.

[He] happily climbed levels of abstraction as if he had already been there.

Those who share with me the unique experience of having benefited from his "advice" know how strong and illuminating it was. The weekly half-day sitting at his side and scribbling on parallel or common sheets is something I'll never forget. I was amazed by the way he discovered (saw!) things as they came along, happily climbing levels of abstraction as if he had already been there. I did not view him as I did other great mathematicians I have met in my career, who I felt were made of the same fabric as I—better fabric, to be sure, as they were brighter, faster, harder workers. Grothendieck always seemed essentially different; he was an "alien".

After my thesis, in fall 1964, I became a professor at the University of Strasbourg, and with the distance, my relation to Grothendieck weakened. The SGA seminar went its way (actually SGA3 was a parenthesis and did not really belong to the SGA mainstream), and I was geographically unable to follow it. Two years later I joined Université Paris-Sud in Orsay, with new interests and new responsibilities.

I must say I never felt really at ease with his view of mathematics. At the time when I had contact with him, I could not put this uneasiness into words. I understand it better now. There are two components.

Rereading *Récoltes et Semailles* and also his correspondence with Serre, I find the first component of my uneasiness centers on the question: What, after all, is

Michel Demazure was at the École Polytechnique from 1976–1999 and was director of the Palais de la Découverte from 1992–1998. He was president of the Cité des Sciences et de l'Industrie de La Villette from 1998 until his retirement in 2002. His email address is `michel@demazure.com`.

[8] *The French verb* impliquer *can be understood in two ways: simply as "imply", as in "A implies B", or as "implicate", as in "A is implicated in the crime against B".*

mathematics about? Of course, I am really pleased when I see (or in a few cases contribute to) the perfection of a general tool, but the pleasure is much greater when I see what such tools say in specific situations, where there is not enough room for those tools (size, dimension,...) or when they collide. I remember Robert Steinberg saying, “It is a pity there are so few simple Lie groups and that most of them are classical.” He would have been happy (and so would I) had the number of exceptional Lie groups been larger! I think this pleasure in exceptions was foreign to Grothendieck.

The second component centers on the question of “computation”, which takes a large place in Grothendieck’s correspondence with (and controversy about) Ronald Brown. I always liked to compute (I even spent the summer of 1955, before entering École Normale Supérieure, in the first computer company in France). For me, a complete mathematical theory should lead to effective computations. Grothendieck did not like computations (and hated computers!). He wrote to Brown: “The question you raised ‘how can such a formulation lead to computations’ doesn’t bother me in the least!” It is striking to compare this to what Voevodsky, whom I see as Grothendieck’s true continuator, wrote thirty years later: “It soon became clear that the only real long-term solution to the problems that I encountered is to start using computers in the verification of mathematical reasoning.”

What I have written might give a wrong impression and hide how much I owe to Grothendieck—as well as to Serre and Tits—and how intellectually enriched I have been by him. One cannot get rid of the “Grothendieck way”. For years, when I was stuck while struggling with a problem, I used to ask myself, what would Grothendieck say? Most certainly: If you just had stated the problem in the right way, you’d see the answer in the question.

If there is something like a “space of mathematics”, I see Grothendieck as an extremal point, and maybe so extremal as to be felt outside. In the “space-time of mathematics” there was a time interval in which I came into contact with that extremal point. That was a crucial period of my life, when I was, to use his wording, “implicated” by him and with him.

Marvin Jay Greenberg

Memories of Alexandre Grothendieck

My 1959 thesis, which proved a conjecture in arithmetic algebraic geometry by Serge Lang, introduced a technique that seemed complicated. When I learned, from notes by Dieudonné, about Grothendieck’s new foundation for algebraic geometry based on schemes, I rewrote my thesis for publication using that language. Moreover, that foundation showed that I had discovered a very natural, useful functor (Grothendieck later incorporated that functor as part of his general theory of “descent”). While teaching at Berkeley, I heard that Grothendieck would visit Harvard in fall 1961, so I obtained a fellowship to learn from him there and also at the IHES in Paris in spring 1962.

My first impression on seeing Grothendieck lecture was that he had been transported from an advanced alien civilization in some distant solar system to visit ours in order to speed up our intellectual evolution. His shaven head, his rapid, intense, commanding mode of speaking, plus the new concepts and generality of his view of algebraic geometry conveyed that impression.

I recall a lecture he gave at Harvard about Hilbert schemes, at the end of which he suddenly announced that he could develop a certain topic much more generally. Professor Oscar Zariski, who was in the audience, stopped Grothendieck from speaking overtime, asking him to “please exercise a little self-control.”

In Paris I attended Grothendieck’s lectures that were later published as SGA. The lectures were overwhelming, and I was also somewhat intimidated by his forceful personality.

I told him about an excellent symphony I had attended that cost me only a few francs. His firm response…“Ah, but it also cost you your time!”

Nevertheless, during an intermission in one of his presentations, I approached him and attempted to informally chat with him. I told him about an excellent symphony concert I had attended the night before that had cost me only a few francs. His firm response was, “Ah, but it also cost you your time!” Grothendieck evidently worked so hard on mathematics that he spent very little time on anything else.

Feeling utterly out of place attempting to relate to such a formidable person, I was subsequently surprised and elated when he invited me to dine with him and his wife at his home. It was a working-class, unpretentious abode. His wife was busy caring for their young baby. Grothendieck wasted very little time making small talk. With paper and pen at hand, he spent nearly the entire time sketching ways to use the functor I had found. I couldn’t follow what he was suggesting. He also urged me to work on presenting, within the framework of schemes, A. Neron’s important minimal models theory, which had been written in the old language of Weil’s foundations. I did begin studying Neron’s publications. Three years later I was able to push through a little of what Grothendieck had suggested. With the help of Michael Artin, I took one result in Neron’s work, expressed it in the language

Marvin Jay Greenberg is professor emeritus of mathematics at the University of California at Santa Cruz. He is the author of Euclidean and Non-Euclidean Geometries: Development and History *(4th edition 2008, W. H. Freeman, NY, NY) and of the Lester Ford Award-winning article "Old and New Results in the Foundations of Elementary Euclidean and Non-Euclidean Geometries"* (Amer. Math. Monthly) *March, 2010, 198–219. His email address is* mjg0@pacbell.net.

of schemes, and proved a new version of it in much greater generality. Grothendieck arranged to have this work appear in the *Publications IHES*.

I had no further direct interaction with Grothendieck after that publication, but other connections to him did arise. For example, I taught a course at Crown College, UC Santa Cruz, called The Quest for Enlightenment, in part presenting the teachings of J. Krishnamurti. Many years later Grothendieck, in his *Récoltes et Semailles*, listed Krishnamurti as one of eighteen enlightened masters of our age.

Grothendieck's copious output and originality in mathematics demonstrated a level of intellectual achievement I never imagined was possible by one man. I will forever be grateful that he took a little time to kindly inspire me to contribute a bit. There seems to be a consensus that Grothendieck went mad in his later years. I strongly disagree with that consensus. It is the madness of ordinary society that eventually drives geniuses like Grothendieck (and more recently Grigory Perelman) to withdraw.

Robin Hartshorne

Reflections on Grothendieck

After majoring in math at Harvard, I spent a year at the École Normale Supérieure in Paris. I had courses with Cartan, Serre, and Chevalley and learned some general topology and sheaf theory. After becoming a graduate student at Princeton, I started reading Serre's article "Faisceaux algébriques cohérents" and thought I would like to study algebraic geometry. At that time there was no algebraic geometry at Princeton, so the fall of 1961 found me back at Harvard, and there was Grothendieck.

He gave a lecture course on local properties of morphisms, which later became part of EGA IV, and he gave two seminars, one on local cohomology and one on construction techniques—the Hilbert scheme, the Picard scheme, and so forth. I could see that his was "the right way" to do things and jumped headlong into his world. In 1963 I finished my thesis, which was on the connectedness of the Hilbert scheme. While Grothendieck was not my official advisor, nor did I discuss the work in progress with him, I am sure it was the stimulating atmosphere of discovery he created that provided the context for me to be able to do this work.

I sent a draft copy of my thesis to Grothendieck. He responded with a long letter, containing a few sentences of appreciation for the result and then many pages of further questions about the Hilbert scheme, most of which are still unanswered today. Each new result he encountered gave rise to a myriad of further questions to investigate.

A couple of years later I offered to run a seminar at Harvard on his theory of duality, which he had hinted at in his ICM talk in 1958 but had not yet developed. He agreed and sent me about 250 pages of "prenotes" for the seminar. My job was to digest them, fill in details, give the seminar talks, and then write up the notes. This was quite a challenge, as it included the first occurrence of Verdier's theory of the derived category and Grothendieck's use of it in developing the duality theory for a morphism of schemes. I regard this period as my "apprenticeship" with Grothendieck. We had a constant interchange of letters, as I sent him drafts of the seminar talks, and he returned them covered with red ink. In this way I learned the craft of exposition in his style. After the lecture notes were published (*Residues and Duality*, Springer Lecture Notes 20, 1966), I did not see him so often. But some time later he did ask me, "Well, those lecture notes were a good rough account, but when are you going to write the *book* on duality?" I did not answer that because I was already moving in other directions.

The last time I saw Grothendieck in person was in Kingston, Ontario, in 1971. He had so withdrawn from engagement in mathematics that he devoted equal time in his talks to his new brainchild "Survivre". I could appreciate the sincerity of his beliefs but felt he was hopelessly naïve about political action.

When I finished my book *Algebraic Geometry* in 1977, which is basically an introduction to Grothendieck's way of thinking using schemes and cohomology, I sent him a copy together with a note of thanks and appreciation for all that I had learned from him. He sent a polite card in reply, saying, "It looks like a nice book. Perhaps if one day I again teach a course on algebraic geometry, I will look at the inside."

Near the beginning of his rambling reflections *Récoltes et Semailles*, Grothendieck mentions "les héritiers et le bâtisseur," the heirs and the constructor. As an heir of the master builder Grothendieck, I am now happily inhabiting several of the rooms he built and using his tools to refine my understanding of classical geometry. I owe him the inspiration for my life work.

Robin Hartshorne is professor emeritus of mathematics at the University of California, Berkeley. His email address is robin@math.berkeley.edu.

Luc Illusie

Alexander Grothendieck was a professor at the IHÉS from 1959 until 1970. In the seminars he led—the famous SGA—a team of students coalesced around him, exploring the new territories that "the Master" had discovered. We were many, coming from various corners of the world, to participate in this adventure, which constituted a sort of golden age of algebraic geometry.

The seminars took place at the IHÉS on Tuesday afternoons and spread out over a year or two. They were held in a former music pavilion that had been

Luc Illusie is professeur honoraire at Université de Paris-Sud. His email address is Luc.Illusie@math.u-psud.fr.

This is a slightly edited translation of a piece, "Grothendieck était d'un dynamisme impressionnant", which appeared in CNRS, Le journal (https://lejournal.cnrs.fr/billets/grothendieck-etait-dun-dynamisme-impressionnant), *and is an excerpt of a longer article "Alexandre Grothendieck, magicien des foncteurs", which appeared in the online publication of the Mathematical Institute of the CNRS* (www.cnrs.fr/insmi/spip.php?article1093). *The* Notices *thanks the editors of both publications for permission to include this piece here.*

transformed into a library and lecture hall, with large picture windows onto the Bois-Marie park. Occasionally before the lectures, the Master took us for a walk in the woods to tell us about his latest ideas. The seminars were mainly about his own work. There were also related results, of which he sometimes entrusted the exposition to students or colleagues. For instance, he asked Deligne, in the seminar SGA7, to transpose into the setting of étale cohomology the classical Picard-Lefschetz formula, whose proof he confessed to me he had not understood. This étale analogue was later to play a key role in the proof, by Deligne, of the Riemann hypothesis over finite fields. At the blackboard Grothendieck had an impressive dynamism but was always clear and methodical. No black boxes, no sketches—everything was explained in detail. Occasionally he omitted a verification that he considered purely routine (but that could turn out to be more delicate than it had appeared). After the lecture, the audience went to have tea in the administrative building. This was an opportunity to discuss various points from the seminar and exchange ideas.

[He] was always clear and methodical. No black boxes, no sketches.

Grothendieck liked to ask his students to write up his lectures. In this way, they learned their craft. When it came to editing, he was tough and demanding. My typed manuscripts, which might reach fifty pages, would be blackened all over with his critiques and suggestions. I remember reviewing them one by one over the course of long afternoons at his home. The results had to be presented in their natural framework, which usually meant the most general possible. Everything had to be proved. Phrases like "it is clear" and "one easily sees" were banished. We discussed the mathematics point by point, but also punctuation and the order of words in a sentence. Length was not an issue. If a digression looked interesting, it was welcome. Very often we were not finished before 8 o'clock in the evening. He would then invite me to share a simple dinner with his wife, Mireille, and their children. After the meal, as a form of recreation, he would explain to me bits of mathematics he had been thinking about lately. He would improvise on a white sheet of paper, with his large pen, in his fine and rapid hand, stopping occasionally at a certain symbol to once again run his pen over it in delight. I can hear his sweet and melodious voice, punctuated from time to time with a sudden "Ah!" when an objection came to mind. Then he would see me off at the station, where I would take the last train back to Paris.

Nicholas Katz

There is no need, I hope, to discuss the mathematical achievements and the mathematical vision of Grothendieck. What is perhaps less known to people who did not interact with Grothendieck personally was his incredible charisma. We thought of him (as he did of himself, as he says in *Récoltes et Semailles*) as the boss (*patron*) of a construction site (*chantier*). When he asked someone to carry out some work that would be part of this, the person asked felt that he or she had been honored to have been asked, was proud to have been asked, and was delighted to undertake the task at hand (which might take many years to complete). Combined with this charisma, Grothendieck had an uncanny sense of whom to ask to do what. One sees this in looking at the long list of people whose work became an essential part of Grothendieck's *chantier*.

Steven L. Kleiman

The first time I saw Grothendieck was in September 1961 at Harvard. I was an eager new graduate student; he, a second-time visitor teaching a course. He started by explaining he'd cover some preliminaries to appear in [4]: the course would be elementary; the prerequisites, just the basics.

Soon I found my three terms of graduate algebra as an MIT undergraduate hadn't prepared me for Grothendieck's course. So I dropped it and skipped his two weekly seminars developing the Picard scheme and local cohomology. I believe he assigned no homework and gave no exams in the course; at the end he unexpectedly collected the notebooks of the registered students and assigned grades.

Grothendieck began each meeting of the course by erasing the board and then writing $X \xrightarrow{f} Y$ vertically. One time before Grothendieck arrived, John Fogarty, another graduate student, erased the board and wrote $X \xrightarrow{f} Y$ vertically. When Grothendieck arrived, he looked at the board and silently erased it. Then he began his lecture, writing $X \xrightarrow{f} Y$ vertically.

Fogarty had considerable skill as a caricature artist. One day he drew a large, lovingly detailed cartoon on the blackboard in the common room. It showed a side view of Grothendieck with a quiver of arrows on his back, looking ahead where he'd written $X \xrightarrow{f} Y$ vertically.

Thus Fogarty satirized one of Grothendieck's signature insights: it pays off in better understanding and in greater flexibility to generalize *absolute* properties of objects X to *relative* properties of maps $X \xrightarrow{f} Y$.

Grothendieck's paper [2] was highly regarded in the student Algebraic Geometry Seminar, which I joined in fall 1962. Grothendieck had upgraded sheaf cohomology: he found enough injectives to resolve sheaves and yield

Nicholas Katz is professor of mathematics at Princeton University. His email address is nmk@math.princeton.edu.

Steven L. Kleiman is emeritus professor of mathematics at the Massachusetts Institute of Technology. His email address is kleiman@math.mit.edu.

their higher cohomology groups as derived functors. Thus he demoted Čech cohomology: taken as the definition in Jean-Pierre Serre's [9], it became just a computational device.

Later Grothendieck went further. He generalized the very notion of topology! In an open covering of U by U_i for $i \in I$, the maps $U_i \to U$ needn't be inclusions, just members of a suitable class. For example, they could be *étale,* his generalization of the local isomorphisms of analytic spaces.

Michael Artin began the systematic development of Grothendieck topology in a Harvard seminar in spring 1962. I didn't attend but did lecture from [1] in a student seminar at Woods Hole, July 1964. The ensuing work of Grothendieck and collaborators (especially Artin, Jean-Louis Verdier, and Pierre Deligne) culminated in the resolution of André Weil's celebrated conjectures, just as Grothendieck [3, p. 104] had predicted.

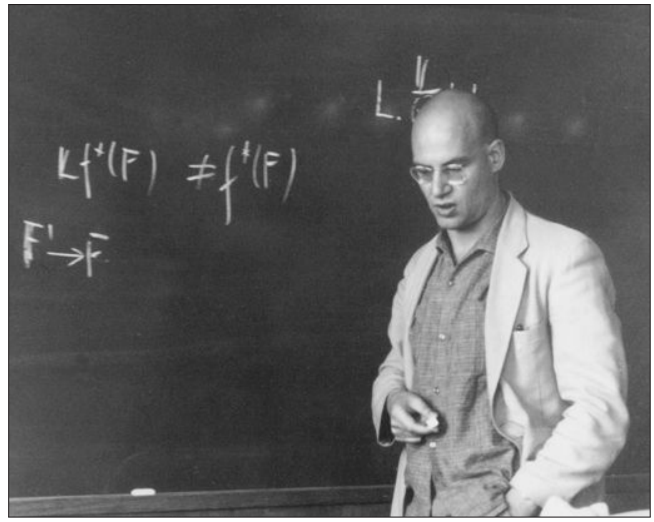
I learned more of Grothendieck's innovations in David Mumford's course, spring 1964, published as [7]. It was devoted to Grothendieck's proof of completeness of the characteristic system of a good complete algebraic system of curves on a smooth projective complex surface F. It is the *first algebraic proof* of a theorem with a long, rich, and colorful history (see [6]).

"The key," Mumford wrote on p. viii, "... is the systematic use of nilpotent elements" to handle higher-order infinitesimal deformations. That's another of Grothendieck's signature insights. Yet another is to use flatness to formalize the notion of algebraic system. Moreover, Grothendieck proved a complete algebraic system is parameterized by a component H of his Hilbert scheme of F; namely, H classifies all systems via maps into H.

Grothendieck showed the Theorem of Completeness simply provides conditions for H to be smooth at a given point. To prove the conditions work, he used his Picard scheme P and the map $H \to P$. In an ingenious sense, P *classifies families* of line bundles: its *functor of points,* that is, functor of maps $T \to P$, isn't equal to the naive functor of line bundles on $T \times F$, but rather to its associated sheaf in the étale Grothendieck topology.

Mumford sent a preliminary copy of [7] to Grothendieck, who commented in a letter [8, pp. 693–6] dated August 31, 1964. Mumford's numerical characterization of *good* systems reminded Grothendieck about his conjectural *numerical theory of ampleness.* In particular, on an n-fold, just as on a surface, a divisor should be pseudoample if it meets every curve nonnegatively. More generally, the ample divisors should form the interior of the polar cone of the numerical cone of curves. In [8, p. 701], Mumford replied he didn't know if the conjectures are true, even on a 3-fold, but he'd ask me.

Shortly afterwards, I proved Grothendieck's first conjecture; in January, the second. Then I used the second to prove Chevalley's conjecture: a complete smooth variety is projective if any finitely many points lie in some affine subset. In April, Mumford suggested I write to Grothendieck. Grothendieck replied with comments and said, "I would appreciate knowing a simple proof," of the key ingredient, the Nakai–Moishezon criterion.



Grothendieck around 1965.

Photo courtesy of the estate of Friedrich Hirzebruch.

I sent Grothendieck a reprint of my first paper [5], where I had simplified Nakai's proof and extended it to a *nonprojective* n-fold as announced by Moishezon. He replied with more comments, and in a PS he gave his opinion on the history of the development of the criterion. The body was typewritten, but the PS, handwritten, showing he'd thought more about it and really wanted everyone to receive proper credit, not because it's due, but to indicate how ideas develop.

Grothendieck's letters show impressive clarity and thoroughness. They pose questions, indicating a wish to continue the discussion. They suggest being generous with ideas while acknowledging their provenance. His letters are complimentary and encouraging. This is the way to do collaborative mathematics!

Grothendieck agreed to supervise my NATO postdoc 1966–67, and I returned to his institute, the IHES, the summers of 1968 and 1969 and the spring of 1970. In [6], I discussed my mathematical experiences.

Socially, Grothendieck had me and others over to his house for dinner several times. The last time, in spring 1970, I brought my new wife. Beverly remembers "a feeling of trepidation, as he was a living legend. However, the minute we entered his home, it was apparent that he was an exceptional person, gracious and attentive. Not for an instant did I feel my deficiency in mathematics and French was something that even occurred to him. His genuine interest and participation in conversation, the general atmosphere of inclusion, is something I've always remembered."

Spring 1970 was hard on Grothendieck, as his era at the IHES ended. Outwardly, he didn't show his feelings, but people did talk about what was happening. I never saw him again and heard from him only once more when he sent me his four volumes of *Récoltes et Semailles,* with this inscription opposite a picture of himself as an adorable six(?)-year-old: "To Steven Kleiman, with my friendly regards, Oct. 1985, Alexander Grothendieck."

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How Grothendieck Simplified Algebraic Geometry

Photo courtesy of the IHES.

Colin McLarty

> The idea of *scheme* is childishly simple—so simple, so humble, no one before me dreamt of stooping so low.…It grew by itself from the sole demands of simplicity and internal coherence.
>
> [A. Grothendieck, *Récoltes et Semailles* (R&S), pp. P32, P28]

Algebraic geometry has never been really simple. It was not simple before or after David Hilbert recast it in his algebra, nor when André Weil brought it into number theory. Grothendieck made key ideas simpler. His schemes give a bare minimal definition of space just glimpsed as early as Emmy Noether. His derived functor cohomology pares insights going back to Bernhard Riemann down to an agile form suited to étale cohomology. To be clear, étale cohomology was no simplification of anything. It was a radically new idea, made feasible by these simplifications.

Grothendieck got this heritage at one remove from the original sources, largely from Jean-Pierre Serre in shared pursuit of the Weil conjectures. Both Weil and Serre drew deeply and directly on the entire heritage. The original ideas lie that close to Grothendieck's swift reformulations.

Generality As the Superficial Aspect

Grothendieck's famous penchant for generality is not enough to explain his results or his influence. Raoul Bott put it better fifty-four years ago describing the Grothendieck-Riemann-Roch theorem.

Riemann-Roch has been a mainstay of analysis for one hundred fifty years, showing how the topology of a Riemann surface affects analysis on it. Mathematicians from Richard Dedekind to Weil generalized it to curves over any field in place of the complex numbers. This makes theorems of arithmetic follow from topological and analytic reasoning over the field $\mathbb{F}_p$ of integers modulo a prime p. Friedrich Hirzebruch generalized the complex version to work in all dimensions.

Grothendieck proved it for all dimensions over all fields, which was already a feat, and he went further in a signature way. Beyond single varieties he proved it for suitably continuous families of varieties. Thus:

> Grothendieck has generalized the theorem to the point where not only is it more generally applicable than Hirzebruch's version but it depends on a simpler and more natural proof. (Bott [6])

This was the first concrete triumph for his new cohomology and nascent scheme theory. Recognizing that many mathematicians distrust generality, he later wrote:

> I prefer to accent "unity" rather than "generality." But for me these are two aspects of one quest. Unity represents the profound aspect, and generality the superficial. [16, p. PU 25]

Colin McLarty is Truman P. Hardy Professor of Philosophy and professor of mathematics, Case Western Reserve University. His email address is `colin.mclarty@case.edu`.

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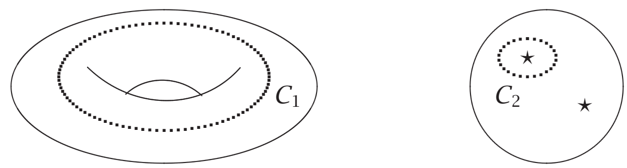
The Beginnings of Cohomology

> Surfaces with holes are not just an amusing pastime but of quite fundamental importance for the theory of equations. (Atiyah [4, p. 293])

The Cauchy Integral Theorem says integrating a holomorphic form ω over the complete boundary of any region of a Riemann surface gives 0. To see its significance look at two closed curves C on Riemann surfaces that are *not* complete boundaries. Each surrounds a hole, and each has $\int_C \omega \neq 0$ for some holomorphic ω.

Cutting the torus along the dotted curve C_1 around the center hole of the torus gives a tube, and the single curve C_1 can only bound one end.



The *punctured sphere* on the right has stars depicting punctures, i.e. holes. The regions on either side of C_2 are unbounded at the punctures.

Riemann used this to calculate integrals. Any curves C and C' surrounding just the same holes the same number of times have $\int_C \omega = \int_{C'} \omega$ for all holomorphic ω. That is because C and the reversal of C' form the complete boundary of a kind of collar avoiding those holes. So $\int_C \omega - \int_{C'} \omega = 0$.

Modern cohomology sees holes as *obstructions* to solving equations. Given ω and a path $P : [0,1] \to S$ it would be great to calculate the integral $\int_P \omega$ by finding a function f with $\mathrm{d}f = \omega$, so $\int_P \omega = f(P_1) - f(P_0)$. Clearly, there is not always such a function, since that would imply $\int_P \omega = 0$ for every closed curve P. But Cauchy, Riemann, and others saw that if $U \subset S$ surrounds no holes there are functions f_U with $\mathrm{d}f_U = \omega$ all over U. Holes are obstructions to patching local solutions f_U into one solution of $\mathrm{d}f = \omega$ all over S.

This concept has been generalized to algebra and number theory:

> Indeed one now instinctively assumes that all obstructions are best described in terms of cohomology groups. [32, p. 103]

Cohomology Groups

> With homologies, terms compose according to the rules of ordinary addition. [24, pp. 449-50]

Poincaré defined addition for curves so that $C + C'$ is the union of C and C', while $-C$ corresponds to reversing the direction of C. Thus for every form ω:

$$\int_{C+C'} \omega = \int_C \omega + \int_{C'} \omega \qquad \text{and} \qquad \int_{-C} \omega = -\int_C \omega.$$

When curves $C_1, \ldots, C_k$ form the complete boundary of some region, then Poincaré writes $\Sigma_k C_k \sim 0$ and says their sum is *homologous to* 0. In these terms the Cauchy

Integral Theorem says concisely:

$$\text{If } \Sigma_k C_k \sim 0, \text{ then } \int_{\Sigma_k C_k} \omega = 0 \quad \text{for all holomorphic forms } \omega.$$

Poincaré generalized this idea of homology to higher dimensions as the basis of his *analysis situs*, today called topology of manifolds.

Notably, Poincaré published two proofs of *Poincaré duality* using different definitions. His first statement of it was false. His proof mixed wild non sequiturs and astonishing insights. For topological manifolds M of any dimension n and any $0 \le i \le n$, there is a tight relation between the i-dimensional submanifolds of M and the $(n-i)$-dimensional. This relation is hardly expressible without using homology, and Poincaré had to revise his first definition to get it right. Even the second version relied on overly optimistic assumptions about triangulated manifolds.

Topologists spent decades clarifying his definitions and theorems and getting new results in the process. They defined homology groups $H_i(S)$ for every space S and every dimension $i \in \mathbb{N}$. In each H_1 the group addition is Poincaré's addition of curves modulo the homology relations $\Sigma_k C_k = 0$. They also defined related cohomology groups $H^i(S)$ such that Poincaré duality says $H_i(M)$ is isomorphic to $H^{n-i}(M)$ for every compact orientable n-dimensional manifold M.

Poincaré's two definitions of homology split into many using simplices or open covers or differential forms or metrics, bringing us to the year 1939:

> Algebraic topology is growing and solving problems, but nontopologists are very skeptical. At Harvard, Tucker or perhaps Steenrod gave an expert lecture on cell complexes and their homology, after which one distinguished member of the audience was heard to remark that this subject had reached such algebraic complication that it was not likely to go any further. (MacLane [21, p. 133]

Variable Coefficients and Exact Sequences

> In his Kansas article (1955) and *Tôhoku* article (1957) Grothendieck showed that given any category of sheaves a notion of cohomology groups results. (Deligne [10, p. 16])

Algebraic complication went much further. Methods in topology converged with methods in Galois theory and led to defining cohomology for groups as well as for topological spaces. In the process, what had been a technicality to Poincaré became central to cohomology, namely, the choice of coefficients. Certainly he and others used integers, rational numbers or reals or integers modulo 2 as coefficients:

$$a_1C_1 + \cdots + a_mC_m \quad a_i \in \mathbb{Z} \text{ or } \mathbb{Q} \text{ or } \mathbb{R} \text{ or } \mathbb{Z}/2\mathbb{Z}.$$

But only these few closely related kinds of coefficients were used, chosen for convenience for a given calculation. So topologists wrote $H^i(S)$ for the ith cohomology group of space S and left the coefficient group implicit in the context.

In contrast group theorists wrote $H^i(G, A)$ for the ith cohomology of G with coefficients in A, because many kinds of coefficients were used and they were as interesting as the group G. For example, the famed Hilbert Theorem 90 became

$$H^1(\mathrm{Gal}(L/k), L^\times) \cong \{0\}.$$

The Galois group $\mathrm{Gal}(L/k)$ of a Galois field extension L/k has trivial 1-dimensional cohomology with coefficients in the multiplicative group $L^\times$ of all nonzero $x \in L$. Olga Taussky[33, p. 807] illustrates Theorem 90 by using it on the Gaussian numbers $\mathbb{Q}[i]$ to show every Pythagorean triple of integers has the form

$$m^2 - n^2,\ 2mn,\ m^2 + n^2.$$

Trivial cohomology means there is no obstruction to solving certain problems, so Theorem 90 shows that some problems on the field L have solutions. Algebraic relations of $H^1(\mathrm{Gal}(L/k), L^\times)$ to other cohomology groups imply solutions to other problems. Of course Theorem 90 was invented to solve lots of problems decades before group cohomology appeared. Cohomology organized and extended these uses so well that Emil Artin and John Tate made it basic to class field theory.

Also in the 1940s topologists adopted *sheaves* of coefficients. A sheaf of Abelian groups $\mathcal{F}$ on a space S assigns Abelian groups $\mathcal{F}(U)$ to open subsets $U \subseteq S$ and homomorphisms $\mathcal{F}(U) \to \mathcal{F}(V)$ to subset inclusions $V \subseteq U$. So the sheaf of holomorphic functions $\mathcal{O}_M$ assigns the additive group $\mathcal{O}_M(U)$ of holomorphic functions on U to each open subset $U \subseteq M$ of a complex manifold. Cohomology groups like $H^i(M, \mathcal{O}_M)$ began to organize complex analysis.

Leaders in these fields saw cohomology as a unified idea, but the technical definitions varied widely. In the Séminaire Henri Cartan speakers Cartan, Eilenberg, and Serre organized it all around *resolutions*. A resolution of an Abelian group A (or module or sheaf) is an *exact sequence* of homomorphisms, meaning the image of each homomorphism is the kernel of the next:

$$\{0\} \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \dots.$$

It quickly follows that many sequences of cohomology groups are also exact. That proof rests on the *Snake Lemma* immortalized by Hollywood in a scene widely available online: "A clear proof is given by Jill Clayburgh at the beginning of the movie *It's My Turn*" [35, p. 11].

Fitting a cohomology group $H^i(X, \mathcal{F})$ into the right exact sequence might show $H^i(X, \mathcal{F}) \cong \{0\}$, so the obstructions measured by $H^i(X, \mathcal{F})$ do not exist. Or it may prove some isomorphism, $H^i(X, \mathcal{F}) \cong H^k(Y, \mathcal{G})$. Then the obstructions measured by $H^i(X, \mathcal{F})$ correspond exactly to those measured by $H^k(Y, \mathcal{G})$.

Group cohomology uses resolution by *injective* modules I_i. A module I over a ring R is injective if for every R-module inclusion $j\colon N \rightarrowtail M$ and homomorphism

$f : N \to I$ there is some $g : M \to I$ with $f = gj$. The diagram is simple:

$$\begin{tikzcd} N \arrow[r, tail, "j"] \arrow[dr, "f"'] & M \arrow[d, dashed, "g"] \\ & I \end{tikzcd}$$

This works because every R-module A over any ring R embeds in some injective R-module. No one believed anything this simple would work for sheaves. Sheaf cohomology was defined only for sufficiently regular spaces using various, more complicated topological substitutes for injectives. Grothendieck found an unprecedented proof that sheaves on all topological spaces have injective embeddings. The same proof later worked for sheaves on any *Grothendieck topology*.

Tôhoku

> Consider the set of all sheaves on a given topological space or, if you like, the prodigious arsenal of all "meter sticks" that measure the space. We consider this "set" or "arsenal" as equipped with its most evident structure, the way it appears so to speak "right in front of your nose"; that is what we call the structure of a "category." [16, p. P38]

We will not fully define sheaves, let alone *spectral sequences* and other "drawings (called "diagrams") full of arrows covering the blackboard" which "totally escaped" Grothendieck at the time of the Séminaire Cartan [R&S, p. 19]. We will see why Grothendieck wrote to Serre on February 18, 1955: "I am rid of my horror of spectral sequences" [7, p. 7].

The Séminaire Cartan emphasized how few specifics about groups or modules go into the basic theorems. Those theorems only use diagrams of homomorphisms. For example, the sum $A + B$ of Abelian groups A, B can be defined, uniquely up to isomorphism, by the facts that it has homomorphisms $i_A : A \to A + B$ and $i_B : B \to A + B$ and any two homomorphisms $f : A \to C$ and $g : B \to C$ give a unique $u : A + B \to C$ with $f = ui_A$ and $g = ui_B$:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "f"'] & A + B \arrow[d, dotted, "u"] & B \arrow[l, "i_B"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

The same diagram defines sums of modules or of sheaves of Abelian groups.

Grothendieck [13, p. 127] took the basic patterns used by the Séminaire Cartan as his *Abelian category* axioms. He added a further axiom, AB5, on infinite colimits. Theorem 2.2.2 says if an Abelian category satisfies AB5 plus a set-theoretic axiom, then every object in that category embeds in an injective object. These axioms taken from module categories obviously hold as well for sheaves of Abelian groups on any topological space, so the conclusion applies.

People who thought this was just a technical result on sheaves found the tools disproportionate to the product. They were wrong on both counts. These axioms also simplified proofs of already-known theorems. Most especially they subsumed many useful spectral sequences (*not* all) under the *Grothendieck spectral sequence* so simple as to be Exercise A.3.50 of (Serre [11, p. 683]).

Early editions of Serge Lang's *Algebra* gave the Abelian category axioms with a famous exercise: "Take any book on homological algebra, and prove all the theorems without looking at the proofs given in that book" [20, p. 105]. He dropped that when homological algebra books all began using axiomatic proofs themselves, even if their theorems are stated only for modules. David Eisenbud, for example, says his proofs for modules "generalize with just a little effort to [any] nice Abelian category" [11, p. 620].

Injective resolutions in any Abelian category give *derived functor cohomology* of that category. This was obviously general beyond any proportion to the then-known cases. Grothendieck was sure it was the right generality: For a cohomological solution to any problem, notably the Weil conjectures, find the right Abelian category.

The Weil Conjectures

> This truly revolutionary idea thrilled the mathematicians of the time, as I can testify at first hand. [30, p. 525]

The Weil Conjectures relating arithmetic to topology were immediately recognized as a huge achievement. Weil knew that just conceiving them was a great moment in his career. The cases he proved were impressive. The conjectures were too beautiful not to be true and yet nearly impossible to state fully.

The conjectures were too beautiful not to be true and yet nearly impossible to state fully.

Weil [37] presents the topology using the nineteenth-century terminology of Betti numbers. But he was an established expert on cohomology and in conversations:

> At that time, Weil was explaining things in terms of cohomology and Lefschetz's fixed point formula [yet he] did not want to predict [this could actually work]. Indeed, in 1949–50, nobody thought that it could be possible. (Serre quoted in [22, p. 305].)

Lefschetz used cohomology, relying on the continuity of manifolds, to count fixed points $x = f(x)$ of continuous functions $f : M \to M$ on manifolds. Weil's conjectures deal with spaces defined over finite fields. No known version of those was continuous. Neither Weil nor anyone knew what might work. Grothendieck says:

> Serre explained the Weil conjectures to me in cohomological terms around 1955 and only in these terms could they possibly "hook" me. No one had any idea how to define such a cohomology

> and I am not sure anyone but Serre and I, not even Weil if that is possible, was deeply convinced such a thing must exist. [R&S, p. 840]

Exactly What Scheme Theory Simplified

> Kronecker was, in fact, attempting to describe and to initiate a new branch of mathematics, which would contain both number-theory and algebraic geometry as special cases. (Weil [38, p. 90])

Riemann's treatment of complex curves left much to geometric intuition. So Dedekind and Weber [8, p. 181] proved a Riemann-Roch theorem from "a simple yet rigorous and fully general viewpoint," over any algebraically closed field k containing the rational numbers. They note k can be the field of algebraic numbers. They saw this bears on arithmetic as well as on analysis and saw all too well that their result is "very difficult in exposition and expression" [8, p. 235].

Meromorphic functions on any compact Riemann surface S form a field $M(S)$ of transcendence degree 1 over the complex numbers $\mathbb{C}$. Each point $p \in S$ determines a function e_p from $M(S)$ to $\mathbb{C} + \{\infty\}$: namely $e_p(f) = f(p)$ when f is defined at p, and $e_p(f) = \infty$ when f has a pole at p. Then, if we ignore sums $\infty + \infty$:

$$e_p(f+g) = e_p(f) + e_p(g),$$
$$e_p(f \cdot g) = e_p(f) \cdot e_p(g),$$
$$e_p(\frac{1}{f}) = \frac{1}{e_p(f)}.$$

Dedekind and Weber define a general *field of algebraic functions* as any transcendence degree 1 extension L/k of any algebraically closed field k. They define a *point* p of L to be any function e_p from L to $k + \{\infty\}$ satisfying those equations. Their Riemann-Roch theorem treats L as if it were $M(S)$ for some Riemann surface.

Kronecker [19] achieved some "algebraic geometry over an absolutely algebraic ground-field" [38, p. 92]. These fields are the finite extensions of $\mathbb{Q}$ or of finite fields $\mathbb{F}_p$. They are not algebraically closed. He aimed at "algebraic geometry over the integers" where one variety could be defined over all these fields at once, but this was far too difficult at the time [38, p. 95].

Italian algebraic geometers relied on an idea of *generic points* of a complex variety V, which are ordinary complex points $p \in V$ with no apparent special properties [26]. For example, they are not points of singularity. Noether and Bartel van der Waerden gave abstract generic points which actually have only those properties common to all points of V. Van der Waerden [34] made these rigorous but not so usable as Weil would want. Oscar Zariski, trained in Italy, worked with Noether in Princeton, and later with Weil, to give algebraic geometry a rigorous algebraic basis [23, p. 56].

Weil's bravura *Foundations of Algebraic Geometry* [36] combined all these methods into the most complicated foundation for algebraic geometry ever. To handle varieties of all dimensions over arbitrary fields k, he uses algebraically closed field extensions L/k of infinite transcendence degree. He defines not only points but also subvarieties $V' \subseteq V$ of a variety V purely in terms of fields of rational functions. Raynaud [25] gives an excellent overview. We list three key topics.

(1) Weil has generic points. Indeed, a variety defined by polynomials over a field k has infinitely many generic points with coordinates transcendental over k, all conjugate to each other by Galois actions over k.
(2) Weil defines *abstract algebraic varieties* by data telling how to patch together varieties defined by equations. But these do not exist as single spaces. They only exist as sets of concrete varieties plus patching data.
(3) Weil could not define a variety over the integers, though he could systematically relate varieties over $\mathbb{Q}$ to others over the fields $\mathbb{F}_p$.

Serre Varieties and Coherent Sheaves

Then Serre [27] temporarily put generic points and non-closed fields aside to describe the first really penetrating cohomology of algebraic varieties:

> This rests on the use of the famous Zariski topology, in which the closed sets are the algebraic sub-varieties. The remarkable fact that this coarse topology could actually be put to genuine mathematical use was first demonstrated by Serre and it has produced a revolution in language and techniques. (Atiyah [3, p. 66])

Say a *naive variety* over any field k is a subset $V \subseteq k^n$ defined by finitely many polynomials $p_i(x_1, \dots, x_n)$ over k:

$$V = \{\, \vec{x} \in k^n \mid p_1(\vec{x}) = \cdots = p_h(\vec{x}) = 0 \,\}.$$

They form the closed sets of a topology on k^n called the Zariski topology. Even their infinite intersections are defined by finitely many polynomials, since the polynomial ring $k[x_1, \dots, x_n]$ is Noetherian. Also, each inherits a Zariski topology where the closed sets are the subsets $V' \subseteq V$ defined by further equations.

These are very coarse topologies. The Zariski closed subsets of any field k are the zero-sets of polynomials over k: that is, the finite subsets and all of k.

Each naive variety has a *structure sheaf* $\mathcal{O}_V$ which assigns to every Zariski open $U \subseteq V$ the ring of regular functions on U. Omitting important details:

$$\mathcal{O}_V(U) = \{ \frac{f(\vec{x})}{g(\vec{x})} \text{ such that when } \vec{x} \in U \text{ then } g(\vec{x}) \neq 0 \,\}.$$

A *Serre variety* is a topological space T plus a sheaf $\mathcal{O}_T$ which is locally isomorphic to the structure sheaf of a naive variety. Compare the sheaf of holomorphic functions $\mathcal{O}_M$ of a complex manifold. The sheaf apparatus lets Serre actually paste varieties together on compatible patches, just as patches of differentiable manifolds are pasted together. Weil could not do this with his abstract varieties.

Certain sheaves related to the structure sheaves $\mathcal{O}_T$ are called *coherent*. Serre makes them the coefficient sheaves of a cohomology theory widely used today with schemes. The close tie of coherent sheaves to structure sheaves makes this cohomology unsuitable for the Weil conjectures. When variety (or scheme) V is defined over a finite field $\mathbb{F}_p$, its coherent cohomology is defined modulo p and can count fixed points of maps $V \to V$ only modulo p. Still:

> The principal, and perhaps only, external inspiration for the sudden vigorous launch of scheme theory in 1958 was Serre's (1955) article known by the acronym FAC. [R&S, p. P28]

Schemes

> The point, *grosso modo*, was to rid algebraic geometry of parasitic hypotheses encumbering it: base fields, irreducibility, finiteness conditions. (Serre [29, p. 201])

Schemes overtly simplify algebraic geometry. Where earlier geometers used complicated extensions of algebraically closed fields, scheme theorists use any ring. Polynomial equations are replaced by ring elements. Generic points become prime ideals. The more intricate concepts come back in when needed, which is fairly often, but not always and not from the start.

In fact this perspective goes back to unpublished work by Noether, van der Waerden, and Wolfgang Krull. Prior to Grothendieck:

> The person who was closest to scheme-thinking (in the affine case) was Krull (around 1930). He used systematically the localization process, and proved most of the nontrivial theorems in Commutative Algebra. (Serre, email of 21/06/2004, Serre's parentheses)

Grothendieck made it work. He made every ring R the coordinate ring of a scheme $\mathrm{Spec}(R)$ called the *spectrum* of R. The points are the prime ideals of R, and the scheme has a structure sheaf $\mathcal{O}_R$ on the Zariski topology for those points, like the structure sheaf on a Serre variety. It follows that the continuous structure preserving maps from $\mathrm{Spec}(R)$ to another affine scheme $\mathrm{Spec}(A)$ correspond exactly to the ring homomorphisms in the other direction:

$$A \xrightarrow{f} R \qquad\qquad \mathrm{Spec}(R) \xrightarrow{\mathrm{Spec}(f)} \mathrm{Spec}(A)\ .$$

The points can be quite intricate: "When one has to construct a scheme one generally does not begin with the set of points" [10, p. 12].

For example, the ring $\mathbb{R}[x]$ of real polynomials in one variable is the natural coordinate ring for the real line, so the spectrum $\mathrm{Spec}(\mathbb{R}[x])$ is the scheme of the real line. Each nonzero prime ideal is generated by a monic irreducible real polynomial. Those polynomials are $x - a$ for $a \in \mathbb{R}$ and $x^2 - 2bx + c$ for $b, c \in \mathbb{R}$ with $b^2 < c$. The first kind correspond to ordinary points $x = a$ of the real line. The second kind correspond to pairs of conjugate complex roots $b \pm \sqrt{b^2 - c}$. The scheme $\mathrm{Spec}(\mathbb{R}[x])$ automatically includes both real and complex points, with the nuance that a single complex point is a conjugate pair of complex roots.

A polynomial equation like $x^2 + y^2 = 1$ has many kinds of solutions. One could think of rational and algebraic solutions as kinds of complex solutions. But solutions modulo a prime p, such as $x = 2$ and $y = 6$ in the finite field $\mathbb{F}_{13}$, are not complex numbers. And solutions modulo one prime are different from those modulo another. All these solutions are organized in the single scheme

$$\mathrm{Spec}(\mathbb{Z}[x, y]/(x^2 + y^2 - 1)).$$

The coordinate functions are simply integer polynomials modulo $x^2 + y^2 - 1$. The nonzero prime ideals are not simple at all. They correspond to solutions of this equation in all the absolutely algebraic fields by which Weil explicated Kronecker's goal, including all finite fields. Indeed, the closest Grothendieck comes to defining schemes in *Récoltes et Semailles* is to call a scheme a "magic fan" (*éventail magique*) folding together varieties defined over all these fields (p. P32). This is algebraic geometry over the integers.

Now consider the ideal $(x^2 + y^2 - 1)$ consisting of all polynomial multiples of $x^2 + y^2 - 1$ in $\mathbb{Z}[x, y]$. It is prime, so it is a point of $\mathrm{Spec}(\mathbb{Z}[x, y])$. And schemes are not Hausdorff spaces: their points are generally not closed in the Zariski topology. The closure of this point is $\mathrm{Spec}(\mathbb{Z}[x, y]/(x^2 + y^2 - 1))$. This ideal is the *generic point* of the closed subscheme

$$\mathrm{Spec}(\mathbb{Z}[x, y]/(x^2 + y^2 - 1)) \rightarrowtail \mathrm{Spec}(\mathbb{Z}[x, y]).$$

The irreducible closed subschemes of any scheme are, roughly speaking, given by equations in the coordinate ring, and each has exactly one generic point.

In the ring $\mathbb{Z}[x, y]/(x^2 + y^2 - 1)$ the ideal $(x^2 + y^2 - 1)$ appears as the zero ideal, since in this ring $x^2 + y^2 - 1 = 0$. So the zero ideal is the generic point for the whole scheme $\mathrm{Spec}(\mathbb{Z}[x, y]/(x^2 + y^2 - 1))$. What happens at this generic point also happens *almost everywhere* on $\mathrm{Spec}(\mathbb{Z}[x, y]/(x^2 + y^2 - 1))$. Generic points like this achieve what earlier algebraic geometers sought from their attempts.

Schemes vindicate more classical intuitions as well. Ancient Greek geometers debated whether a tangent line meets a curve in something more than a point. Scheme theory says yes: a tangency is an infinitesimal segment around a point.

The contact of the parabola $y = x^2$ with the x-axis $y = 0$ in $\mathbb{R}^2$ is plainly given by $x^2 = 0$. As a variety it would just be the one point space $\{0\}$, but it gives a nontrivial scheme $\mathrm{Spec}(\mathbb{R}[x]/(x^2))$. The coordinate functions are real polynomials modulo x^2 or, in other words, real linear polynomials $a + bx$.

Intuitively $\mathrm{Spec}(\mathbb{R}[x]/(x^2))$ is an infinitesimal line segment containing 0 but no other point. This segment is big enough that a function $a + bx$ on it has a slope b but is too small to admit a second derivative. Intuitively a scheme map v from $\mathrm{Spec}(\mathbb{R}[x]/(x^2)$ to any scheme S is an infinitesimal line segment in S, i.e. a tangent vector with base point $v(0) \in S$.

Grothendieck's signature method, called the *relative viewpoint*, also reflects classical ideas. Earlier geometers would speak of, for example, $x^2 + t \cdot y^2 = 1$ as a quadratic equation in x, y with parameter t. So it defines a conic section E_t which is an ellipse or a hyperbola or a pair of lines depending on the parameter. More deeply, this is a cubic equation in x, y, t defining a surface E bundling together all the curves E_t. Over the real numbers this gives a map of varieties

$$E = \{\langle x, y, t\rangle \in \mathbb{R}^3 | x^2 + t \cdot y^2 = 1\} \longrightarrow \mathbb{R} \qquad \langle x, y, t\rangle \mapsto t\ .$$

Each curve E_t is the *fiber* of this map over its parameter $t \in \mathbb{R}$. Classical geometers used the continuity of the family of curves E_t bundled into surface E but generally left the cubic surface implicit as they spoke of the variable quadratic curve E_t.

Grothendieck used rigorous means to treat a scheme map $f : X \to S$ as a single scheme simpler than either one of X and S. He calls f a *relative scheme* and treats it roughly as the single fiber $X_p \subset X$ over some indeterminate $p \in S$.[1]

Grothendieck had this viewpoint even before he had schemes:

> Certainly we're now so used to putting some problem into relative form that we forget how revolutionary it was at the time. Hirzebruch's proof of Riemann-Roch is very complicated, while the proof of the relative version, Grothendieck-Riemann-Roch, is so easy, with the problem shifted to the case of an immersion. This was fantastic. [18, p. 1114]

What Hirzebruch proved for complex varieties Grothendieck proved for suitable maps $f : X \to S$ of varieties over any field k. Among other advantages this allows reducing the proof to the case of maps f, called immersions, with simple fibers.

The method relies on *base change* transforming a relative scheme $f : X \to S$ on one base S to some $f' : X' \to S'$ on some related base S'. Fibers themselves are an example. Given $f : X \to S$ each point $p \in S$ is defined over some field k, and $p \in S$ amounts to a scheme map $p : \mathrm{Spec}(k) \to S$. The fiber X_p is intuitively the part of X lying over p and is precisely the relative scheme $X_p \to \mathrm{Spec}(k)$ given by pullback:

$$\begin{tikzcd} X_p \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[r, "p"'] & S \end{tikzcd}$$

Other examples of base change include extending a scheme $f : Y \to \mathrm{Spec}(\mathbb{R})$ defined over the real numbers into one $f' : Y' \to \mathrm{Spec}(\mathbb{C})$ over the complex numbers by pullback along the unique scheme map from $\mathrm{Spec}(\mathbb{C})$ to $\mathrm{Spec}(\mathbb{R})$:

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{R}) \end{tikzcd}$$

Other changes of base go along scheme maps $S' \to S$ between schemes S, S' taken as parameter spaces for serious geometric constructions. Each is just a pullback in the sense of category theory, yet they encode intricate information and express operations which earlier geometers had only begun to explore. Grothendieck and Jean Dieudonné took this as a major advantage of scheme theory:

> The idea of "variation" of base ring which we introduce gets easy mathematical expression thanks to the functorial language whose absence no doubt explains the timidity of earlier attempts. [17, p. 6]

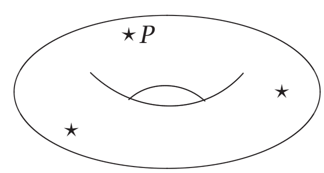
Étale Cohomology

In the Séminaire Chevalley of April 21, 1958, Serre presented new 1-dimensional cohomology groups $\tilde{\mathrm{H}}^1(\mathrm{X}, \underline{\mathrm{G}})$ suitable for the Weil conjectures: "At the end of the oral presentation Grothendieck said this would give the Weil cohomology in all dimensions! I found this very optimistic" [31, p. 255]. That September Serre wrote:

> One may ask if it is possible to define higher cohomology groups $\tilde{\mathrm{H}}^q(\mathrm{X}, \underline{\mathrm{G}})$…in all dimensions. Grothendieck (unpublished) has shown it is, and it seems that when G is finite these furnish "the true cohomology" needed to prove the Weil conjectures. On this see the introduction to [14]. [28, p. 12]

Grothendieck later described that unpublished work of 1958, saying, "The two key ideas crucial in launching and developing the new geometry were those of scheme and of topos. They appeared almost simultaneously and in close symbiosis." Specifically he framed "the notion of *site*, the technical, provisional version of the crucial notion of *topos*" [R&S, pp. P31 and P23n]. But before pursuing this idea into higher-dimensional cohomology he used Serre's idea to define the fundamental group of a variety or scheme in a close analogy with Galois theory.

Notice that Zariski topology registers punctures much more directly than it registers holes like those through the center of the torus or inside the tube.

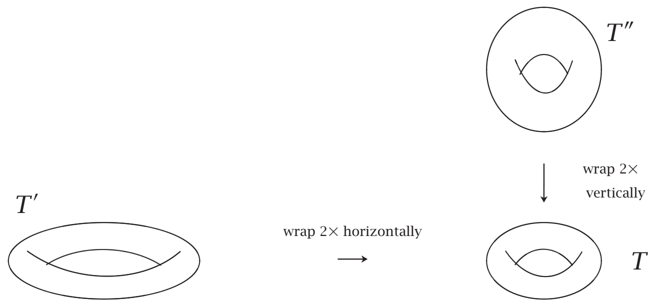


Zariski closed subsets are (locally) the zero-sets of polynomials, so a nonempty Zariski open subset of the torus is the torus minus finitely many punctures (possibly none). Such a subset might or might not be punctured at some point P itself, so the Zariski opens themselves distinguish between having and not having that puncture. But every nonempty Zariski open subset surrounds the hole through the torus center and the one through the torus tube. These subsets by themselves cannot distinguish between having and not having those holes. Coherent cohomology registers those holes by using coherent sheaves, which cannot work for the Weil conjectures, as noted above.

So Serre used many-sheeted covers. Consider two different 2-sheeted covers of one torus T. Let torus T' be

[1] *In 1942 Oscar Zariski urged something like this to Weil* [23, p. 70]. *Weil took the idea much further without finally making it a working method* [38, p. 91ff].

twice as long as T, with the same tube diameter. Wrap T' twice around T along the tube:



Let torus T'' be as long as T with twice the tube diameter. Wrap it twice around the tube. The difference between these two covers, and both of them from T itself, reflects the two holes in T.

Riemann created Riemann surfaces as analogues to number fields. As $\mathbb{Q}[\sqrt{2}]$ is a degree 2 field extension of the rational numbers $\mathbb{Q}$, so $T' \to T$ is a degree 2 cover of T. As $\mathbb{Q}[\sqrt{2}]/\mathbb{Q}$ has a two element Galois group where the nonidentity element interchanges $\sqrt{2}$ with $-\sqrt{2}$, so $T' \to T$ has a two-element symmetry group over T where the nonidentity symmetry interchanges the two sheets of T' over T.

Serre consciously extended Riemann's analogy to a far-reaching identity. He gave a purely algebraic definition of *unramified covers* $S' \to S$ which has the Riemann surfaces above as special cases, as well as Galois field extensions, and much more. Naturally, in this generality some theorems and proofs are a bit technical, but over and over Serre's unramified covers make intuitions taken from Riemann surfaces work for all these cases. Grothendieck used these to give the first useful theory of the fundamental group of a variety or a scheme, that is, the one-dimensional *homotopy*. He also worked with a slight generalization of unramified covers, called *étale maps*, which include all algebraic Riemann covering surfaces.

Serre had not calculated cohomology of sheaves but of *isotrivial fiber spaces*. Over a torus T those are roughly spaces mapped to T which may twist around T but can be untwisted by lifting to some other torus $T''' \to T$ wrapped some number of times around each hole of T. While Grothendieck [12] also used fiber spaces for one-dimensional cohomology, he found his *Tôhoku* methods more promising for higher dimensions. He wanted some notion of sheaf matching Serre's idea.

During 1958 Grothendieck saw that instead of defining sheaves by using open subsets $U \subseteq S$ of some space S, he could use étale maps $U \to S$ to a scheme. He published this idea by spring 1961 [15, §4.8, p. 298]. Instead of inclusions $V \subseteq U \subseteq S$, he could use commutative triangles over S:

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr] & & U \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} U \times_S V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & S \end{tikzcd}$$

In place of intersections $U \cap V \subseteq S$ he could use pullbacks $U \times_S V$ over S. Then an *étale cover* of a scheme S is any set of étale maps $U_i \to S$ such that the union of all the images is the whole of S. Sites today are often called *Grothendieck topologies*, and this site may be called the *étale topology* on S.

There are two basic ways to solve a problem locally in the étale topology on S. You could solve it on each of a set of Zariski open subsets of S whose union is S, or you could solve it in a separable algebraic extension of the coordinate ring of S. The first gives an actual, global solution if the local solutions agree wherever they overlap. The second gives a global solution if the local solution is Galois invariant—like first factoring a real polynomial over the complex numbers, then showing the factors are actually real. Étale cohomology would measure obstructions to patching actual solutions together from combinations of such local solutions.

In 1961 Michael Artin proved the first higher-dimensional geometric theorem in étale cohomology [1, p. 359]. According to David Mumford this was that the plane with origin deleted has nontrivial H^3; in the context of étale cohomology that means the coordinate plane punctured at the origin, $k^2 - \{0\}$, for any field of coordinates k. Weil's conjectures suggest that, when k is absolutely algebraic, this cohomology should largely agree with the classical cohomology of the complex case $\mathbb{C}^2 - \langle 0, 0\rangle$. That space is topologically $\mathbb{R}^4$ punctured at its origin. It has the classical cohomology of the 3-sphere S^3, and that is nontrivial in H^3. So Artin's result needed to hold in any Weil cohomology. Artin proved it does hold in the derived functor cohomology of sheaves on the étale site. Today this is *étale cohomology*.

In short, Artin showed the étale site yields not only some sheaf cohomology but a good usable one. Classical theorems of cohomology survive with little enough change. Grothendieck invited Artin to France to collaborate in the seminar that created *Théorie des topos et cohomologie étale* [2]. The subject exploded, and we will go no further into it.

Toposes are less popular than schemes or sites in geometry today. Deligne expresses his view with care: "The tool of topos theory permitted the construction of étale cohomology" [10, p. 15]. Yet, once constructed, this cohomology is "so close to classical intuition" that for most purposes one needs only some ordinary topology plus "a little faith/*un peu de foi*" [9, p. 5]. Grothendieck would "advise the reader nonetheless to learn the topos language which furnishes an extremely convenient unifying principle" [5, p. VII].

We close with Grothendieck's view of how schemes and his cohomology and toposes all came together in étale cohomology, which indeed in his hands and Deligne's gave the means to prove the Weil conjectures:

> The crucial thing here, from the viewpoint of the Weil conjectures, is that the new notion of space is vast enough, that we can associate to each scheme a "generalized space" or "topos" (called the "étale topos" of the scheme in question). Certain "cohomology invariants" of this topos (as "babyish" as can be!) seemed to have a good chance of offering "what it takes" to give the

conjectures their full meaning, and (who knows!) perhaps to give the means of proving them. [16, p. P41]

Courtesy of Patricia Princehouse.

Colin McLarty with a Great Pyrenees, from the region where Grothendieck spent much of his life.

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COMMUNICATION

Meeting Grothendieck, 2012

Katrina Honigs

Photo courtesy of Michael Moewe.

Katrina Honigs enjoys travel. In addition to having met Alexander Grothendieck, she also once touched the nose of a marmot in the Swiss Alps. Both experiences were very thrilling, though in different ways.

I met Alexander Grothendieck on January 2, 2012. I had read a bit of his mathematical work and felt a sense of connection with it. I was at a point in my third year of graduate school where I was not only not making progress on solving any problems but miserably unengaged by my work. Despite the burnout, Grothendieck's work remained an island of enjoyment in an otherwise featureless sea. So when I was in France for a conference, I sought him out.

The village of Lasserre is small and remote. My appearance in a rental car was a strange enough event that as soon as I parked, a friendly man came out of a nearby house to ask if I needed any help. It turned out Grothendieck's house was not fifty feet away.

I had purchased some *galettes du roi* that morning in preparation for apologizing to Grothendieck for entering his yard. After clearing the fence, I stepped furtively across the slightly ramshackle yard, which had many plants and terra cotta pots in various degrees of wholeness, and walked up the steps. I knocked on the door and then shouted "Monsieur Grothendieck!" and waited, but there was no response.

Suddenly I realized that a figure with a large white beard and a brown robe over his clothes had appeared utterly silently quite nearby on my left. In one hand, he held a short pitchfork loosely at his side. It reminded me of his doodle of devils with pitchforks around the Grothendieck-Riemann-Roch formula. His free hand rose, brandishing an admonitory finger. "Il ne faut pas entrer," he said, advancing slowly toward me. I tried to form some sentences about visiting, but Grothendieck did not react. He continued to walk slowly toward me, wagging his finger, telling me that I shouldn't be in here disturbing him. I tried to give him the galettes, but he told me again to leave.

Once he had seen me leave his yard, we studied each other from opposite sides of the gate for a moment. We were a similar height, and his blue eyes were alert and focused. Grothendieck asked me not angrily, but a bit sternly, in French how I knew his address and how I had gotten there. He told me again that I should not have come in and should not have disturbed him in his "cloître", which reinforced the impression given by the brown robe that he thought of himself, in some sense, as a monk. When I was given the address, I had said I wouldn't tell Grothendieck how I came by it, so I just watched him silently during this monologue, looking shocked.

Then, he asked me my name and explained that he could not hear very well anymore and so I must shout into his ear. After I said my name, I started to spell it, but he stopped me partway, since he had already recognized it: a couple of weeks before I had sent what I now realize was a very enthusiastic fan letter. He then switched to English and, irritably,

Katrina Honigs is research assistant professor at the University of Utah. Her email address is honigs@math.utah.edu.

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asked me why I had included a French translation. "To be polite? La politesse?" Nothing by him that I had read had been in English. But of course he knew English, and I had offended him a little.

He told me he had responded to my letter, explaining that my reasons for contacting him were not sufficient and that I should not visit. I felt a bit deflated but also couldn't help but be a little amused. Trust a mathematician to tell me that my reasons for writing were "not sufficient."

After Grothendieck discovered that I had not received his response to my letter, he seemed to decide that this partly explained my presence. But he was still dissatisfied and asked again why I had visited. Clearly a bit suspicious that I had some unsavory motive, he said he thought my visit must indicate that I wanted something. I told him that maybe he didn't realize, but he is very famous and I just wanted to meet him. He shrugged and said again that I didn't have any satisfactory reason for visiting, but I could tell that he was a bit amused by being told that he was famous, and he relaxed a bit.

He told me that he could see from my face that I didn't have any bad intentions and that he would never want to harm anyone. I saw then that the pitchfork was no longer in his hand but propped against the fence. If you had received my reply, he said, you would have understood that I am not taking visitors and you should not disturb someone in their retirement. He expected, though, to receive a letter from me soon explaining how I got his address. "C'est la moindre des choses," he intoned, switching back to French for a moment. He used to receive all his visitors, he said, but he had had two very bad experiences and no longer did it, though he was very sorry that I came such a long way to not be invited in and sorry for himself as well that he was not able to invite me in. He took my hand and shook it. He told me that he thought we would meet again, very soon, though not in this life. He told me he thought that he would die within the year, though this prediction was made with a practiced air that suggested this was not the first time he had made it.

After these heavy declarations, he turned his attention back to my visit. For all his bluster about not wishing to be disturbed, a part of him was curious about his visitor. How did I get here? On a train? No, in a car. Am I rich? No. Am I poor? Of course I'm poor! I'm a graduate student! I laughed, and he chuckled good-naturedly. Am I alone? Yes. Didn't I have something for him? The bakery box reemerged, and I opened it to show him the contents. He looked at the pastry inside. What is it? Galettes. What? Galettes! Did you make them? No, I bought them. What? I bought them! Oh, thank you for making them. He took the box from me and said he wanted to get something for me too and then went back into his house. I was glad for my instinct to bring baked goods. They smooth everything over in the American Midwest, where I'm from.

I was not able to discuss math with him at all. At one point, when I tried to make our conversation more detailed by writing on a piece of paper, he waved it away. But we had spoken more than I had thought we might. When he came back out of his house he presented me with a tomato and a packet of almond paste. The tomato was large and fresh and came from his garden—impressive for January—and he told me to eat it in good health. He also said I should remember that it was his friend (likely something was lost in translation). The packet of almond paste was very large. A kilo.

"He told me we would meet again, very soon, though not in this life."

After the exchange of gifts was over, it seemed we were finished. Grothendieck wished me well, shook my hand again, and, after entreating me once more to write a letter telling him how I came to know his address, told me goodbye and walked back to his house. I said goodbye as well, but his back was already turned to me, and I realized right after I spoke that he likely didn't hear me.

My experience of the rest of the day was odd and heightened. The drive back through the countryside. The primary colors of the public transit train in Toulouse. The tomato, when I ate it later that day. As the days and weeks went on, the visit was something I reflected on with enjoyment. My burnout faded, and I got more excited about my work again.

A little while after my visit, I did write Grothendieck again, but my letters were returned. Although my fantasies of having some magical conversation about math with him had to be swept aside, I am grateful to have had the chance to meet him.

A fuller version of this essay is at Katrina Honigs's personal website, `math.utah.edu/~honigs/Grothendieck.pdf`.

CALL FOR CONFERENCE PROPOSALS

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The American Mathematical Society invites individuals and groups of individuals to serve as organizers of the conferences of the Mathematics Research Communities (MRC) program to be held in Snowbird, Utah, during the summer of 2018.

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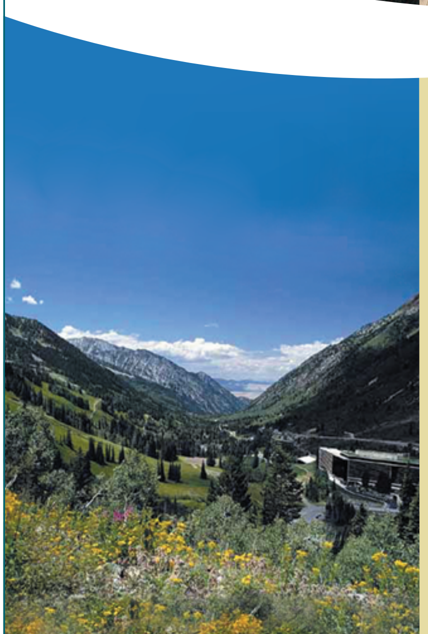

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COMMUNICATION

Who Would Have Won the Fields Medal 150 Years Ago?

Jeremy Gray

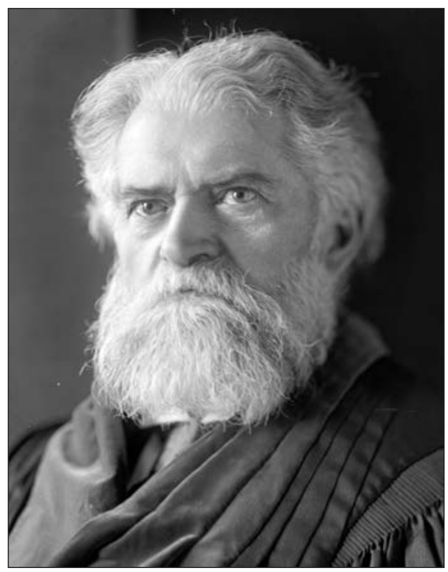
Communicated by Thomas Garrity

Introduction

Hypothetical histories are a way of shedding light both on what happened in the past and on our present ways of thinking. This article supposes that the Fields Medals had begun in 1866 rather than in 1936 and will come to some possibly surprising conclusions about who the first winners would have been. It will, I hope, prompt readers to consider how mathematical priorities change—and how hindsight can alter our view of the mathematical landscape.

Let us imagine that in 1864 the Canadian-American astronomer Simon Newcomb had seen past the horrors of Antietam and Gettysburg to a better world in which mathematics would take its place among other cultural values in the new republic. Let us further suppose that Newcomb, in his optimism for the future, had decided that there should be a prize awarded regularly to young mathematicians who had done exceptional work and that the first awards should be made in August 1866.

What might have motivated Newcomb to thus recognize exemplary mathematics research? Since 1861 Newcomb had worked at the Naval Observatory in Washington, DC, as astronomer and professor of mathematics. There Newcomb helped fill the void left when other academics, uncomfortable at being employed by a military institution during a time of war, resigned their positions at the observatory.



Simon Newcomb (1835–1909)

The US government tasked Newcomb with determining the positions of celestial objects. It was challenging work for Newcomb, who had had no formal education as a child and had learned mostly from his father before attending the Lawrence Scientific School at Harvard University and studying under Benjamin Pierce. It was this work which gave Newcomb his appreciation of mathematics.

To give the mathematics prize credibility, Newcomb would have realised that he would have to select a panel of sufficiently eminent judges. To find them, he would have to travel to Europe, so let us follow him to Paris in early 1865, the year he turned thirty.

Jeremy Gray is professor emeritus at the Open University and honorary professor at the University of Warwick. His email address is `j.j.gray@open.ac.uk`.

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The French were then feeling an unaccustomed inferiority. For two generations the French had dominated mathematics. Laplace, in the five-volume *Traité de Mécanique Céleste* (1798–1825), had given conclusive reasons to believe that Newtonian gravity rules the solar system and could explain all apparently discordant observations. Lagrange had been the true successor of Euler in many fields, and what little he had set aside Legendre had taken up. These luminaries had been succeeded by Cauchy, Fourier, and Poisson, and Paris had drawn in many of the best young mathematicians from abroad, Abel and Dirichlet among them.

But these great figures had all died by 1865, and news from the German states had the French looking uncomfortably second rate. The influence of Gauss, who had died in 1855, the year before Cauchy, was ever more apparent. Gauss had revivified the theory of numbers—"the higher arithmetic," as he had called it—making it more substantial than Euler or Lagrange had managed to. Closely intertwined with Jacobi's elliptic function theory, the field had been further extended by Dirichlet. Although Jacobi and Dirichlet had both died by 1865, the University of Berlin was flourishing, and, it had to be admitted, the École Polytechnique was not. Apparently the glory days of Napoleon had taught better lessons to those whose countries he had conquered than to the French themselves, who found themselves with a complacent government unable or unwilling to keep up.

Newcomb would make it his business to meet Joseph Liouville, who was the founder and editor of the important *Journal de Mathématiques Pures et Appliquées* and used its pages to keep the French mathematical community aware of what was happening abroad. Liouville had been involved in bringing to France Kummer's discovery of the failure of the prime factorisation of cyclotomic integers and his attempts to redefine "prime" to deal with this unexpected setback. Kummer had subsequently won the Grand Prix des Sciences Mathématiques of the Paris Academy of Sciences in 1857 for his work on the subject. As a successful editor, Liouville was the best person to ask about new and exciting mathematicians. And he would be fifty-six in 1865, too old to be considered for the prize himself.

Liouville had done important work on the theory of differential equations (Sturm–Liouville theory), potential theory, elliptic and complex functions, the shape of the earth, and other subjects. But Newcomb might well have been discouraged on consulting the most recent issues of the *Journal*, because they were full of Liouville's interminable and shallow investigations of the number theory of quadratic forms in several variables. In a sense, what Liouville's *Journal* and the *Journal* of the

Joseph Liouville (1809–1882)

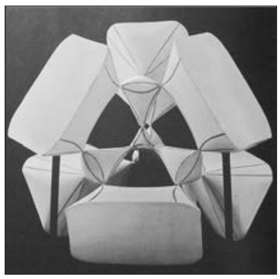
École Polytechnique showed was that mathematics in France was at a low ebb.

Liouville would surely have praised Charles Hermite highly, had Newcomb prompted him to suggest potential prizewinners. Hermite had studied with Liouville, and together they had come to a number of insights about elliptic functions. It was in this context that Liouville had discovered the theorem that still bears his name: a bounded complex analytic function that is defined everywhere in the plane is a constant. Hermite had gone on to explore the rich world of elliptic functions and in 1858 had used that theory and algebraic invariant theory to show that the general polynomial equation of degree five has solutions expressible in terms of elliptic modular functions. This work had drawn the attention of Kronecker and Brioschi and remained an insight that rewarded further attention. Galois's discovery that the general quintic equation was not solvable by radicals sat in the context of equations that are solvable by precise classes of analytic functions.

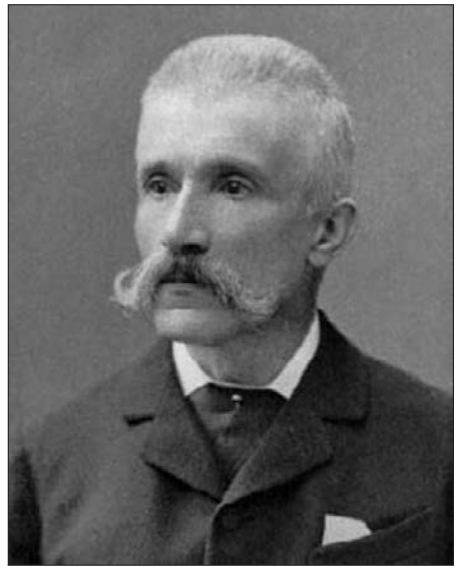
But Hermite was forty-two. Newcomb wanted to recognize younger mathematicians to point the way to the future, and he was beginning to realise that he needed more advisors than Liouville alone.

Kummer was the obvious choice. He was less than a year younger than Liouville, and he had worked on a variety of topics before deciding that Gaussian number theory was the rock on which to build a career. He had written on the hypergeometric equation (another Gaussian topic) and as recently as 1863 on a quartic surface with sixteen nodal points. Kummer had succeeded Dirichlet at Berlin in 1855, when Dirichlet moved to Göttingen to succeed Gauss, and had swiftly arranged for Weierstrass to be hired at Berlin. Weierstrass, who had just turned forty then, had only recently burst onto the mathematical scene with his theory of hyperelliptic functions and was now, with Kummer, establishing Berlin as the preeminent place to study mathematics. Furthermore, Kummer, as proof of his administrative ability, had just become dean of the University of Berlin. Newcomb would choose Kummer as the second member of his prize committee.

Ernst Eduard Kummer (1810–1893)



A quartic surface with sixteen nodal points



Francesco Brioschi (1824–1897)

By now Newcomb was becoming aware that the new state of Italy (unified only in 1861) was also producing important mathematicians. The Italian figures comparable to the German judges Kummer and Liouville were Enrico Betti and Francesco Brioschi, who had both just turned forty. Both had been actively involved in the unification of Italy and now led political lives: Betti had been elected to the Italian parliament in 1862, and Brioschi was to become a senator in 1865. Betti had just become the director of the Scuola Normale Superiore; Brioschi was the founder and director of the Istituto Tecnico Superiore in Milan. Both men were devoted to raising the standard of mathematics in Italy in both schools and universities; both were active in research. Betti, who had become a close friend of Riemann's when he stayed in Italy, was interested in extending Riemann's topological ideas and also worked on mechanics and theoretical physics. Brioschi had done important work in algebra, the theory of determinants, elliptic and hyperelliptic function theory, and he had taught many of the next generation of Italian mathematicians: Casorati, Cremona, and Beltrami among them. Newcomb, we shall suppose, would have decided that Brioschi was the man to keep him informed of the latest developments in the emerging domain of Italian mathematics.

Should that be enough, or should Newcomb make a trip to Britain? In pure mathematics this meant a visit to Cayley; in more applied fields there were several people at Cambridge who might be consulted. The mathematicians Newcomb had already met spoke well of Cayley and respected him as an inventive and well-read mathematician who spoke several languages. He was, along with his friend Sylvester, best appreciated for his exhaustive, and sometimes exhausting, investigations into invariant theory. But it did not seem that there was anyone in England in 1865 who could be considered for the prize, now that Cayley was in his early forties and thus ineligible himself.

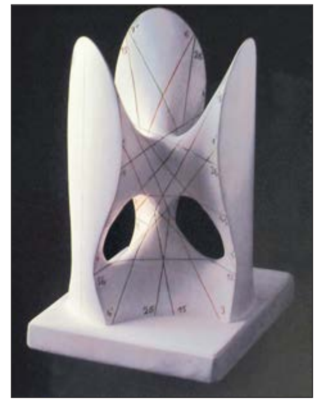
Newcomb decided that three judges were already enough: Liouville, Kummer, and Brioschi. It was time to select a prizewinner.

Newcomb already knew one name. Everyone he spoke to told him about Bernard Riemann, a truly remarkable former student of Gauss in Göttingen. Riemann had published a remarkable paper on abelian functions in 1857 which was so innovative that Weierstrass had withdrawn a paper of his own on the subject, saying that he could not proceed until he had understood what Riemann had to put forward. That same year, Riemann had published a very difficult paper on the distribution of the primes in which he made considerable use of the novel and fundamentally geometric theory of complex analytic functions that he had developed and which was essential to the paper on abelian functions. There was also a paper in real analysis where he had developed a theory of trigonometric series to explore the difficult subject of nondifferentiable functions, and there was talk of a paper in which he was supposed to have completely rewritten the subject of geometry.

Riemann would turn thirty-nine in 1865, so Newcomb could agree that he was still, officially,

Bernhard Riemann (1826–1866)

Luigi Cremona (1830–1903)



A cubic surface with 27 lines

young. There was the disturbing problem of Riemann's health, however. He suffered from pleurisy and was said to have collapsed in 1862 and to be recuperating in Italy. Newcomb would have to stay informed.

As for the generation born in the 1830s, Liouville, Kummer, and Brioschi might well have given different reports.

Liouville, to his regret, would have had no one to suggest. Kummer, too, would have struggled to name a nominee. His former student Leopold Kronecker had just turned forty, and although there were some promising younger mathematicians at Berlin—Lazarus Fuchs sprang to mind—they had yet to do anything remarkable.

Brioschi, on the other hand, would have been optimistic. He could have suggested the names of Cremona, Casorati, and Beltrami. Cremona was already known for his work on projective and birational geometry, including the study of geometric (birational) transformations, and Brioschi could have assured the panel that Cremona was writing a major paper on the theory of cubic surfaces (it was to share the Steiner Prize in 1866—Kummer was one of the judges—and was published as (Cremona 1868)). Casorati was perhaps the leading complex analyst the Italians had produced, and Beltrami was emerging as a differential geometer in the manner of Riemann.

Over the summer of 1865 Newcomb would have faced a difficult decision. No one disputed Riemann's brilliance, although Kummer reported that Weierstrass was hinting that not all of Riemann's claims were fully established, and Liouville was saying that Hermite was hoping for direct proofs of some results that presently relied on Riemannian methods.

The problem was to decide who else might get the prize. There were several bright young mathematicians, but none of the highest calibre. Should Newcomb announce the existence of the prize, call for nominations, and risk disappointment? Or should he postpone it and give the younger mathematicians a better chance to shine?

And then there was the worrisome matter of Riemann's deteriorating health. Weak and prone to illness, he had spent the summer recuperating near Lake Maggiore and in Genoa and had returned to Göttingen in early October.

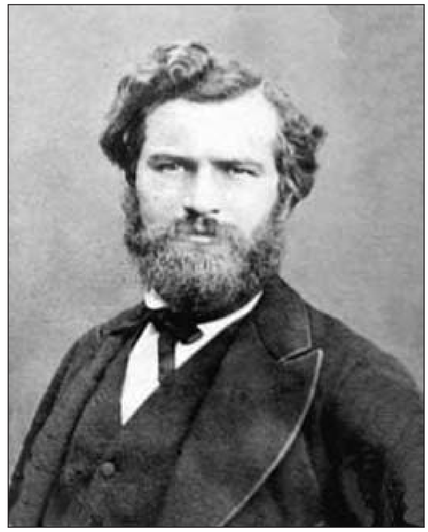
Let's suppose Newcomb decided to postpone the competition for four years.

Riemann died on July 20, 1866, within a month of returning to Lake Maggiore. He was thirty-nine. Among the papers published shortly after his death, the one entitled "The hypotheses that lie at the foundations of geometry" was to inspire Beltrami (born 1835) to publish his "Saggio", in which non-Euclidean geometry was described rigorously in print for the first time. The leading German physicist, Hermann von Helmholtz, was independently converted to the possibilities of spherical geometry and in correspondence with Beltrami came to advocate the possibilities of non-Euclidean ("hyperbolic") geometry as well.

Asked about candidates from Germany, Kummer could now offer three or four. The first was Rudolf Clebsch, who, with his colleague Paul Gordan, had devised an obscure but effective notation for invariants that had led to many new results, and he had applied himself successfully to the study of plane curves. He also had showed that elliptic functions can be used to parameterise cubic curves and in 1864 had begun to extend such ideas to curves of higher genus. Then, in 1868, he had opened the way to extending Riemann's ideas to the study of complex surfaces by defining the (geometric) genus of an algebraic surface.

The second was Lazarus Fuchs, a former student of Kummer's, who was now attached to Weierstrass and seemed poised to extend Riemann's ideas. So too was the third candidate, Hermann Amandus Schwarz, who was using Weierstrass's representations of minimal surfaces to tackle the Plateau

Eugenio Beltrami (1835–1900)



Camille Jordan (1838–1922)

Rudolf Clebsch (1833–1872)

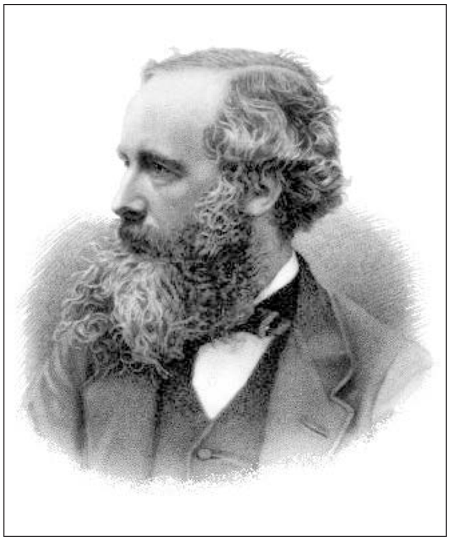
problem. He was also beginning to think about the Dirichlet problem.

Then there was the fourth candidate, Richard Dedekind, who was emerging as a number theorist in the tradition of Gauss and Dirichlet. But Kummer, and still more his colleague Kronecker, had their doubts about the highly abstract and not always explicit character of Dedekind's approach. It would seem appropriate to wait.

Liouville, too, would have had a new candidate to put forward as the 1860s came to an end: Camille Jordan. Jordan had published a series of papers that he was now drawing together in his book *Théorie des Substitutions et des Équations Algébriques*. In these papers and again in the book, he set out a theory of groups of substitutions (permutation groups) of great generality and showed how to use it to derive all of Galois's results systematically. He had then gone on to use it in a number of geometrical settings, finding, for example, the groups of the twenty-seven lines on a cubic surface and Kummer's surface with its sixteen nodal points. He outlined in over three hundred pages a programme to find all finite groups. Not everyone was convinced of the need for such a big new idea, but it was bold and rich in applications to topics that were known to be interesting.

And what of James Clerk Maxwell? Could his best work even be called mathematics? He had written on many subjects, but his major 1864 paper "A dynamical theory of the electromagnetic field" and a 1866 paper in which he suggested that electromagnetic phenomena travel at the speed of light (thus implying that light is just such a phenomenon) displayed a considerable mastery of the difficult mathematics they involved. He had also published his second major paper on the dynamical theory of gases, which did much to establish the statistical approach to physics.

So Newcomb would have had four candidates: Beltrami, Clebsch, Jordan, and Maxwell. Under pressure, Brioschi would have had to admit that Beltrami had published only one remarkable result amid a stream of good ones, mostly in differential geometry. But at least his work was independent of Riemann's, as Cremona would attest. Clebsch was another heir of Riemann's, but he worked in a tradition that was opposed in some ways to Kummer's way of doing things. Jordan was the youngest, and his advocacy of substitution group theory was controversial. Some found it a fine addition to the geometrical way of thinking, and some were to see in it the way to rewrite Galois theory the way Galois might have meant it, but others were to find it needlessly abstract and



James Clerk Maxwell

almost unnecessary even for Galois theory (and preferred the language of field extensions).

As for Maxwell, electricity and magnetism had been the major topics of mathematical physics for the preceding fifty years, but no one in continental Europe understood Maxwell's ideas. In particular, they found it incomprehensible that electric current was a discontinuity in a field and not the passage of a (possibly mysterious) substance. It was too risky to choose Maxwell.

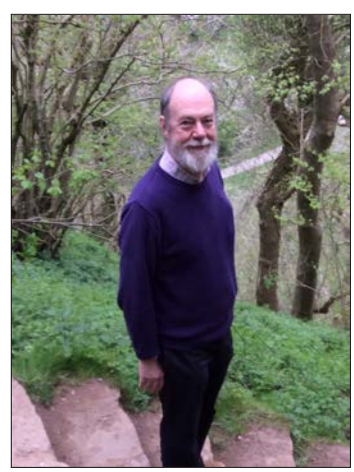
What might Brioschi, Kummer, Liouville, and Newcomb then decide? From today's perspective, it seems they should have rewarded (1) conclusive proof of the existence of a new, possible geometry of space (when the work of Bolyai and Lobachevskii was almost completely forgotten), and (2) a new and highly abstract structure (the group) that seemed to have many applications, although many important details remained to be published. Beltrami and Jordan might well be our modern choices.

But I suggest that Newcomb and his advisors would have chosen differently. Kummer had a great sympathy for the study of algebraic surfaces and a high opinion of Cremona's work. Brioschi could have agreed that it offered a way into the general study of algebraic geometry, to which Clebsch had also made a contribution, and that the new, non-Euclidean geometry, however remarkable, had yet to lead to new results. If Cremona represented the opening up of a subject that had long challenged mathematicians, then Jordan could embody the spirit of radical innovation, one that also led to insights into geometry. But such a decision would be strongly opposed by Kronecker and Hermite, who were powerful advocates of invariant theory, and their views would be well known to Kummer and Liouville. That would have made Clebsch a contender, not so much for his Riemannian work as for his development of invariant theory.

Newcomb, who was to publish a paper in 1877 on spaces of constant positive curvature, might have pushed hard for Beltrami. But in the 1860s his own work was firmly in mathematical astronomy, so I suppose that he would have let himself be guided by the counsel of his chosen judges. He surely would have wanted to see that Riemann's ideas were being carried forward, but the citation for Clebsch, the leading advocate of Riemann's ideas, would take care of that. The prizes, I conclude, would have gone to Cremona and Clebsch.

End Note

The real Fields Medal was established in the bequest of John Charles Fields, a Canadian mathematician who had been active in the International Mathematical Union. It was his wish that the prize be awarded every four years to two young mathematicians, and although he did not define "young," this has come to mean under forty, and that guideline has therefore been preserved here. The first Fields Medals were awarded in 1936 to Lars Ahlfors and Jesse Douglas.



Jeremy Gray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Can Math Education Research Improve the Teaching of Abstract Algebra?

Tim Fukawa-Connelly, Estrella Johnson, and Rachel Keller

Teaching matters. It is arguably the most important factor affecting student learning. Efforts to improve teaching have led to reform initiatives being proposed and tested throughout the college mathematics curriculum. Abstract algebra specifically has been the subject of such reform, including new curricula and pedagogies, since at least the 1960s, yet there is little evidence that these change initiatives have widely influenced the way abstract algebra is taught. We conducted a survey of abstract algebra instructors to investigate typical teaching practices and, more specifically, faculty knowledge, goals, and orientation towards teaching and learning. Results revealed that a majority of respondents appear quite content to lecture. Even among those who indicated a willingness to consider a change of pedagogical strategy, there is very little usage of existing reform materials or interaction with pedagogical research results. There appears to be an impermeable barrier between the pedagogical researchers' findings and recommendations and practitioners who might implement them.

Tim Fukawa-Connelly is assistant professor in the Department of Teaching and Learning at Temple University. His email address is `tim.fc@temple.edu`.

Estrella Johnson is assistant professor of mathematics at Virginia Tech. Her email address is `strej@vt.edu`.

Rachel Keller is a PhD student in mathematics education at Virginia Tech. Her email address is `rakeller@vt.edu`.

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Research Questions

There is essentially no research that helps us understand why some mathematicians adopt reform practices in their teaching and some do not [3]. There has been little research attempting to explore these issues from the perspective of the instructors who are the ones being asked to change practice. We investigated the following research questions: (1) What pedagogical practices do abstract algebra professors report using in their classrooms and why? (2) What encouragement and constraints on their use of nonlecture practices do they perceive?

Methods and Data Analysis

To create an instrument designed to measure the knowledge, goals, and teaching/learning orientation of mathematicians, we adapted questions from both Henderson and Dancy's physics education survey [1] and the Characteristics of Successful Programs in College Calculus survey (see `www.maa.org/cspcc` for more information about the CSPCC project). In addition to basic demographic information, the survey[1] questions asked the professors to rate the

[1] *Survey available at* `pcrg.gse.rutgers.edu/algebra-survey`.

importance of various sources of information and to list factors that influenced their teaching decisions. In an attempt to understand their beliefs about teaching and learning, we asked them to describe and characterize their classroom practices, including the motivation behind those choices. Finally, we asked questions designed to test the claim, found in the education literature, that instructors are reluctant to change and, if such resistance was identified, to elucidate the reasons why. Survey requests were sent to departmental administrators at approximately two hundred institutions, targeting instructors who teach undergraduate abstract algebra. Our intention was to survey instructors at Master's- and PhD-granting institutions; however, a small portion of our respondents (9 percent) did come from schools that offer only a Bachelor's degree in mathematics. In total, we received 131 completed surveys. On the whole, the respondents (92 percent tenure-stream faculty) had significant experience both with teaching in general (81 percent reporting 6+ years) and with abstract algebra specifically and were most likely to be teaching an undergraduate groups-first course designed for a mixed (i.e., education, physics, engineering majors commingled with pure math majors) audience. (See Figure 1.)

After compiling the demographic information, we focused our attention on instructor satisfaction in order to determine if any impetus for change existed. To address the first research question, we examined the self-reported teaching practices of the respondents by asking how frequently per class period they engaged in various practices, e.g., using visual or physical representations of groups, having students discuss or work together on problems, having students question one another. Allowable responses were: zero times, one or two times, three or more times. We compared these responses to instructors' self-reported satisfaction with outcomes and their extent of agreement with a series of statements designed to measure teaching/learning orientation. Some examples of those statements are: *I think lecture is the best way to teach; I think students learn better when they struggle with the ideas prior to me explaining the material to them; I think that all students can learn advanced mathematics.* Respondents indicated their level of agreement on a four-point scale.

In our discussion, we highlight areas where the respondents appear to hold beliefs that should lead to certain pedagogical actions but they themselves do not report engaging in those actions. To address the second research question, we categorize instructor reports on implementation of nonlecture reform practices in terms of perceived constraints and viable supports, and we compare these with those cited in the literature.

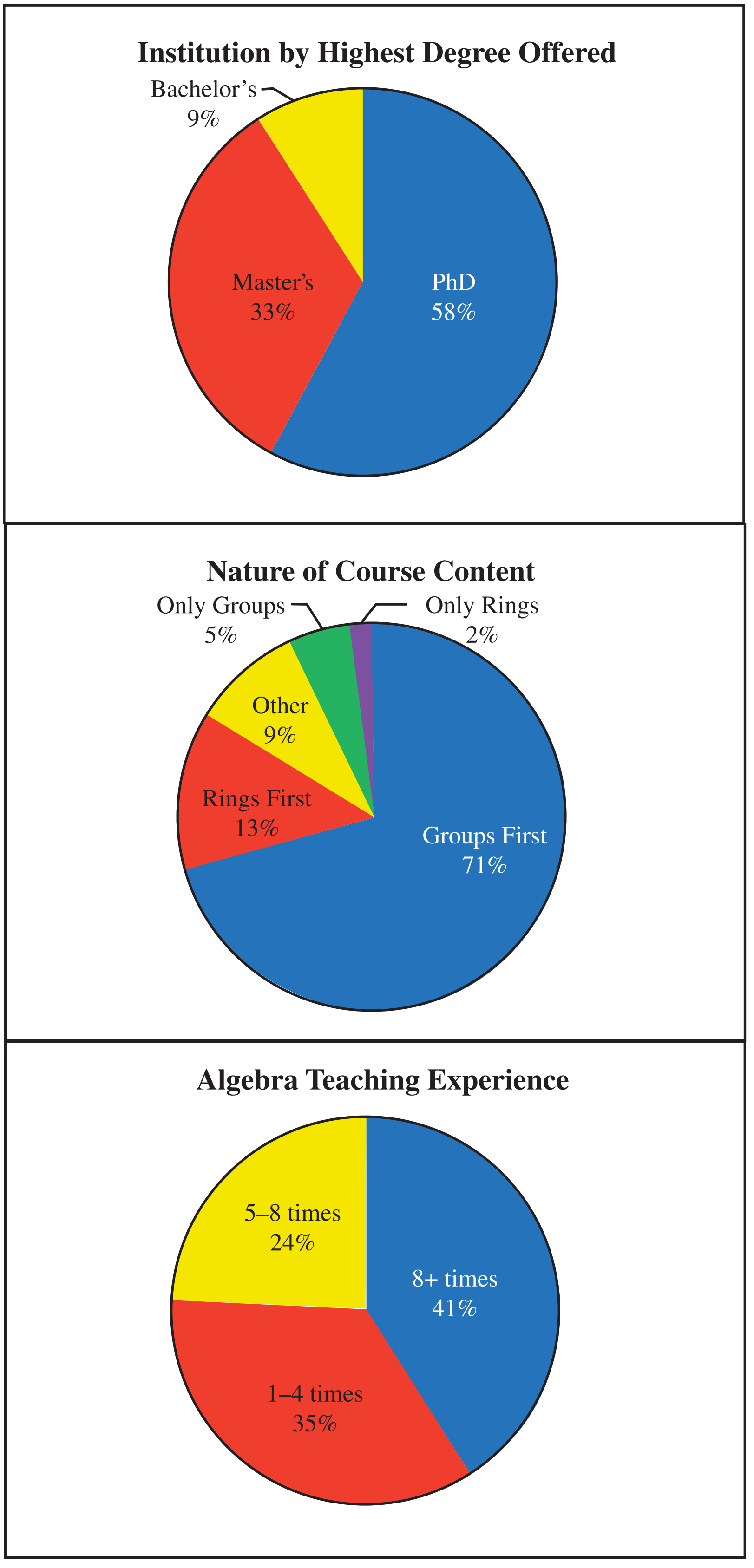

Figure 1. Information about Survey Respondents.

Results

Satisfaction

When measuring satisfaction, several dimensions were considered. For this report, we choose to discuss two in particular: textbook and student learning outcomes. The questions, asked separately in open-ended format (*How satisfied are you with your textbook/students' learning? Please give some explanation.*), were analyzed and categorized by the research team in terms of level of satisfaction: *Satisfied, Mixed, Dissatisfied.* Collectively, 87.6 percent of respondents indicated that they were satisfied with the textbook they used. Instructor comments indicated that the satisfactory rating stemmed from the breadth, depth, and sequencing of content. Even amongst the satisfied, however, complaints about pricing and frequency of new editions were rampant.

When reporting on satisfaction with student learning outcomes, an overwhelming majority of the classifiable responses fell into the *Mixed* (44 of 89) or *Satisfied* (23) categories; fewer than one-quarter gave responses we categorized as *Dissatisfied* (22). The responses were organized by domain (student engagement, student preparation, student performance, student understanding, curriculum issues) and level of satisfaction, allowing us to look for common themes. In summary, instructors that we interpreted as reporting *Mixed* satisfaction indicated (unsurprisingly) that students learned most of the important content and worked reasonably hard. The courses might be in need of a little reorganization or supplemental materials, but major pedagogical overhauls were considered neither warranted nor desired. The comments of the instructors we characterized as *Dissatisfied* were complaints about the unsatisfactory work ethic, motivation, and ability of the students. In contrast, the *satisfied* instructors were less likely to mention the students; rather, instructors who reported high levels of satisfaction were the most likely to comment on the format and curriculum of their courses, with nearly 40 percent (9/23) of them indicating belief that their course was different from most traditional abstract algebra courses due to the use of some form of inquiry-based learning (increased use of examples, student research, Modified Moore Method, etc.).

While the groups did vary widely in typical responses, it was interesting to note that there were two common themes that emerged across all levels of satisfaction. The first observation was a general frustration with students' lack of prerequisite proof skills and poor proof-writing ability. The other common opinion was that it was both difficult and inappropriate to design and teach a course for different constituencies (most often cited was the commingling of math and math education majors). The consensus was that neither population was being adequately served by teaching them simultaneously. A surprising finding of our research was that despite this mixed sense of satisfaction with student learning outcomes, the overall grade distribution was actually quite agreeable. Of those instructors surveyed, the combined passing rate of the students was a whopping 87.82 percent (33.37 percent A, 33.85 percent B, 20.55 percent C), with only 12.18 percent receiving D/F/W grades.

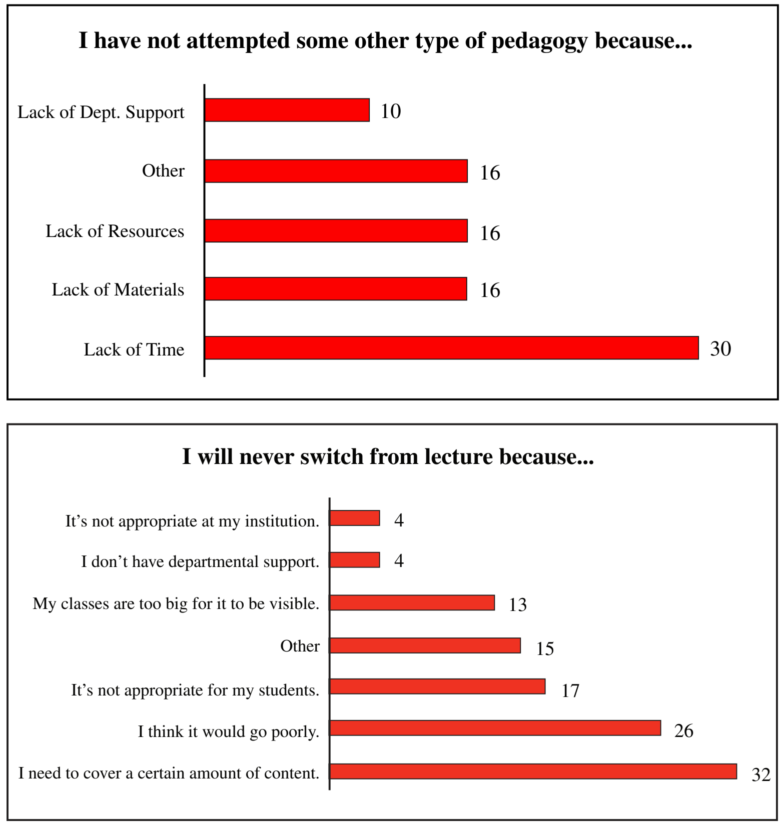

Figure 2. Perceived constraints on the use of nonlecture practices.

…there exists a mismatch between beliefs about student learning and actual teaching practice.

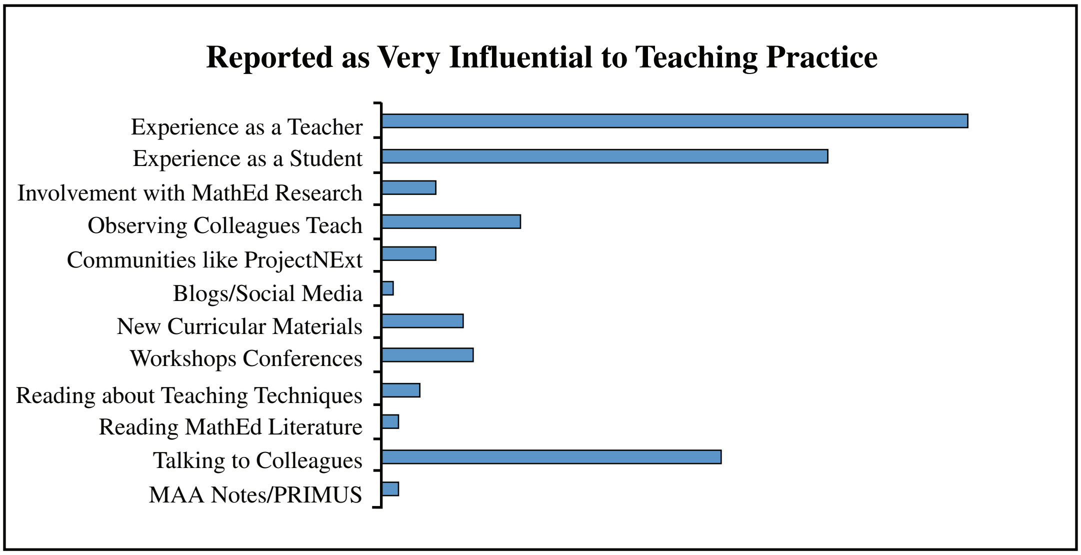

Figure 3. Resources reported as Very Influential by respondents.

Teaching Methods

Lecture was the most common pedagogical practice, with 85 percent of respondents claiming that they currently lecture to teach abstract algebra. This includes the 8 percent of instructors who report returning to lecture after trying some other method. Of the 23 percent who either now or in the past used nonlecture pedagogy and curricular materials, most (fifteen respondents) created it themselves without formal support (typically drawing on a mixture of texts and problem-sets). There were only two respondents who cited use of a particular established curriculum (*Teaching Abstract Algebra for Understanding*, Larsen, 2013; *Learning Abstract Algebra with ISETL*, Dubinsky and Leron, 1994). The others used their own experiences with inquiry-based classes, collaboration with other instructors practicing IBL, or participation in the Academy of Inquiry-Based Learning as a guide to develop their materials and shape their practice.

Of the 85 percent who are currently teaching with lecture, 56 percent of them say that they would consider teaching with nonlecture practices (the remaining 44 percent say they would never do so). The reasons instructors provided for not yet attempting other pedagogy and the explanations offered for why they would never change their habits can be seen in Figure 2.

In short, the two main themes in the comments related to the effort and support needed to revise and teach such a class and concerns about covering the appropriate amount of material. Of the thirty-two instructors who stated coverage as a reason not to adopt a nonlecture format, twenty-three of them answered in the negative when asked: *Do you feel pressure from your department to cover a fixed set of material in your abstract algebra course?* It appears, therefore, that concerns about coverage might be internally situated rather than stemming from an actual source of external pressure.

One of the most interesting findings was the apparent contradiction that emerged when comparing the responses to the following prompts. 82 percent of respondents agreed with the statement *Lecture is the best way to teach.* However, 56 percent agreed (and 26 percent more slightly agreed) with the statement *I think students learn better when they do mathematical work (in addition to taking notes and attending to the lecture) in class.* This result suggests that faculty support the use of nonlecture class activities, yet when asked what students do in class besides taking notes (given a list of options), the only things that instructors claimed that students did in class, even at a rate of once per month, was doing calculations, working with examples, or working with applications. Moreover, 63 percent reported that students never spent time working on mathematics problems in class. So it would appear that what instructors think is best for student learning (students doing mathematical work in class) is not happening with any frequency. Thus, we argue that there exists a mismatch between beliefs about student learning and actual teaching practice. One could argue that this mismatch might be explained by a perceived lack of time to make adjustments to their teaching practices on the part of the instructors; however, the data indicate otherwise. When asked if they believe that they would have time to plan and redesign their courses in a way that would be supported and valued in terms of formal review, nearly all (100/129) respondents reported this as a possibility (42 *yes* and 58 *maybe*). Therefore, in general, it does not appear to be the case that time

Photo by Lawrence Academy.

Photo by David Plaxco, 2014.

Tim Fukawa-Connelly (top) and Estrella Johnson (bottom) teaching.

constraints alone account for the discrepancy between how instructors say they want to teach and how they actually teach.

Influences on Instruction

When asked to identify the primary influences on their teaching practice (*How influential are the following on your teaching? Very/Somewhat/Not at all*), the respondents overwhelmingly identified three sources of inspiration. In decreasing order of significance, the participants reported that their experiences as a teacher (84 percent) and experiences as a student (64 percent) were far and away the most significant. Participants also reported that talking to colleagues about how to teach specific content was important (49 percent). Little importance was assigned to the normal means that grant-supported projects use to disseminate new teaching ideas: Project NExT (8 percent), MathFest, MAA mini-courses or other workshops (13 percent), or publications about teaching such as the MAA Notes series or PRIMUS (2 percent). From these numbers, it appears that most mathematicians have few influences on their teaching that are external to their own universities. To be fair, it should be noted that we do not know the distribution of those individuals who read the literature and attend professional development opportunities. Unless there exist mechanisms of which we are unaware, it appears that the majority of math departments might be closed to outside influences on teaching.

This lack of outside influences is likely to be especially prohibitive for the 59 of 106 respondents who would consider trying something other than lecture but have not because they haven't had time to redesign their course (30/59), haven't found materials that they like (16/59), or don't know where to start (16/59). So, it would appear that the very resources designed to alleviate some of these challenges are failing to meet that objective. Again, looking only at the 59 of 106 participants who state that they would consider not lecturing, only one finds PRIMUS or the MAA Notes series *very influential*; only 1 finds mathematics education research literature *very influential*; only 6 find talks, workshops, or conferences about teaching (e.g., MathFest mini-courses) *very influential*; and only 4 find participating in communities like Project NExT *very influential*. It is our belief that this is not because the materials themselves are not useful, but rather that those who most need them are not utilizing them.

Conclusions

There are four major findings that we highlight. First, lecture is the predominant mode of instruction (97/126), and even those who have tried other pedagogies appear to switch back to lecturing at surprisingly high rates (10/29). Moreover, given the significant amount of time, money, and energy spent developing, testing, promoting, and training mathematicians to use new curricula and pedagogies, there is almost no uptake. Those using nontraditional materials are far more likely to have developed their own materials than to have adopted NSF-supported curricula.

The second major finding relates to the factors that influence pedagogical decisions. In decreasing order of significance, the participants reported that their experiences as a teacher and student were far and away the most significant influence, followed by talking to colleagues about how to teach specific content; the least significant source of influence was grant-supported distribution methods such as publications and workshops. If mathematicians essentially give no weight to the traditional means of dissemination of new pedagogical ideas and techniques (and evidence of their effectiveness), reformers have few means of promoting change other than individual conversation. This alone suggests why reforming undergraduate mathematics, and abstract algebra in particular, is difficult.

Third, while faculty claim they have the ability to change their courses, the reported satisfaction levels indicate they do not have the desire to do so. Furthermore, the majority of dissatisfaction stems from perceived problems with the students and not the course materials. Given the strong content focus and high belief in the efficacy of (and preference for) lecture, it appears that as a collective, the abstract algebra teaching faculty has little interest in adopting new pedagogical approaches at this time.

We propose two concurrent research directions. First, we need to better explore the reasons that mathematicians appear to strongly believe in their current practice, the types of evidence that they hold as dispositive, and what means of dissemination of new approaches achieve meaningful penetration. Second, we need to further explore the types of changes to the practice of lecture that mathematicians would adopt. There appears to be a conflict between the stated goals of policy boards and national organizations and the way that faculty, on the ground, think about their courses. Math educators are responding to the claims of the stated goals of changing undergraduate courses to include more student-active work, but if mathematicians have different perceived needs, as our work shows, these new ideas won't gain traction. Thus, we want to have a conversation about what is understood as practical and feasible in the eyes of those charged with delivering the instruction.

Finally, for us and mathematics education researchers generally, we wonder how best to propose new strategies about teaching and how to receive feedback from the mathematical community as to their interest and feasibility. Basically, if the only people that mathematics instructors ever talk to are their colleagues, it is a closed circle with no obvious entry point for new ideas. As an example, a major source of dissatisfaction revealed in this survey was instructor frustration with students' poor proof-writing abilities, an area that has received significant attention from mathematics education researchers and has produced practical suggestions for improving proof comprehension. These ideas are heavily researched and, given the comprehensive pedagogical supports available, often do not require the extensive time commitment often incorrectly assumed of nontraditional methodology. But without open communication between researchers and practitioners, the validity and viability of these ideas go unappreciated. We mathematics education researchers have spent significant time, literally decades, trying to understand how students learn mathematics in general and specific content areas in particular. Help us to help you. If you are dissatisfied with your current practice or results, if you are frustrated by lecture-dominated classes, if you are looking for inspiration—we might have the answer. All you have to do is ask!

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Elisenda Grigsby Interview

Photo courtesy Lee Pellegrini.

Elisenda Grisby *is associate professor at Boston College. She won an NSF CAREER grant in 2012, and more recently the inaugural AWM-Joan and Joseph Birman Research Prize.*

Diaz-Lopez: When did you know you wanted to be a mathematician?

Grigsby: Not until pretty late. I always knew I liked science, but when I first got to college I planned to major in biochemistry. I ended up taking this amazing seminar-style course led by David Layzer (an astrophysicist) and Cynthia Friend (a chemist), consisting of twice-weekly meetings to discuss together how we'd tackle some really hard (but fun!) problems.

After that course, I decided to major in physics and ended up taking a number of math courses to go along with it. But I burned out halfway through my junior year and took a semester off from school to do something completely different. I got a plane ticket to London and a six-month work permit. I spent the semester waitressing, and ended up reading topology textbooks in my spare time.

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When I came back, I decided to switch my major to math. At that point, though, I was completely outside of the math community at my school—a lot of the majors had gone through the standard math sequence during their first year, and I didn't know any of those people, so I felt a bit isolated. By the time I graduated, I had decided not to pursue a PhD. I liked math, but I still had the impression that I wasn't good enough at it to actually do it for a living.

Instead I worked for a year doing decision analysis/operations research at a company in Silicon Valley. This was a great experience, but the main thing it taught me was how much more fun I was having back when I was doing math. A public lecture by Brian Greene that I attended that fall at Stanford clinched it. I applied for grad school that winter, and felt very lucky to be accepted by Berkeley—which ended up being the perfect place for me.

Diaz-Lopez: Who encouraged or inspired you?

Grigsby: So many people that I can't possibly list them all without accidentally leaving off someone important, so I'll just list the obvious ones. David Layzer was the first: his Chem 8/9 course was really a turning point for me, and he went out of his way to convince me that I belonged, at a time when I really didn't think I did. Then my undergraduate thesis advisor, Ken Fan: he knew how to encourage and direct me without being overbearing about it. As a graduate student: of course my two advisors Rob Kirby and Peter Ozsváth. Their styles of doing mathematics are so different from each other, but it was a pleasure to watch both of them at work and to get to talk to them so often.

If you don't find a problem interesting, it's really hard to expend the energy required to solve it.

I think this is the wonderful thing about the mathematical community: once you have become interested enough in a subject that you are able to start making your own dent in it, most people are excited to talk to you and to help you succeed. Of course, you have to show that you're willing to work and think on your own. But the more you put in, the more you get out.

Diaz-Lopez: How would you describe your research to a graduate student?

Grigsby: I study objects in low-dimensional topology (e.g., 3-manifolds, 4-manifolds, links, and braids) using "homology-type" invariants. Briefly, one associates an abstractly-defined chain complex to some collection of data describing a topological object (e.g., a Heegaard diagram for a 3-manifold, a diagram of a link). The homology of this chain complex ends up being an invariant of the topological object (i.e., does not depend on the choices involved in its definition). These homology-type invariants are inspired by ideas in physics—in particular, quantum field theory and gauge theory.

Diaz-Lopez: What theorem are you most proud of and what was the most important idea that led to this breakthrough?

Grigsby: I'm partial towards some work I did recently with Tony Licata and Stephan Wehrli, but I think one is always biased towards recent work. This is what I'm thinking about these days, because this is what I like best!

We proved that a particular homology-type invariant (sutured Khovanov homology) associated to an n-cable of a knot (an n-cable of a knot is, roughly-speaking, n parallel copies of the knot, see Figure 1) admits commuting actions of the Lie algebra sl_2 and the symmetric group S_n. In the case of the trivial knot (aka the "unknot"), these actions agree with the classical commuting actions of sl_2 and S_n on the n-th tensor power of the defining representation of sl_2: one instance of a so-called "Schur-Weyl representation." So one can think of this construction as giving "knotted" Schur-Weyl representations.

As for the breakthrough(s) that led to the work: Stephan and I had been studying this particular homology-type invariant for a while (ever since 2008), and we got to talk to Tony about it while we were at MSRI [Mathematical Sciences Research Institute in Berkeley] in 2010. Tony is a geometric representation theorist, so he had a very different perspective on things, and Stephan and I both found it useful and interesting to chat with him periodically, but nothing happened at the time. In the summer of 2012, he told us that there ought to be an action of sl_2 on the homology-type invariants we were studying. The construction was pretty abstract, and there were details we weren't quite sure how to fill in, so we didn't know what to do with it. But the idea that there could be such an action was really intriguing.

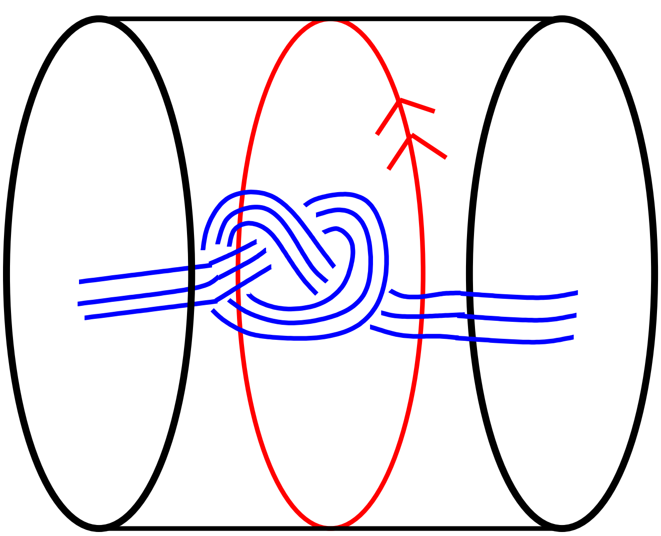


Figure 1. A 3-cable of the right-handed trefoil knot. Licata, Wehrli, and Grigsby recently found commuting actions of the Lie algebra *sl(2)* and the symmetric group S_n on a variant of the Khovanov homology of the n-cable of any knot.

Image courtesy of E. Grigsby.

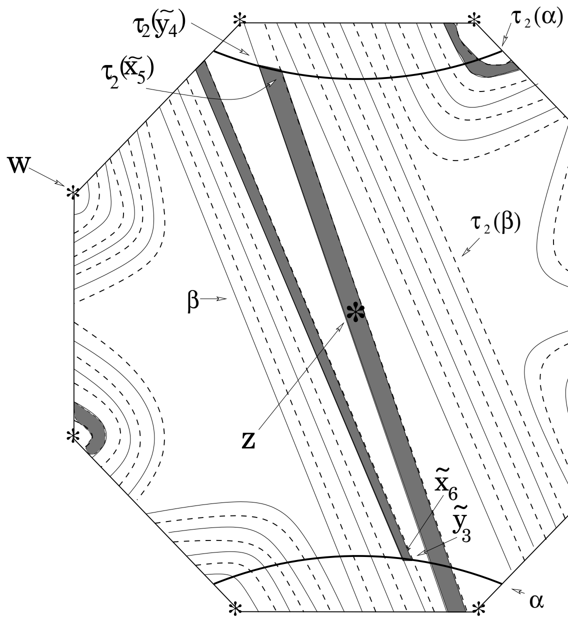


Image courtesy of E. Grigsby.

Figure 2. Identifying the boundary arcs of this octagon gives a genus 2 surface. The curves indicate how to attach handlebodies to form a 3-manifold, and the starred points specify a knot in the 3-manifold. Ozsváth and Szabó showed how to use such data to define a homology-type invariant of the knot. This particular manifold has a Z_2 symmetry covering the 3-sphere with fixed point set the knot.

Then one morning (probably about 2 months later!) right after I woke up, I realized that there was a completely obvious, completely concrete, sl_2 action right at the chain complex level that was staring us in the face the whole time. It also turned out that Stephan had worked out the details of an S_n action on the homology of n-cables way back in 2005. With a little further thought, it was clear that his action commuted with the sl_2 action. Once we started realizing things, everything came together in just a few weeks.

But (as is the case with most results), that few weeks was several years in the making! You're spinning your wheels most of the time, thinking you're not moving. But secretly you are.

Diaz-Lopez: What advice do you have for graduate students?

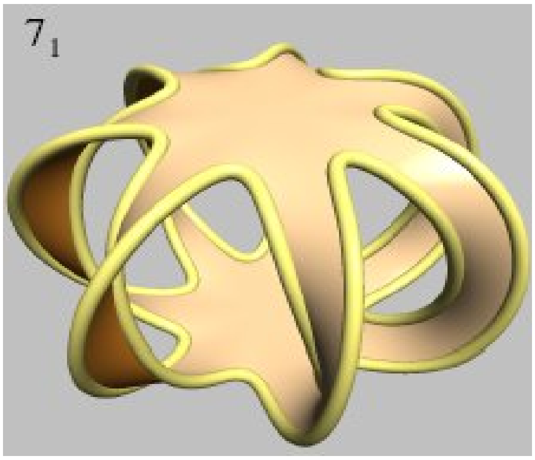


Image from "Visualization of the Genus of Knots," by Jarke J. van Wijk and Arjeh M. Cohen, in *Proceedings IEEE Visualization 2005*, edited by C. Silva, E. Gröller, H. Rushmeier, IEEE CS Press.

Figure 3. A smoothly imbedded, oriented surface in the 3-sphere bounded by the (7, 2) torus knot. A conjecture of Milnor proved by Kronheimer-Mrowka states that this surface has the smallest possible genus among smoothly imbedded surfaces in the 4-ball with this knot as boundary.

Grigsby: Talk to as many people as you can. That's the only way to find out not only what the community finds interesting but also what you find interesting. And if you don't find a problem interesting, it's really hard to expend the energy required to solve it. Having a good "nose" for problems and the tenacity to keep chewing on them—within reason—is a much more important trait than being quick. Of course, quickness helps, but depth is better.

Diaz-Lopez: All mathematicians feel discouraged occasionally. How do you deal with discouragement?

Grigsby: Not very well. But after you've been around for a while, you eventually realize that it's all part of the process. As mathematicians, it nags at us if we don't understand something as well as we'd like, and I've often found myself unable to get up from my desk when I'm stuck. But usually the right thing to do when you're stuck is to step as far away from your desk as possible. Otherwise you'll never get out of the particular rabbit hole you're in. Breakthroughs often come in the first five minutes after taking a break.

Also, Paul Melvin once told me (and I think this is right on): Periodically remind yourself how much more you know today than you did a year ago today. That will raise your spirits.

Diaz-Lopez: You have won several honors and awards. Which one has been the most meaningful and why?

Practically speaking, the CAREER [from the National Science Foundation]. Who knows if I'd have tenure now if I hadn't gotten that. Putting together the proposal also gave me a chance to reflect on my goals, research-wise and otherwise.

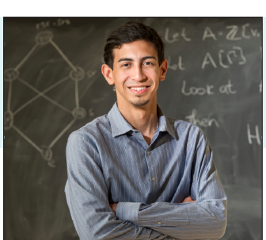
The Birman Prize [from the Association for Women in Mathematics] was also very personally meaningful, mostly because I have the deepest respect for Joan Birman—both her mathematics and her life story are amazing. I think the interview she gave in the *Notices* of the AMS back in 2006 should be required reading for all graduate students. She's a giant in the field, and she didn't even begin her PhD studies until after the age of thirty.

Diaz-Lopez: If you were not a mathematician, what would you be?

Grigsby: If I were eighteen again, I'd probably want to get an engineering degree, working on developing alternative energy sources. Either that or machine learning and A.I.

Diaz-Lopez: If you could recommend one lecture to graduate students, what would it be?

Grigsby: I'm topologically-biased. I loved Bott and Tu's *Differential Forms in Algebraic Topology* back when I was a grad student. Also anything by Milnor: e.g., *Topology from the Differentiable Viewpoint*, *Morse Theory*, and *Singular Points of Complex Hypersurfaces*.



Alexander Diaz-Lopez is a PhD student at the University of Notre Dame. Diaz-Lopez is the first graduate student member of the *Notices* Editorial Board.

WHAT IS...

an Anabelian Scheme?

Kirsten Wickelgren

Communicated by Cesar E. Silva

Suppose we are given a set of polynomial equations that we wish to solve. A scheme is an object which records the solutions to these polynomials as the domain of the variables ranges over many rings. Allowing the domain to vary helps solve equations even over a fixed ring. Schemes were invented around 1960 by Alexander Grothendieck as an early step in his reworking of algebraic geometry, and they had a revolutionary impact on the subject. As he put it in his *Récoltes et Semailles*, schemes "represent a metamorphosis of the old notion of 'algebraic variety'."

In an Anabelian Scheme, the solutions are controlled not by the usual algebraic manipulations but rather by using the loops on the complex solutions together with a Galois group. This difference opens up new possibilities for understanding the solutions, which is one reason to care about Anabelian Schemes. Before giving precise definitions, let's look at an example.

Let $f(x)$ be a polynomial with coefficients in $\mathbb{Q}$, or to be even more specific, let's suppose $f(x) = \prod_{n=0}^{5}(x-n)$. The solutions

$$X(\mathbb{C}) = \{(x,y) \in \mathbb{C}^2 : y^2 = f(x)\}$$

are drawn in Figure 1 (see p. 286). Note that these solutions form a genus 2 surface with two punctures. Replacing $\mathbb{C}$ by any $\mathbb{Q}$-algebra R yields a corresponding set of solutions, also called R-points,

$$X(R) = \{(x,y) \in R^2 : y^2 - f(x) = 0\}.$$

Kirsten Wickelgren received an AB-AM in Mathematics from Harvard University in 2003 and a PhD in Mathematics from Stanford University in 2009. She is currently an assistant professor at the Georgia Institute of Technology. She likes French literature and running. Her research interests lie in the intersection of algebraic topology, algebraic geometry, and number theory. Her email address is kwickelgren3@math.gatech.edu.

Editor's Note*: This month's installment of the "WHAT IS ...?" column, providing as it does an unusually daring peek into some technical and abstract mathematics, seems a perfect accompaniment to this issue's tribute to Grothendieck.*

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For example, while $X(\mathbb{C})$ is a surface, $X(\mathbb{R})$ consists of three circles with two points removed from one of them, and $X(\mathbb{Q})$ is only a finite set of points. All of these solution sets together determine a scheme X.

Note that $(0,0)$, and $(1,0)$ are both in $X(\mathbb{R})$, and view $(0,0)$ as a designated base point. If we travel along a path γ from $(0,0)$ to $(1,0)$ in $X(\mathbb{C})$ and then travel backwards along the path $\overline{\gamma}$ given by taking the complex conjugates of the coordinates of points of γ, we get a loop $\gamma\overline{\gamma}^{-1}$ from $(0,0)$ to itself. The point $(1,0)$ in $X(\mathbb{R})$ is controlled by analogues of the loop $\gamma\overline{\gamma}^{-1}$. Conjecturally, all points of $X(k)$ are controlled by analogous loops formed from paths from $(0,0)$ to (x,y) when k is a finite extension of $\mathbb{Q}$. This is the way in which the solutions to the polynomials defining an Anabelian Scheme are controlled by the loops.

To be more precise, we need a generalization of the loops on $X(\mathbb{C})$ which also incorporates field automorphisms, such as complex conjugation. This generalization is called the étale fundamental group $\pi_1^{\text{ét}}$ and its definition uses the classification of covering spaces by the fundamental group to define a notion of fundamental group given a notion of covering space. This process was discovered by Grothendieck and $\pi_1^{\text{ét}}$ records information both about topological fundamental groups and Galois groups. For example, suppose a scheme X is such that all of its defining polynomials have coefficients in k and the only $X(R)$ considered are those where R is a k-algebra. Such a scheme is said to be *over* k. Under mild hypotheses, there is a short exact sequence

$$1 \to \pi_1(X(\mathbb{C}))^\wedge \to \pi_1^{\text{ét}} X \to G \to 1,$$

where $\pi_1(X(\mathbb{C}))^\wedge$ denotes the inverse limit of finite quotients of $\pi_1(X(\mathbb{C}))$, and G denotes the absolute Galois group of the number field k.

The procedure given two paragraphs above associating the point $(1,0)$ to the loop $\gamma\overline{\gamma}^{-1}$ on $X(\mathbb{C})$ generalizes to give a map

$$X(k) \to \operatorname{Map}_G^{\text{out}}(G, \pi_1^{\text{ét}} X), \tag{1}$$

where $\operatorname{Map}_G^{\text{out}}(G, \pi_1^{\text{ét}} X)$ denotes the outer continuous group homomorphisms from G to $\pi_1^{\text{ét}} X$ which respect

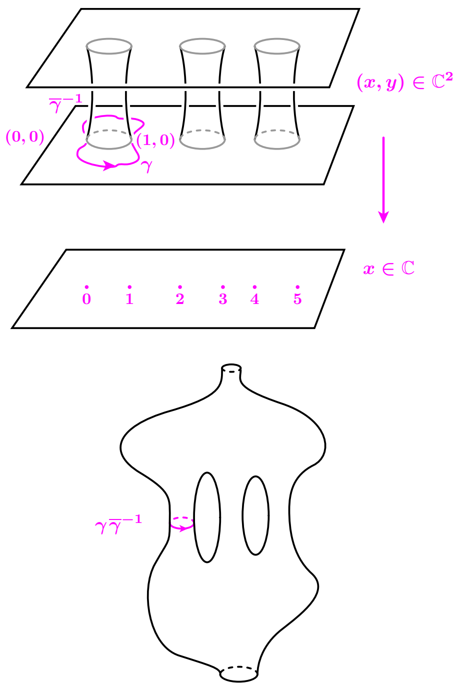

Figure 1. The map $f : X(\mathbb{C}) \to \mathbb{C}$ defined by $(x, y) \mapsto x$ is a branched covering map; over every point x of $\mathbb{C}$ such that $f(x) \neq 0$ there are two points of $X(\mathbb{C})$, and over every point x in $\mathbb{C}$ such that $f(x) = 0$ there is one point in $X(\mathbb{C})$. We partition the zeros of $f(x)$ into pairs and cut a slit running from each point of a pair to the other. The inverse image under f of the complex plane $\mathbb{C}$ minus the slits S is two disjoint copies of $\mathbb{C} - S$ because any loop in the base must wrap around an even number of zeros of f, which causes a lift of that loop to stay on the same sheet of the covering space. These two copies of $\mathbb{C} - S$ are attached along the inverse images of the slits. When a loop on the base passes through a slit, the lift of the loop must change sheets. Since the associated gluing would result in self-intersections, it is easier to see the shape of the solutions if we flip the bottom copy of $\mathbb{C} - S$ over the real axis. We then glue or add small cylinders and can see that the solutions form a genus 2 surface with two punctures.

the map $\pi_1^{\text{ét}} X \to G$. More precisely, $\operatorname{Map}_G^{\text{out}}(G, \pi_1^{\text{ét}} X)$ denotes the set of equivalence classes of continuous group homomorphisms $G \to \pi_1^{\text{ét}} X$ such that the diagram

$$\begin{tikzcd} G \arrow[rr] \arrow[dr, "1"'] & & \pi_1^{\text{ét}} X \arrow[dl] \\ & G & \end{tikzcd}$$

commutes and where two group homomorphisms $f_1, f_2 : G \to \pi_1^{\text{ét}} X$ are considered equivalent if there is γ in $\pi_1(X(\mathbb{C}))^\wedge$ such that $f_2(g) = \gamma f_1(g) \gamma^{-1}$. The purpose of considering outer homomorphisms instead of homomorphisms is to eliminate the dependency on the choice of base point. More generally still, there is a map

$$\text{(2)} \qquad \operatorname{Map}(Y, X) \to \operatorname{Map}_G^{\text{out}}(\pi_1^{\text{ét}} Y, \pi_1^{\text{ét}} X)$$

for any scheme Y over k.

Roughly speaking, Anabelian schemes are a conjectural type of scheme for which maps similar to (2) and (1) are bijections, which is to say that the solutions to the polynomial equations underlying X correspond to maps of étale fundamental groups. Grothendieck gave specific examples of schemes he predicted to be anabelian in this way, and for definiteness, let's use that as a definition. Let k be a finitely generated field, and we'll also assume characteristic 0 as a precaution.

Definition. A finite type scheme[1] X over k is said to be *anabelian* if it can be constructed by successive smooth fibrations of curves with negative Euler characteristic.

Armed with this definition, let's give two theorems saying that Anabelian Schemes behave as Grothendieck predicted. The first is due to Neukirch and Uchida and says that any isomorphism of absolute Galois groups of number fields $\operatorname{Gal}(\overline{L}/L) \cong \operatorname{Gal}(\overline{k}/k)$ corresponds to an isomorphism of fields. For example, it follows that for any a,b in $\mathbb{Q}^*$, if $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{a}]) \cong \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{b}])$, we must have that the fields $\mathbb{Q}[\sqrt{a}]$ and $\mathbb{Q}[\sqrt{b}]$ are themselves equal, or equivalently, that $a = b$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, the rational numbers modulo their squares. Using the identification of Galois groups of fields with étale fundamental groups of the corresponding schemes, the Neukirch–Uchida theorem can be restated to say that an analogue of (2) where isomorphisms replace maps is a bijection.

To state our second theorem saying that Anabelian Schemes behave as Grothendieck predicted, we need the notion of a dominant map between schemes. A map between schemes whose image is dense is called *dominant*. One can refine the map (2) to a map

$$\text{(3)} \qquad \operatorname{Map}^{\text{dom}}(Y, X) \to \operatorname{Map}_G^{\text{out,open}}(\pi_1^{\text{ét}}(Y), \pi_1^{\text{ét}}(X)),$$

from the set of dominant maps from Y to X to the subset

$$\operatorname{Map}_G^{\text{out,open}}(\pi_1^{\text{ét}}(Y), \pi_1^{\text{ét}}(X)) \subset \operatorname{Map}_G^{\text{out}}(\pi_1^{\text{ét}}(Y), \pi_1^{\text{ét}}(X))$$

consisting of triangles such that $\pi_1^{\text{ét}}(Y) \to \pi_1^{\text{ét}}(X)$ has open image. Grothendieck conjectured that for X and Y anabelian, the map (3) is bijective. Shinichi Mochizuki proved an impressive case of this conjecture.

Theorem. *(Mochizuki 1999) For Y any smooth scheme and X a smooth curve with negative Euler characteristic,* (3) *is bijective.*

The prediction that (2) is a bijection when $Y = \operatorname{Spec} k$ is called the Section Conjecture and is a major open problem in the field.

[1]Finite type *is a mild technical assumption on a scheme, which can be thought of as the requirement that the scheme be a finite union of subschemes of affine space, and isn't terribly important here.*

The AMS Graduate Student Blog, by and for math graduate students, includes crossword puzzles, and a variety of interesting columns and excerpts. This month's blog section offers the following excerpts from `blogs.ams.org/mathgradblog`.

One excerpt from the November 2015 AMS Graduate Student Blog.

Prelims and Master's Exams Tips

by Shelby Heinecke, the University of Illinois, Chicago

Photo Courtesy of Lisa Heinecke.

Shelby Heinecke is a math PhD student at University of Illinois-Chicago exploring mathematical computer science.

1) Make a study plan at least several weeks prior to your exam date. Include topics you need to review on which days. Check off the days as you complete the study assignments as this will help to motivate you and build a sense of accomplishment.

2) Prioritize prelim/master's exam courses as you take them. Don't take shortcuts in these courses as your success on the prelims and master's exams depends on a deep understanding of these topics. Stay organized in these courses and make an effort to take excellent notes so you can study from them when preparing for your exam.

3) Schedule full-length, timed practice exams.

4) Find a study group to meet with regularly at least several weeks prior to the exam.

5) Complete all learning of exam topics at least a couple of weeks before the exam.

My Professor

Artwork by Sam White.

AMERICAN MATHEMATICAL SOCIETY

Director of the Washington Office

The American Mathematical Society (AMS) invites applications for the position of Director of the Washington Office.

The Washington Office is one of seven divisions of the AMS. It works to connect the mathematics community with Washington decision makers who impact science and education funding. The Director has high visibility and a profound effect on the way in which the AMS serves the broad mathematical community.

Responsibilities of the Director focus on government relations and programs and include:

- serving as liaison with federal agencies, legislative members and their staffs, and other professional groups regarding activities related to the mathematical sciences
- providing advice to the AMS leadership on issues and strategies related to federal science and education policy and funding
- overseeing AMS projects and programs related to the activities of the Washington Office (e.g., recruitment of AMS Congressional Fellow and representation of the AMS on various public policy coalitions)
- communicating with the AMS members and disseminating information related to the mathematical sciences and federal science and education policy

For further information regarding specific activities of the Washington Office, please see **www.ams.org/government.**

The Director reports to the Executive Director of the Society. In carrying out the responsibilities of the position, the Director works with the AMS Board of Trustees, Council, committees, and staff; government agencies; Congress; corporations; foundations; other professional and scientific organizations; and mathematicians from throughout the world.

The Society is seeking a candidate who is aware of the concerns of the mathematical sciences research community and understands the need for involvement of mathematicians in federal science and education policy decisions. Such a candidate should have an earned Ph.D. in one of the mathematical sciences, the ability to work effectively with mathematicians and non-mathematicians, an understanding of national issues and activities that impact mathematics and the mathematics profession, the ability to communicate effectively with a wide audience that includes government policymakers, mathematicians, and the general public.

Nominations of outstanding candidates are encouraged.

This is a full-time position at the AMS office in Washington, DC. The initial appointment will be for three to five years, with possible renewal, and will commence in late 2016. The starting date and length of term are negotiable. Applications are welcome from individuals taking leaves of absence from another position. Salary is negotiable and will be commensurate with experience.

Applications (including a curriculum vitae, a letter explaining interest in the position and relevant experience, and the names and contact information for at least three references) should be sent to:

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Applications received by March 28, 2016 will receive full consideration.

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2016 Class of the Fellows of the AMS

Fifty mathematical scientists from around the world have been named Fellows of the American Mathematical Society (AMS) for 2016.

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics. Among the goals of the program are to create an enlarged class of mathematicians recognized by their peers as distinguished for their contributions to the profession and to honor excellence.

The 2016 class of Fellows was honored at a dessert reception held during the Joint Mathematics Meetings in Seattle, Washington. Names of the individuals who are in this year's class, their institutions, and citations appear below.

The nomination period for Fellows is open each year from February 1 to March 31. For additional information about the Fellows program, as well as instructions for making nominations, visit the web page `www.ams.org/profession/ams-fellows`.

AMS President Robert Bryant greeting Professor William A. Massey, Princeton University and Professor Terrance Pendleton, Iowa State University.

Jim Agler, University of California, San Diego
For contributions to operator theory and the theory of analytic functions of several complex variables.

Noga Alon, Tel Aviv University
For contributions to combinatorics, theoretical computer science, combinatorial geometry, information theory, and related areas.

Shiferaw Berhanu, Temple University
For contributions to complex analysis and partial differential equations, and for service to the global mathematical community.

Alexandru Buium, University of New Mexico
For contributions to number theory and algebraic geometry, particularly the development of a theory of arithmetic differential equations.

Eric Anders Carlen, Rutgers The State University of New Jersey, New Brunswick
For contributions to functional analysis, mathematical physics, and probability.

Sun-Yung Alice Chang, Princeton University
For contributions to geometric analysis, nonlinear partial differential equations, and harmonic analysis.

Vyjayanthi Chari, University of California, Riverside
For contributions to the theory of quantum groups and affine Lie algebras, and for service to the mathematical community.

J. Brian Conrey, American Institute of Mathematics
For contributions to research and exposition in number theory, and for service to the profession.

Steven Dale Cutkosky, University of Missouri-Columbia
For contributions to algebraic and analytic geometry and to commutative algebra, and for exposition.

Mihalis Dafermos, Princeton University
For contributions to general relativity and partial differential equations.

Lisette de Pillis, Harvey Mudd College
For contributions to mathematical oncology and immunology research, leadership in mathematical biology education, and for service to the mathematical community.

William Duke, University of California, Los Angeles
For contributions to analytic number theory and the theory of automorphic forms.

John Erik Fornaess, Norwegian University of Science and Technology
For contributions to several complex variables and to complex dynamics.

Alexander Furman, University of Illinois at Chicago
For contributions to dynamical systems, ergodic theory, and Lie groups.

Andrei Gabrielov, Purdue University
For contributions to real algebraic and analytic geometry, and the theory of singularities, and for contributions to geophysics.

Martin Hairer, University of Warwick
For contributions to the theory of stochastic partial differential equations, in particular introducing a theory of regularity structures for such equations.

(Continued on next page)

Fellows of the AMS

Patricia Hersh, North Carolina State University
For contributions to algebraic and topological combinatorics, and for service to the mathematical community.

Olga V. Holtz, University of California, Berkeley
For contributions to numerical linear algebra, numerical analysis, approximation theory, theoretical computer science, and algebra.

Martin Kassabov, Cornell University
For contributions to the theory of discrete groups and their growth and expansion properties.

Ju-Lee Kim, Massachusetts Institute of Technology
For contributions to the representation theory of semisimple groups over nonarchimedean local fields and for service to the profession.

Alexander Kleshchev, University of Oregon
For contributions to the representation theory of finite groups, Hecke algebras, and Kac-Moody algebras, and for exposition.

Nancy Kopell, Boston University
For contributions to dynamical systems, applications to neuroscience, and leadership in mathematical biology.

Joachim Krieger, École Polytechnique Fédérale de Lausanne (EPFL)
For contributions to nonlinear hyperbolic equations.

Tao Li, Boston College
For contributions to low-dimensional topology, especially the topology of three-manifolds.

Francois Loeser, Université Pierre et Marie Curie (Paris VI)
For contributions to algebraic and arithmetic geometry and to model theory.

Valery Lunts, Indiana University, Bloomington
For contributions to algebraic and arithmetic geometry and to model theory.

Svitlana Mayboroda, University of Minnesota-Twin Cities
For contributions to harmonic analysis, partial differential equations, and applications to mathematical physics.

Ralph Mckenzie, Vanderbilt University
For contributions to universal algebra and for mathematical exposition.

Cristopher Moore, Santa Fe Institute
For contributions to randomized algorithms and quantum computing, bridging mathematics, statistical physics, and theoretical computer science.

Yiannis N. Moschovakis, University of California, Los Angeles
For contributions to mathematical logic, especially set theory and computability theory, and for exposition.

Lee Mosher, Rutgers The State University of New Jersey, New Brunswick
For contributions to geometric group theory.

Hagey Family Professor of Mathematics Cesar E. Silva, Williams College and Lingurn H. Burkhead Professor Erica Flapan, Pomona College.

János Pach, Ecole Polytechnique Fédérale de Lausanne (EPFL) and Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
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COMMUNICATIONS

Using Mathematics at AIM to Outwit Mosquitoes

József Z. Farkas, Stephen A. Gourley, Rongsong Liu, and Abdul-Aziz Yakubu

Wolbachia (Figure 1) is a reproductive parasite that infects arthropod species, including mosquitoes, all over the world. Only infected females can pass on Wolbachia infection to their offspring, and therefore Wolbachia has evolved to maximise its spread by manipulating reproductive processes to enhance the production of infected females. These manipulations include feminisation (resulting in genetic males developing as females), cytoplasmic incompatibility (which prevents Wolbachia-infected males from successfully mating with females that do not have the same Wolbachia type), and male killing (which results in increased food availability for surviving female progeny). However, it is also known that Wolbachia can block or reduce replication of viruses of mosquito-borne diseases such as dengue fever and West Nile virus (WNv). What if Wolbachia infection could be used as a biological control tool to fight mosquito-borne diseases such as WNv?

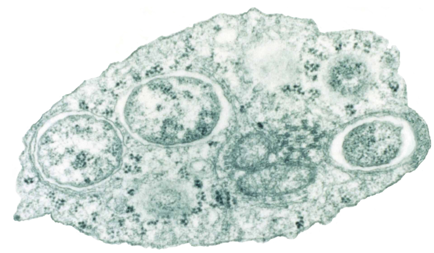
The four authors met as an AIM SQuaRE (Structured Quartet Research Ensemble), in which groups of four to six mathematicians spend a week at AIM in San Jose, California, for up to three consecutive years. Using Wolbachia to control vector-borne diseases is a well-established idea, with field trials already under way. The main goal of our project was to theoretically investigate this possibility for West Nile virus by introducing and analysing a 12-dimensional dynamical system. As the first building block, we derived from basic principles a sex-structured model for a mosquito population infected with Wolbachia, capturing most of the key reproductive effects of the Wolbachia infection together, including male killing, in one model. The four differential equations, variables, parameters, and coefficient functions appearing in the model are given below:

$$M'(t) = -\mu_m M + \frac{\lambda(F_{total})}{N}(MF + (1-\beta)(1-\tau) \cdot (MF_w + M_w F_w) + (1-q)M_w F),$$

$$F'(t) = -\mu_f F + \frac{\lambda(F_{total})}{N}(MF + (1-\beta)(1-\tau) \cdot (MF_w + M_w F_w) + (1-q)M_w F),$$

$$M_w'(t) = -\mu_{mw} M_w + \frac{\lambda(F_{total})}{N}(1-\beta)\tau(1-\gamma) \cdot (MF_w + M_w F_w),$$

$$F_w'(t) = -\mu_{fw} F_w + \frac{\lambda(F_{total})}{N}(1-\beta) \cdot \tau(MF_w + M_w F_w).$$



Public Library of Science, Scott O'Neill

Figure 1. Transmission electron micrograph of Wolbachia within an insect cell.

József Z. Farkas is reader of applied mathematics at the University of Stirling, United Kingdom. His email address is jozsef.farkas@stir.ac.uk.

Stephen A. Gourley is professor of mathematics at University of Surrey, United Kingdom. His email address is s.gourley@surrey.ac.uk.

Rongsong Liu is associate professor of mathematics at the University of Wyoming. Her email address is Rongsong.Liu@uwyo.edu.

Abdul-Aziz Yakubu is professor of mathematics at Howard University. His email address is ayakubu@howard.edu.

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- M, F: numbers of uninfected male, female mosquitoes.
- M_w, F_w: numbers of *Wolbachia*-infected male, female mosquitoes.
- $M_{\text{total}} = M + M_w$, $F_{\text{total}} = F + F_w$, $N = M_{\text{total}} + F_{\text{total}}$: total numbers of male, female, all mosquitoes.
- β: reduction in reproductive output of Wolbachia-infected females.
- τ: maternal transmission probability for Wolbachia infection.
- q: probability of cytoplasmic incompatibility (CI).
- γ: probability of male killing (MK) induced by Wolbachia infection.
- $\lambda(F_{\text{total}})$: average egg-laying rate, which depends on the total number of female mosquitoes.
- μ_m, μ_f: per capita mortality rates for uninfected male, female mosquitoes.
- μ_{mw}, μ_{fw}: per capita mortality rates for Wolbachia-infected male, female mosquitoes.

Our rigorous analysis of the above Wolbachia model revealed, amongst other things, that under certain biologically relevant assumptions, our model has multiple steady states in which Wolbachia-infected mosquitoes could coexist with small numbers of uninfected mosquitoes.

Building on initial results in [2] and the first part of [1], we extended our mosquito population model to include WNv, which is spread by birds and mosquitoes. Our full model takes the form of a 12-dimensional system of nonlinear differential equations. We were motivated by results recently reported by Hussain et al. [3], which suggest that a particular strain of Wolbachia substantially reduces WNv replication in the mosquito species *Aedes aegypti*. We modelled this crucial phenomenon by incorporating a small parameter, the reciprocal of which is proportional to the time spent in the WNv-exposed class for Wolbachia-infected mosquitoes. This enabled us to assess the potential of Wolbachia infection to eradicate WNv via its effect on WNv replication in Wolbachia-infected mosquitoes. Notably, the expression we obtained for the basic reproduction number suggests that Wolbachia infection substantially reduces WNv replication in mosquitoes and that WNv will be eradicated if at the steady state the overwhelming majority of mosquitoes are infected with Wolbachia.

Wolbachia infection in mosquitoes could have a beneficial effect on the control of many other mosquito-borne diseases besides WNv. Our model of Wolbachia infection should be suitable for application to the study of whole classes of these diseases. Our ongoing work focuses on the broad application of our Wolbachia model to other mosquito-borne diseases that affect humans, such as dengue fever.

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I. M. Gelfand and His Seminar—A Presence

A. Beilinson

How nice to be like a fool for then one's Way is grand beyond measure

From a poem of Tainin Kokusen given to his student Ryõkan Taigu, 1790


Photo courtesy of Sergei Gelfand.

Undated photo of I. M. Gelfand, probably from the 1940s.

The mathematical seminar of Israel Moiseevich Gelfand started each year in the beginning of September and ended in the spring when IM would observe that "rivulets (of melting snow) are beginning to flow." The sessions were on Mondays in the big auditorium on the 14th floor of Moscow University's main building, and each consisted of two parts: a preseminar that began at 6pm, and the seminar proper, which began with IM's arrival at around 7pm and ended at 10pm when a cleaning lady entered the room to announce her departure (at which time the floor was to be locked, and those wishing to spend the night at home had to hurry down). During the preseminar dozens of people congregated near the auditorium entrance, chatting and exchanging books and texts of all kinds.[1] The seminar typically began with IM telling some anecdotes and mathematical news, after which would come a talk by an invited speaker.[2] Often there was not enough time to finish, and the talk continued serially, each time beginning from scratch and covering about half of the material from the week before, the speaker gradually fading away and being replaced by a student assigned by IM to explain what the talk was, or should have been, about. Any speaker deemed not to have understood the subject, or to have explained it badly (or if the writing was too small and the voice not clear) was harshly reprimanded.[3]

The seminar started in 1943; I saw its later years, which coincided with the late period of the Soviet Union. After Stalin's death the edifice of the state shrank into itself, and free space teemed with life. The ideology had lost its fulcrum, the show of democracy was simple (a single candidate to vote for, not two equally unpalatable ones), newspapers were mostly used as toilet tissue. The remaining taboos were private commerce[4] and entrepreneurship, and political activity outside the Party's womb. Many people shared the attitude of Pushkin's poem "From Pindemonte"[5] and viewed all matters political as not interesting anyway. The market in a modern sense, this incessant gavage of unneeded things, did not exist. One could quit the tarmac road to look for one's own trail into the woods. If the trail happened to be mathematics, it would surely meet with IM's seminar.

There was a distinct inner music.[6] The air was thin and transparent. One could hear the sound of one's breathing, of snowflakes falling, of hoarfrost's brush decorating the windowpane. Old villages still existed within Moscow limits, such as wonderful Dyakovo, its empty church over an ancient cemetery on a high scarp above Moscow River, wooden houses edged by deep ravines, and vast apple gardens where nightingales sang.[7] Poetry was by far more real than social ranks—poems were rewritten by hand and learned by heart.[8]

I was brought to the seminar and introduced to IM by Alesha Parshin in the fall of 1972; I was then a senior of the second mathematical school (IM taught there years earlier). The precious feeling of being a fool and despite that, or rather thanks to that, being in balance with life's

A. Beilinson is the David and Mary Winton Green University Professor at the University of Chicago. His email address is sasha@math.uchicago.edu.

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Photo courtesy of Anna and Vladimir Gelfand.

Mark Iosifovich Graev (left) and I. M. Gelfand on Volga River in summer 1963.

flow, akin to running on the cracking ice of a river, goes back to these days.

After failing the entrance exams to the math department of Moscow University[9] I found myself in the lovely Pedagogical Institute. This was a benison—going to classes or playing truant in the morning, going to mathematical seminars or taking a train to walk in the woods[10] later in the day; and there were wonderful friends. After a while I managed to be transferred to the University. The mood there was more sombre, but with no desire for higher grades, one could skip all the ideological classes[11] to retain the good measure of idleness and freedom that are so necessary for doing math.

Accidentally, my first result to be published was close to the one found at the same time (the end of 1977) by IM with Osya Bernstein and Serezha Gelfand. IM gave a talk about his work, mentioning that I had obtained a similar theorem. After the talk I approached IM, and he at once ordered me to leave Yuri Ivanovich Manin, who was my supervisor, and be his student. The colée was violent. I refused. When I told YuI about the accolade, he said this had happened to many, e.g., to himself and to Shafarevich. Thereafter I stayed in an outer orbit of IM's influence, and our relationship was excellent.

After the graduation I got a job in a mathematical laboratory at Moscow Cardiological Center; to that end, the noble Vladimir Mikhailovich Alexeev, who was the head of the laboratory, came to the job committee soon after undergoing major cancer surgery. VM died in December of 1980. The new head of the lab, disagreeable on the matter of skiving, was keen on getting rid of me. After IM learned about the situation, he talked to the head of biological sector of the Center; I was transferred there and left to my own devices. The sinecure was better than a graduate school.

In the early 1970s the high winds of the Cold War[12] brought permission for Soviet Jews to emigrate, and many signed up for what, in retrospect, turned to be a verification of the universality of Griboyedov's quip that the place where it is better for us to be is where we are not.[13] The separation from friends was deemed to be permanent (the imminent demise of the SU was anticipated then no more than that of the US is now). Dima Kazhdan, Ilya Iosifovich Piatetski-Shapiro, and Osya Bernstein, with whom we were happily doing math for his last half year in Moscow, were among those who left. No one at the seminar could replace them.

IM loved playing with people (with him mischief was never far away).[14] A common way to engage someone was to explore his feeling of self-importance. IM rarely lost the game; if this happened (which meant that the opponent was more unpredictable than IM himself), he was furious, but the winner got his respect and, perchance, even love. For example, IM could ask you to wait and then disappear for a very long time.[15] A cheap win was to leave after an hour. A master stroke would be different. According to legend, when IM returned to his office after several hours to see how Misha Tsetlin was doing, he found Misha fast asleep on the sofa.[16]

IM appreciated life.[17] Although IM was a very social person, he paid no outward respect to problems caused by a lack of inner happiness (as a result he was often perceived as rude).[18] He did what interested him for its own sake, and not as a part of any grand project.[19] Running seminars (the mathematical and biological[20,21] ones, and, starting from 1986, the one on informatics) was always interesting.

[IM] did what interested him for its own sake, not as part of any grand project.

Then there was the work with physicians, a long attempt to find out how a doctor diagnoses heart disease. While the attempt itself ended in failure,[22] it included several top notch physicians who brought a distinctly new dimension to the life of those around IM. I came then to know three doctors, true masters, who found it impossible to accept any payment for their help.[23] I learned that such an attitude is utterly natural and, in truth, a doctor cannot behave any differently.[24]

IM emphasized the importance of decency.[25] Its two realizations central, in my opinion, to IM's life were severing, after the work on the bomb, his ties with the military (late 1950s)[26] and his becoming a vegan (mid 1990s).[27] Both have to do with overcoming the habit of what is usually called thinking objectively, i.e., paying no regard to violence directed towards others.[28] Arguably, without the first decision the world around IM would have been much less colorful and the seminar quite different. Becoming vegetarian is probably no less essential. It may loosen knots tied hard in one's mind, bringing back an ability to see many things as obvious and simple.

One difference between IM's seminar and other great mathematical seminars was its openness: the talks were not aimed at explaining any distinctive subject, nor were they connected to IM's current work, but rather these were stories that might contain a call from the future. This was in tune with the next feeling: We are used to seeing science's accomplishments as being fundamental. Over time the magical picture switches, and we realize that, in fact, we know almost nothing about the world, and science merely attempts to hide the vast openness. But we are able to wonder and take in new things, and I feel gratitude, only due to the wind that blows through us.

IM often said that he does not consider himself to be clever.[29] A fool's way to see things differs from that of a clever person like peripheral vision differs from central vision. At every moment there are infinitely many possible directions to look at and to choose. A fool retains awareness of that; a clever person moves successfully in one or two directions while forgetting completely the remaining infinity of dimensions. A new understanding or a fresh poem starts with a tiny movement into an unknown dimension, which is the inimitable act of a fool.

Modern mathematics is a unique thrust of conceptual thought: once the right concept (a mathematical structure) and a language to deal with it are found, a whole new world unfolds.[30] So for a mathematician it is very tempting to search for an adequate language as a key for understanding nonmathematical subjects, e.g., biology. This vision was dear to IM.[31] One reason why it has not been realized might be the following:

Science invariably considers reality as if from outside, the objects of study clearly distinct from the observer. But mathematical structures are part of the true reality that can be seen only from inside, the object of study being inseparable from the activity of our brain. It might be that adequate languages are peculiar to exactly this type of seeing. For example, except on the most superficial level, science is blank about the ways animals interact with the world. The animal's vision should be so wonderfully different to the human's that being privy to it might drastically change our understanding of what reality is. It is in such a quest that an adequate language could be ignited. Which is a mere foolish dream as long as we persist in positioning ourselves as separated from other living beings and above them—to the degree of imagining that the earth, the animals, and the trees can be our property. Incidentally, this same delusion underlies the drive for the destruction of the planet (which has accelerated so much since I saw IM the last time).

As I am writing these lines, it is spring, the season when the past does not seem to be that impossibly separated from the future. Great seminars have something of faerie horses in their nature. Bayard is said to be still living somewhere since his escape into the heart of the wild forest of Ardennes.

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This essay would not exist without many walks and discussions with Jesse Ball, Spencer Bloch, Irene(!) and Nicodemus Beilinson, Volodya Drinfeld, Dennis Gaitsgory, Anyuta and Volodya Gelfand, Senya Gindikin, Dima Kazhdan, Dima Leshchiner, Yuri Manin, Oleg Ogievetsky, and Eric Shutt, the request of Slava Gerovitch to write it down, and the interest and help of Allyn Jackson. My deep gratitude to them.

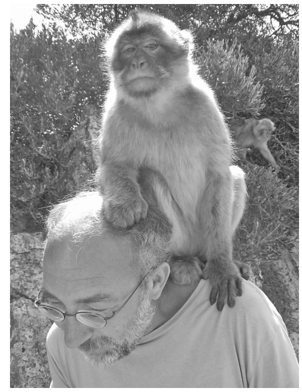


Photo courtesy of Irene and Alexander Beilinson.

A. Beilinson.

Notes

[1]Senya Gindikin: "IM considered these preseminar discussions to be very important. However, he was pathologically disorganized and could not get anywhere on time even if he wished to (e.g., for a meeting with important people)." According to legend, once, on his way upstairs to a meeting with a President of the Academy of Sciences, IM stopped to exchange pleasantries with a cleaning lady; he never reached his destination.

[2]For Misha Shubin's notes of the talks, see `www.mccme.ru/gelfand/notes/`.

[3]At times the scene resembled the koan about Nansen and the cat; see `en.wikipedia.org/wiki/Nanquan_Puyuan`, with no Jōshū in sight.

[4]One exception was Bird's Market in Moscow, where on the weekends all kinds of animals were sold. Once, when I visited it with Don Zagier, a bearded fellow in sheepskins tried to sell him a snow-white goose. The fellow said that he could see Don to be a true gentleman—otherwise he would not offer him the beauty. He said that she would be Don's best friend, going with him everywhere, and sharing his bath. The discussion was in French.

[5]For Nabokov's translation, see `https://ireaddeadpeople.wordpress.com/2014/11/06/alexander-pushkin-to-stroll-in-ones-own-wake/`.

[6]Perhaps not unlike that of another closed country—Japan of the late Edo era. The mores were also not altogether different: e.g., the chief nuclear scientist who dealt with Chernobyl's aftermath killed himself, probably, as an apology for his involvement in the nuclear industry (his superiors practiced the Fukushima era ethics).

[7]Dyakovo was eliminated in the 1980s: first the graves in the cemetery were dug out, then, in a while, the houses were demolished and burned, a single one surviving for over a year.

[8]Two of my friends knew by heart all of Mandelstam's poems. Cf. "An evening of Russian poetry" by Nabokov, `www.sapov.ru/novoe/n00-39.htm`.

[9]The lords of the Moscow mathematical establishment kept it clean from anything Jewish.

[10]For a year I was seeing the woods almost every day.

[11]Officially one had to know the whole contents of the course in order to pass. But the teachers, with the help of the Komsomol leader of the group, revealed on the eve of the exams to each student the exact question he/she would be asked.

[12]Its sole cause was (is) the incompatibility of plutocracy and autocracy; the rest of the US/SU discordances were red herrings (or, if the reader prefers, forget-me-nots of a Kozma Proutkoff fable that was often cited by IM; see www.math.uchicago.edu/~mitya/langlands/nezabudki.html for an English translation).

[13]These departures, the simulations of dreams, had little in common with the high quest of crossing the SU border (in either direction) by one's own free will and with no external purpose, as in Nabokov's *Glory* or as done by Slava Kurilov, see his book *Alone in the Ocean*, rozamira.org/lib/names/k/kurilov_s/kurilov.html (in Russian).

[14]Spencer Bloch: "I am sure I told you my Gelfand story when he came to Paris and was to meet with Serre. He was staying at Ormaille and the people at IHES needed someone to escort him to Paris. I was elected. I suggested we take a train with plenty of time to spare so we would not inconvenience the great Serre. Of course, I did not fully grasp the subtle thinking process of my charge. Suffice it to say that not inconveniencing Serre was rather low on the totem pole of Gelfand's priorities. I arrived at his apartment and he announced that he would instruct me on the Russian technique for making tea. So, of course, we missed the train. But I said no matter, there would be another train along in twenty minutes. But no, Gelfand said that errors had occurred during the making of the tea, and nothing would do except to return to his apartment and make more tea; which we did. So, of course, we missed the next train. And, as was clearly the intent from the beginning, the great Serre was made to wait for the great Gelfand."

[15]Senya Gindikin: "I would think that it was more complicated. IM felt no obligations and at every moment did only what he wished to do at that moment. I don't think he did anything intentionally, he could be distracted for a long while. I have a big personal experience here."

[16]Misha Tsetlin, who for IM was what, probably, Jōshū was for Nansen, died in 1966. About their research in physiology, see sect. 3.1 of M. Latash's book *Synergy*, Oxford University Press, 2008, books.google.ru/books?id=Z45Oj8yCQMIC&pg=PA53. See also V. V. Ivanov's article about Tsetlin, historyofcomputing.tripod.com/essays/CETLINM.HTM (in Russian).

[17]And, just maybe, he admired its beauty to the point where even ugly human deeds do not blot the clarity of vision. I believe that wild animals are not afraid of humans who are able to participate that much in the joy of being.

[18]On the other hand, IM did care a lot when the problem was real: e.g., his help was crucial for saving the lives of Sasha Zamolodchikov and the son of Tolya Kushnirenko after terrible accidents.

[19]IM often said that he abandoned research whenever its subject became too popular.

[20]Volodya Gelfand: "IM knew no biology, but was always able to identify true experts to talk to, and these discussions were often very beneficial for the biologists as well."

[21]IM was fascinated by biology for the mystery that you do not know even how to think about is so immediate there.

[22]Perhaps at the start IM did not recognize medicine as an art (for him, a nonmathematician's project to uncover the way a mathematician proves theorems would be laughable). The work on a simpler problem of diagnosis of meningitis was successful.

[23]The payment for a cab to bring them, after the workday at hospital, to the patient's home included.

[24]A simple criterion to check if a given human society is not dead at its core is the presence in it of such physicians.

[25]Dima Leshchiner: "I recall his favorite saying: 'People do not have shortcomings, but only peculiarities.' It seems to me this has to do with what 'decency' meant in his understanding, namely, that 'decency' is the quality of an action, not of a person."

[26]IM once told me that, back then, he was offered to be the head of any institution of his choice (say, Institute of Applied Mathematics that dealt with military projects) and he refused. Senya Gindikin: "I am not sure if anyone knows how and why he stopped the military activity. To what degree this was initiated by himself. He was extremely cautious. He received a closed Lenin prize around 1960."

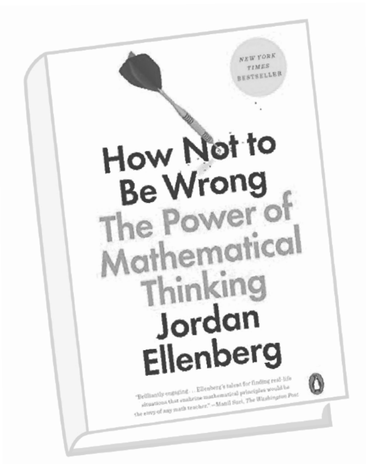
[27]See IM's interview for VITA, israelmgelfand.com/talks/vita.html. In earlier years IM coauthored a series of works on neurophysiology based on grisly experiments on cats.

[28]The trite lament that the cause of the sadness of today's world is that development of technology has overrun our moral development misses the point—for there is no moral development. Common decency now is the same as it was thousands of years ago, and it works well if applied (and if those who apply it are not killed). E.g., having taken it as religious principle (see https://en.wikepedia.ag/wiki/Jainism#Doctrine), the Jains built a reasonable, i.e., nondestructive, society (maybe the only one still in existence). Their cousins in the West, the Good People (referred to as Cathars, "catlovers", by the adversaries), were eliminated in a feat of what now is called "globalization".

[29]"You should not explain to me that I am an idiot: I know this myself." Instructing Oleg Ogievetsky's mother on how to talk to physicians, IM asserted, "No one can revoke your inherent right to be a fool."

[30]A related fact is that in mathematics, unlike elsewhere, wrong notions die off easily. Our capacity for understanding is hampered, foremost, by the inability to dispel false concepts.

[31]See his Kyoto lecture, israelmgelfand.com/talks/kyoto.html, and a birthday party talk, www.math.harvard.edu/conferences/unityofmath_2003/talks/xgelfand-royal-talk.html.



How Not to Be Wrong

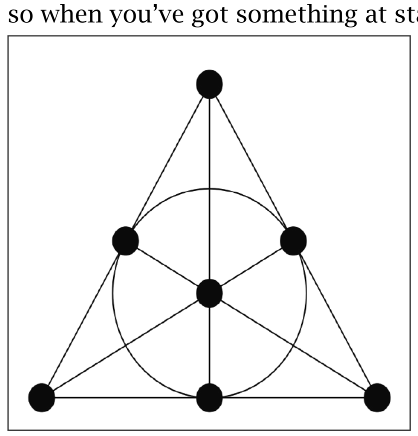
Reviewed by Andrew J. Blumberg

How Not to Be Wrong: The Power of Mathematical Thinking
Jordan Ellenberg
Penguin Books, May 2015
480 pages, US$17.00
ISBN-13: 978-0-1431-2753-6

We are witnessing an interesting historical moment concerning public engagement with mathematics. Not only is daily life permeated by the applications of sophisticated mathematics, but mathematics is increasingly part of our cultural conversation and present in the public sphere. And yet we all know that for too many of our students, we are failing to communicate both the beauty of mathematics and real competency in mathematical reasoning. In particular, the sense of play, exploration, and discovery that is the central experience of the research mathematician is rarely conveyed effectively. It's a frustrating situation: our students are immersed in mathematics, but few appreciate what it is or how they could engage with it. I'm reminded of David Foster Wallace's version of a very old joke, in which an old fish asks some young fish, "How's the water?," to the baffled response of "What's water?"

It is explicitly against the backdrop of this paradoxical state of affairs that Jordan Ellenberg has written his wonderful book *How Not to Be Wrong*. As the title suggests, the book takes as one point of departure a pragmatic view of mathematics as a conceptual framework for organizing and trying to understand the wildly confusing and incomplete data provided by the world. But Ellenberg's text is a multifarious enterprise: it simultaneously advocates for the value of research mathematics, explains what mathematics can do, encourages readers to experiment and view mathematics as something that they too can make their own, and strives to reveal the surprising beauty of mathematical explanation. Remarkably, the book succeeds on all of these fronts.

In the service of "not being wrong", the book emphasizes a basic message: when trying to understand the world, it's easy to fool yourself and very tempting to do so when you've got something at stake. As one would expect, there is a heavy emphasis on probability and statistics: entertaining discussions of regression and linearity fallacies, the problems with statistical tests and p-values, correlation vs. causation, the virtues of Bayesian inference, paradoxes in voting and aggregating public opinion, lotteries


The Fano plane.

Andrew J. Blumberg is associate professor of mathematics at the University of Texas at Austin. His email address is blumberg@math.utexas.edu.

Thanks are due to Deborah, Emily, Olena, and William Blumberg, Daniel Hemel, Mike Hill, and Mike Mandell for critical comments on previous drafts that substantially improved the article.

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and expected value, and so forth. In and of themselves, these are just the sorts of topics one might expect in a book about the use of mathematics aimed at a broad audience. For that matter, many of them parallel a good introductory statistics class. But Ellenberg weaves these together very skillfully and effectively conveys the central message that mathematical tools allow us to answer questions but also have serious limitations. Moreover, a big part of the pleasure of Ellenberg's work comes from the very interesting examples he introduces in the service of his narrative. Some of these are standard: for instance, the stock advice scam involving sending predictions first to 1,000 people, then to the roughly 500 who received correct predictions, until finally you sell your advice to the lucky "survivors". But many of them are more novel, ranging from the entertainingly idiosyncratic (e.g., look for the discussion of the Cat in the Hat when he talks about the risks of bad priors in Bayesian inference) to the remarkably deep.

For an example of the latter, the most surprising and beautiful part of the book is the discussion of the link between error-correcting codes, the lottery, and projective geometry. Since this is such a fantastic example, let me try to convey its flavor. Imagine a lottery game in which the state picks n numbers from the range $[0,m]$ (for $m > n$), and a player buys a ticket with n numbers (of their choosing) on it. The player wins if at least k of the numbers on the ticket match the state's n winning numbers. The question now is the following: if you were going to buy a large number of tickets, what's a good strategy? One possibility is to buy tickets randomly. But it would be better to try to arrange ticket purchases such that all the winning subsets of numbers are covered and appear the same number of times; this minimizes variance in the return.

...the most surprising and beautiful part of the book is the discussion of the link between error-correcting codes, the lottery, and projective geometry.

Trying to generate such a set of tickets by exhaustive search is infeasible due to combinatorial explosion. But Ellenberg turns to a discussion of the projective plane and then finite geometries: finite sets with specified points and lines such that each pair of points lies on a unique line and each pair of lines intersects in a unique point. Using the example of the Fano plane (which has seven points and seven lines, each of which has three points), he observes that in the case when $m = 7$, $n = 3$, and $k = 2$, choosing a ticket for each line results in a set of tickets such that each pair of numbers occurs exactly once. Of course, the Fano plane is too small to be useful; the question of how to find large finite geometries now arises.

Ellenberg then moves on to connect this problem to coding theory; a solution is provided by choosing each ticket to be a codeword in a good error-correcting code (one where the codewords are well separated in the Hamming distance). After explaining coding and something about information theory, he finally also connects this back to geometry via sphere-packing problems and recent progress on optimal packings.

Even when the examples are standard, the explanation is often surprisingly thoughtful. For instance, the most interesting aspect of Ellenberg's discussion of Galton's discovery of the phenomenon of "regression to the mean" is his explanation of the elliptical boundary region for bivariate normal variables. He observes that the ubiquity in applications of simple mathematical objects (e.g., conic sections) arises from the fact that there are comparatively few simple mathematical objects—once you know something has such a description, there aren't so many options!

These discussions give a flavor of the interconnectedness of mathematical ideas. They also communicate the magical nature of understanding. And, indeed, throughout the text there is emphasis on the sheer joy of mathematical discovery and the ease with which one can learn to do this in small examples. As an aspect of this, Ellenberg works to make it clear to ordinary readers why the pursuit of mathematics is important and why even seemingly "nonapplicable" mathematics is potentially important. His remarks in this direction are thoughtful and realistic and work towards demystifying what it is that mathematicians do.

Ellenberg is an engaging and fluid storyteller, and one of the pleasures of the book is the discussion of the mathematical personalities involved in the topics he covers. A particularly attractive aspect of his treatment is an enthusiasm for the compelling stories of mathematical discovery (ranging from satisfying historical melodrama to sheer weirdness) coupled with care to avoid perpetuating a sense that mathematics is the province of "geniuses". In fact, Ellenberg explicitly tries to dispel this notion, and his concrete comments are buttressed by his choice of examples: there is something extremely affirming about the story of the Michigan retirees, Quincy doctors, and MIT undergraduates, all trained to some degree in mathematics but none of them working mathematicians, who independently realized how to beat the Massachusetts lottery.

Which brings me to another important running theme of the book: a focus on a fundamentally progressive and egalitarian view of mathematics, a faith that there is a meaningful sense in which both the beauty and the power of mathematics should be and can be accessible to everybody and that this knowledge changes lives for the better. And from a civic standpoint, Ellenberg powerfully makes the case that a key aspect of mathematical literacy for informed citizens is having enough confidence and understanding to be skeptical of conclusions based on mathematical tools employed either incompetently or solely for rhetorical purposes. The only point I might have wished to be emphasized even further is the persistent risk that the effectiveness of mathematics tempts people to use it to obscure inherent uncertainty.

This is one of the best popular mathematics books I can imagine. There is enough depth for the book to be engaging reading for a working mathematician (especially one without recent formal training in statistics or probability), but of course the real virtue of the book is its value to a nonmathematical audience. It is no exaggeration to say that this book covers as many of the things I would want a numerate nonmathematician to know as a single book of modest length can. There are other topics one could conceivably include, but this is an unimpeachable set, and the style is friendly, warm, and surprisingly funny. Moreover, Ellenberg is aware of the limitations of the medium; he knows full well that no one book is going to change how someone thinks, and thus the text is peppered with the repeated insistence that math can only be learned by practice and that the reader should actually do some math while reading the book.

A possible critique is that the organizational style might seem slightly haphazard: although the chapters link to one another, there isn't a running theme, and earlier topics aren't always reinforced. But I found the structure to be an effective rhetorical choice, especially since many of the topics genuinely do stand on their own. Also, there are a few places (e.g., the discussion of Buffon's needle) where I imagine that a nonmathematician might lose focus (despite the best efforts of the author), but by and large, I suspect Ellenberg's efforts to reach all of his audience will succeed. I think the book will do a service for the mathematical community by explaining what we do and why we do it. And if nothing else, Ellenberg's tone throughout is so reasonable and thoughtful and he seems like such a nice normal guy that he's a good ambassador for the field. It's hard to say that reading this book will necessarily make informed citizens of all of us, but everyone should read it anyway. Even a few more people being just a little bit less wrong would be a good thing. And everyone needs a little more beauty and wonder in their life.

Andrew J. Blumberg in front of his parents' home in Belmont, MA.

Math for the Million$

In early 2005, MIT math major James M. Harvey was casting about for an independent study project. He started analyzing lotteries, eventually zeroing in on a Massachusetts game called Cash WinFall. After some preliminary research, a conversation with a lottery official confirmed what Harvey suspected: The unusual structure of Cash WinFall guaranteed that, at certain times, a US$1 bet would on average be worth US$1.15. Using that fact, Harvey and his fellow MIT alumnus Yuran Lu started buying large numbers of Cash WinFall tickets—and raking in profits. By 2010 they had organized their betting operation into a company, Random Strategies Investments, LLC, named in honor of their MIT dorm Random Hall.

There were other groups of large-scale bettors playing Cash WinFall, including one headed by a medical researcher in Massachusetts and one consisting of a group of retirees in Michigan (where a lottery similar to Cash WinFall had been phased out in 2005). The large-scale betting these groups did was entirely legal and had no adverse impact on the odds of other players winning. The Massachusetts lottery commission knew about the groups' betting and welcomed the increased ticket sales. Nevertheless, a *Boston Globe* investigation in 2011 raised enough questions and controversy to lead to the termination of Cash WinFall.

Cash WinFall differed from most lotteries in that the top prize was capped at US$2 million. When the jackpot hit $2 million, the money would be distributed to the lower-tier prizes, in a process called a "roll-down". So for example, matching 4 out of 6 numbers would ordinarily bring a win of US$150, but during a roll-down, the win became US$807.52. A 2012 report by Massachusetts Inspector General Gregory W. Sullivan calculated that, during a roll-down, buying US$400,000 of Cash WinFall tickets had a 50 percent chance of a bringing in US$425,000 or more.

Jordan Ellenberg's book *How Not to Be Wrong*, and the review here by Andrew Blumberg, give more details about the mathematics of the story (see also "How to Get Rich Playing the Lottery", by *Notices* editorial assistant Katharine Merow, on the website of the Mathematical Association of America, `www.maa.org/meetings/calendar-events/how-to-get-rich-playing-the-lottery`). Indeed, the math was the most reliable part of the high-scale betting operations. Less reliable were the more mundane aspects: Penciling in betting slips by hand, handling unhappy store owners whose employees were tied up processing thousands of orders, and enduring the vagaries of ticket machines that would jam up in humid weather.

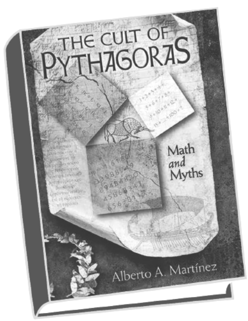
But it was "a lucrative enterprise," Sullivan's report concluded. The report says Harvey's group wagered a total of US$17–18 million on Cash WinFall and estimates the profits were at least US$3.5 million before taxes. Mathematics proved that Cash WinFall could be not just a betting game, but a sound investment. Ironically, that's just what led to the game's demise. As Blumberg put it, "Cash WinFall was discontinued when it became clear that it was less of a scam than most lotteries."

—Allyn Jackson

BOOKSHELF

A man is known by the books he reads. —Emerson

New and Noteworthy Titles on Our Bookshelf
March 2016



The Cult of Pythagoras: Math and Myth, by Alberto A. Martínez (University of Pittsburgh Press, 2013). This book aims to present historically accurate accounts of many of the colorful legends about mathematics and mathematicians. Among the stories that appear here are the cult of the Pythagoreans, Archimedes's solution concerning the weight of a golden crown, and Gauss's childhood feat of a fast and clever way to add all numbers from 1 to 100. The embellishments that have become attached to many of these stories are entertaining. "My Calculus class always chuckles when I tell the story of Archimedes running naked though the streets yelling 'Eureka, Eureka' when he determines how to solve the problem," writes Ellen Ziliak in a review that appeared on the web site of the Mathematical Association of America. "I really enjoyed this book and think it would be a worthwhile read for anyone, but especially a student interested in the history behind the rules mathematicians now take for granted," Ziliak concludes.

In the Dark on the Sunny Side: A Memoir of an Out-of-Sight Mathematician, by Larry W. Baggett (Mathematical Association of America, 2012). Larry Baggett is well known for his work in analysis. What's less well known is that he is blind. His ability to calculate in his head is formidable, though Baggett is modest about it. "My feeling is that sighted mathematicians could do a lot in their heads too, but it's handy to write on a piece of paper," he told the *Notices* in an interview for a November 2002 article on blind mathematicians. In a review of Baggett's book that appeared on the website of the Institute of Mathematics and its Applications (UK), Sean Elvidge writes: "[T]his is a heart-warming and humorous book… Larry's passions and interests are clear—music, maths and his family—and this is a great account of a fascinating life."

In connection with the obituary for Alexandre Grothendieck in this month's *Notices*, we note that Thombooks Press recently put out a book containing translations into English of parts of Grothendieck's memoir *Récoltes et Semailles*. The translation is by Roy Lisker and edited by S. Peter Tsatsanis. The book also includes "The Quest for Alexandre Grothendieck: Report of an Adventurous Journey to Find the World's Most Famous Mathematician/Hermit", by Roy Lisker. Distributed only by `Amazon.com`.

The AMS maintains a comprehensive list of reviews of popular mathematics books on its Reviews page at `www.ams.org/news/math-in-the-media/reviews`. The list highlights current books that have mathematical themes and are aimed at a broad audience potentially including mathematicians, students, and the general public.

Suggestions for books to include on the list should be sent to `notices-booklist@ams.org`.

Mathematics People

Saint-Raymond and Scholze Awarded 2015 Fermat Prizes

Laure Saint-Raymond

Laure Saint-Raymond of Ecole Normale Supérieure and Peter Scholze of the University of Bonn have been awarded Fermat Prizes for 2015. Laure Saint-Raymond was honored for her development of asymptotic theories of partial differential equations, including the fluid limits of rarefied flows, multiscale analysis in plasma physics equations and ocean modeling, and the derivation of the Boltzmann equation from interacting particle systems. Laure shared with the *Notices* her conviction "that research is not a matter of competition but rather a collective challenge…. I like the 'algorithm for discovery' published by [David] Paydarfar and [William] Schwartz in *Science* (April 2001): Slow down to explore… Read but not too much… Pursue quality for its own sake… Cultivate smart friends."

©G. Bergman

Peter Scholze

Peter Scholze was honored for his invention of perfectoid spaces and their application to fundamental problems in algebraic geometry and in the theory of automorphic forms. (See "What is a perfectoid space?" by Bhargav Bhatt in *Notices*, October 2014.)

The Fermat Prize is given every two years for research in fields in which Pierre de Fermat made major contributions: statements of variational principles, foundations of probability and analytic geometry, and number theory. The prize carries a cash value of 20,000 euros (approximately US$22,000).

—From a Fermat Prize announcement

ICMI Klein and Freudenthal Medals Awarded

Alan J. Bishop

Alan J. Bishop of Monash University has been awarded the 2015 Felix Klein Medal of the International Commission on Mathematical Instruction (ICMI) "in recognition of his more than forty-five years of sustained, consistent, and outstanding lifetime achievements in mathematics education research and scholarly development". The prize citation states that "his book *Mathematical Enculturation: A Cultural Perspective on Mathematics Education*, published in 1988, was groundbreaking in that it developed a new conception of mathematics—the notion of mathematics as a cultural product and the cultural values that mathematics embodies…. Bishop has been instrumental in bringing the political, social, and cultural dimensions of mathematics education to the attention of the field." He has done extensive mentoring work with teachers and developed full-time MEd and PhD programs in mathematics education while at Cambridge University. As the citation states, "few researchers can match the way in which his research has improved mathematics education through the connections he has forged between research and practice." Bishop told the *Notices*: "When leaving school I had to make a choice between studying mathematics or music. I played bassoon in the National Youth Orchestra [of Great Britain] at that time, and there was no possibility of a joint degree between the two. I chose to study mathematics but have always enjoyed music. I still play the bassoon as well as other instruments. Is this an example of the famous links between mathematical and musical abilities?"

Jill Adler

Jill Adler of the University of the Witwatersrand has been awarded the Hans Freudenthal Medal for 2015 "in recognition of her outstanding research program dedicated to improving the teaching and learning of mathematics in South Africa—from her 1990s groundbreaking sociocultural research on the inherent dilemmas of teaching mathematics in multilingual classrooms through to her subsequent focus on problems related to mathematical knowledge for teaching and mathematics teacher professional development." According to the citation, "she has played an outstanding leadership role in growing mathematics education research in South Africa, Africa, and beyond. Her development of research teams involving graduate students and postdoctoral fellows, along with the mentoring of numerous PhD and master's students, have all added to the human research capacity she has been instrumental in creating in Southern Africa."

The Klein Medal honors lifetime achievement in mathematics education research. The Freudenthal Medal recognizes a major cumulative program of research.

—From ICMI announcements

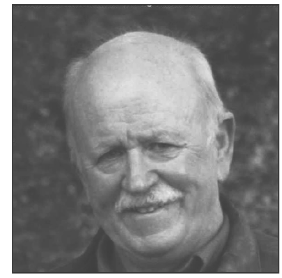
Bergholtz Receives Wallenberg Fellowship

Emil J. Bergholtz of Freie Universität Berlin has been awarded a Wallenberg Academy Fellowship for his work in "developing mathematical theories that may guide the development of a particular form of quantum matter, which has special properties that researchers believe may be a platform for quantum computers". Twenty-nine young researchers in the fields of humanities, medicine, natural sciences, social sciences, and engineering were awarded Wallenberg Academy Fellowships in 2015. Each fellow receives a five-year grant of 5–9 million Swedish krona (approximately US$588,000–1,060,000), depending on the field, and may apply for an additional five years. The program was established by the Knut and Alice Wallenberg Foundation in close cooperation with five learned academies and sixteen Swedish universities to give the most promising young researchers a work situation that enables them to focus on their projects and address difficult research questions over an extended period of time.

—From a Wallenberg Academy announcement

IEEE Control Systems Award Given

The Institute of Electrical and Electronics Engineers (IEEE) awards the Control Systems Award annually. The recipient of the 2016 award is Arthur J. Krener of the Naval Postgraduate School, Monterey, California, "for contributions to the analysis, control, and estimation of nonlinear control systems." The award recognizes an individual's "outstanding contributions to control systems engineering, science, or technology" and considers the seminal nature, depth, and breadth of contributions, as well as singular achievement and practical impact.



Arthur J. Krener

—From an IEEE announcement

Prizes of the Mathematical Society of Japan

The Mathematical Society of Japan (MSJ) has awarded the following prizes for 2015.

The 2015 Autumn Prize was awarded to Koji Fujiwara of Kyoto University for outstanding contributions to work on constructing group actions on quasi-trees.

The 2015 Analysis Prizes were awarded to Mitsuru Sugimoto of Nagoya University for research on harmonic analysis for modulation and related spaces and smoothing estimates for partial differential equations of dispersive type; to Kazunaga Tanaka of Waseda University for work on variational methods for multicluster solutions to a singular perturbation of nonlinear elliptic equations; and to Akimichi Takemura of the University of Tokyo for studies on holonomic gradient method.

The 2015 Geometry Prizes were awarded to Hiroshi Iritani of Kyoto University for the study of quantum cohomology and to Osamu Saeki of Kyushu University for research on stable maps and topology of manifolds. The 2015 Takebe Katahiro Prizes were awarded to Michiaki Onodera of Kyushu University for the study of shapes of solutions of elliptic equations by the evolution equation approach; to Shouhei Honda of Tohoku University for research in geometric analysis on limit spaces of Riemannian manifolds; and to Shunsuke Yamana of Kyoto University for work on automorphic L-functions and theta correspondence.

The 2015 Takebe Katahiro Prizes for Encouragement of Young Researchers were awarded to Masahiro Ikeda of Kyoto University for research on the asymptotic behavior of solutions for nonlinear dispersive wave equations; to Erika Ushikoshi of Tamagawa University for work on the Hadamard variational formula of the Stokes equations; to Motohiro Sobajima of the Tokyo University of Science for the study of second-order elliptic operators with unbounded coefficients; and to Kazuki Morimoto of Kyoto University for research on periods of automorphic forms and special values of L-functions.

—From MSJ announcements

Prizes of the New Zealand Mathematical Society

The New Zealand Mathematical Society (NZMS) has announced several awards for 2015.

Hinke Osinga of the University of Auckland received the NZMS Research Award "for pioneering work on theory and computational methods in dynamical systems and its applications in biology and engineering".

Adam Day of Victoria University of Wellington received the NZMS Early Career Award "for fundamental contributions to the theory of algorithmic randomness and computability, including the solution of the random covering problem".

Andrew Keane of the University of Auckland was awarded the Aitken Prize for the best contributed talk by a student at the NZMS Colloquium for "Bifurcation analysis of a model for the El Niño Southern Oscillation".

Andrus Giraldo of the University of Auckland received the 2015 Australia and New Zealand Industrial and Applied Mathematics (ANZIAM) poster prize for best poster by an early career researcher at the NZMS Colloquium for his poster "To Flip or Not to Flip".

Three mathematicians were chosen as Fellows of the New Zealand Mathematical Society:

- Steven Galbraith, University of Auckland
- Mick Roberts, Massey University
- Charles Semple, University of Canterbury.

—From an NZMS announcement

AAAS Fellows Chosen

The following mathematical scientists have been elected fellows of the Section on Mathematics of the American Association for the Advancement of Science (AAAS):

- Daniel L. Goroff, Alfred P. Sloan Foundation
- Peter Kuchment, Texas A&M University
- Reinhard C. Laubenbacher, University of Connecticut Health Center/Jackson Laboratory for Genomic Medicine
- Howard A. Levine, Iowa State University.

—From an AAAS announcement

Marshall Scholarships Awarded

Two students in the mathematical sciences have been awarded Marshall Scholarships for 2015. Jacob Calvert of the University of Illinois received his BS in bioengineering with a minor in mathematics. He will study mathematical sciences at Bristol University. Rahul Singh of Yale University is a student in economics and mathematics. He will study econometrics and mathematical economics at the London School of Economics and computational statistics and machine learning at University College London. Singh told the *Notices*: "I discovered my love for applied math while teaching financial literacy at charter schools in New Haven and Cleveland. I found joy in explaining the mathematical concepts that underlie economic concepts that in turn affect my students' decisions about how to pay for college."

Rahul Singh

Marshall Scholarships finance young Americans of high ability to study for degrees in the United Kingdom. Up to forty scholars are selected each year to study at the graduate level at a UK institution in any field of study.

—From a Marshall Scholarships announcement

Lesley Sibner (1934–2013)

Lesley Sibner

Used with permission of Deane Yang.

Lesley Sibner died unexpectedly on September 11, 2013. Sibner first pursued a career in acting, studying at the School of Drama at the Carnegie Institute of Technology (now known as Carnegie-Mellon University) and then with Uta Hagen in New York. She returned to college and fell in love with mathematics while taking a required calculus course. She also fell in love with her physics instructor, Robert Sibner, who became her husband and collaborator. Sibner received her PhD from the Courant Institute under the direction of Lipman Bers and Cathleen Morawetz. After a position at Stanford University, she spent most of her career as a professor at Polytechnic Institute of Brooklyn (now known as the New York University Polytechnic School of Engineering). She held visiting positions at the Institute for Advanced Study in Princeton, at Harvard University, at the Institut des Hautes Etudes Scientifiques in Bures-sur-Yvette, at the Max Planck Institute in Bonn, and at the Institut de Mathématiques de Jussieu in Paris. She was a Fulbright Research Scholar at the Institut Henri Poincaré and a Bunting Scholar at the Radcliffe Institute. Sibner was the associate secretary of the AMS for the Eastern Section from 1993 to 2009. She was in the inaugural class of Fellows of the AMS.

Sibner's initial research was on fluid flow and mixed-type PDEs such as the Tricomi equation. Soon afterward, Lesley and her husband Bob began a lifelong collaboration in global analysis, starting with nonlinear Hodge theory on a Riemannian manifold. Later, inspired by encounters with Atiyah and Bott at the Institute for Advanced Study, they began to study the Atiyah-Singer index theorem and *K*-theory, sometimes while sunbathing on nude beaches in the south of France. This led to their best-known work, which was on gauge theories, especially Yang-Mills, and which continued until Lesley passed away.

Cliff Taubes writes: "Lesley was one of the mathematical pioneers of Yang-Mills theory. Her theorems with Bob Sibner about removable singularities taught us how holonomy and finite energy play off each other to allow or prohibit singularities. These theorems led to the applications of instantons by Peter Kronheimer and Tom Mrowka to describe knot polynomials and the like. The theorem I envied most was joint with Bob Sibner and Karen Uhlenbeck, which gave a counterexample to the conjecture that all Yang-Mills fields on the 4-sphere were self-dual or anti-self-dual. Even so, I was so much more delighted, because such a good theorem couldn't have come to nicer people. Lesley and Bob were always so nice to me, and I am sure they probably were to other wet-behind-the-ears youngsters. So sad to see the passing of a good person like Lesley."

Lesley was not a typical mathematician. Karen Uhlenbeck writes: "I have many memories of Lesley, but my favorite was when I met Lesley and Bob for the first time at a summer school in Trieste. I went to a talk on nonlinear Hodge theory. Lesley got up in a purple suede pantsuit and high heels and proceeded to give an interesting and clear talk. I could not believe my eyes! Being a woman in math took on a completely different meaning. Subsequently, she served as an important, inspiring, and kind mentor to me."

—Deane Yang
New York University

Okan Gurel (1931–2015)

Okan Gurel of New York, New York, died on March 14, 2015. Born on August 1, 1931, in Turkey, he received his PhD in applied mechanics from Stanford University in 1961 after receiving his MASc from the University of British Columbia and, before that, his YMüh from Istanbul Technical University in 1954. He was a long-time IBMer working on business-related mathematical problems, such as the marker-layout problem, and on mathematical explanations of phenomena in biology and chemistry, as exemplified by his role as the lead organizer of the New York Academy of Sciences symposium on Bifurcation Theory and Applications in Scientific Disciplines. He was coauthor with Demet Gurel of *Oscillations in Chemical Reactions*, published by Springer-Verlag in English and by Mir in Russian, with an unauthorized version published by Akademie-Verlag in East Germany.

—Ozan Gurel
New York, New York

Mathematics Opportunities

Call for Nominations for 2016 ICTP Ramanujan Prize

The Ramanujan Prize is awarded annually to a researcher from a developing country who is less than forty-five years of age on December 31 of the year of the award and who has conducted outstanding research in a developing country. It is funded by the Department of Science and Technology of the Government of India (DST) and is administered jointly by the International Centre for Theoretical Physics (ICTP), the International Mathematical Union (IMU), and the DST. The deadline for nominations is **March 1, 2016**. See `www.ictp.it/about-ictp/media-centre/news/2015/11/ramanujan-call.aspx`.

—From an ICTP announcement

Project NExT 2016–2017

Project NExT (New Experiences in Teaching) is a professional development program for new and recent PhDs in the mathematical sciences (including pure and applied mathematics, statistics, operations research, and mathematics education). Each year a number of faculty members from colleges and universities throughout the country are selected to participate in a workshop preceding the Mathematical Association of America (MAA) summer meeting, in activities during the summer MAA meetings and the Joint Mathematics Meetings in January, and in an electronic discussion network. Applications are invited for the 2016-2017 fellowship year. The deadline for applications is **April 15, 2016**. See `archives.math.utk.edu/projnext/`.

—From an MAA announcement

Call for Nominations for Graham Wright Award

The 2015 Graham Wright Award for Distinguished Service of the Canadian Mathematical Society (CMS) recognizes individuals who have made sustained and significant contributions to the Canadian mathematical community and, in particular, to the Canadian Mathematical Society. The deadline for nominations is **March 31, 2016**. See `cms.math.ca/Prizes/dis-nom`.

—From a CMS announcement

Intensive Research Programs at CRM

The Centre de Recerca Matemática (CRM) invites proposals for intensive research programs consisting of one to five months (from September to July) of intensive research in any branch of mathematics and the mathematical sciences. Researchers work on open problems and analyze the state and perspectives of their areas. Proposals are sought for programs to be organized preferably after August 2017. The deadline for a preliminary proposal is **February 29, 2016**, and for the final proposal, **April 29, 2016**. Guidelines and application instructions can be found at `www.crm.cat/en/Host/SciEvents/IRP/Pages/CallApplication.aspx`.

—From a CRM announcement

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CALIFORNIA

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Institute for Pure and Applied Mathematics

UCLA's Institute for Pure and Applied Mathematics (IPAM) is seeking its next Director, to begin a five-year term in July 2017 or 2018. Salary will be commensurate with the Director's education and experience. Candidates may come from mathematics, statistics, computer science or related fields, and possess some of the following qualifications:

- Scientific distinction sufficient to be offered a tenured faculty position at UCLA.
- Scientific and mathematical interest and vision, and the ability to interact with a wide range of researchers and research topics.
- Experience and capability to manage IPAM, including programs, staff, finances and administration.
- Ability to reach out to a broad range of constituents, including the math and science communities, the National Science Foundation, and the public, as well as to engage in fund-raising.
- A commitment to diversity in the broadest sense, especially the participation of women and under-represented minorities in research.

For a detailed job description and application instructions, go to `www.ipam.ucla.edu/director`. Applications will receive fullest consideration if received by June 1, 2016. UCLA is an Equal Opportunity/Affirmative Action Employer.

000011

MINNESOTA

University of Minnesota
School of Mathematics

The School of Mathematics of the University of Minnesota is seeking outstanding candidates for up to 3 TENURE-TRACK or TENURED faculty positions starting fall semester 2016. Candidates should have a PhD or equivalent degree in mathematics or a closely related field and excellent records in both research and teaching. Applications and all supporting materials must be submitted electronically through: `www.mathjobs.org`. No paper submission is needed unless the candidate is unable to submit electronically, in which case letters should be sent to the following address: Peter J. Olver, Professor and Head School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church Street S.E. Minneapolis, MN 55455. Applicants must include the following: Cover letter, curriculum vitae, at least 4 letters of recommendation, one of which should address teaching ability, and a research and teaching statement. Reference letter writers should be asked to submit their letters online through `mathjobs.org`. The review process will start in March 2016, and will continue for as long as positions are available.

Any offer of employment will be contingent on a successful criminal background check. The University of Minnesota is an Equal Opportunity Employer/Educator.

000010

KOREA

KOREA INSTITUTE FOR ADVANCED STUDY (KIAS)
Assistant Professor & Research Fellow in Pure and Applied Mathematics

The School of Mathematics at the Korea Institute for Advanced Study (KIAS) invites applicants for the positions at the level of KIAS Assistant Professor and Postdoctoral Research Fellow in pure and applied mathematics. KIAS, founded in 1996, is committed to the excellence of research in basic sciences (mathematics, theoretical physics, and computational sciences)

Suggested uses for classified advertising are positions available, books or lecture notes for sale, books being sought, exchange or rental of houses, and typing services. The publisher reserves the right to reject any advertising not in keeping with the publication's standards. Acceptance shall not be construed as approval of the accuracy or the legality of any advertising.

The 2016 rate is \$3.50 per word with a minimum two-line headline. No discounts for multiple ads or the same ad in consecutive issues. For an additional \$10 charge, announcements can be placed anonymously. Correspondence will be forwarded.

Advertisements in the "Positions Available" classified section will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted.

There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

Upcoming deadlines for classified advertising are as follows: April 2016—January 29, 2016; May 2016—March 2, 2016; June/July 2016—April 29, 2016; August 2016—May 30, 2016; September 2016—June 28, 2016; October 2016—July 29, 2016; November 2016—August 29, 2016; December 2016—September 29, 2016.

US laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. "Positions Available" advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws. Details and specific wording may be found on page 1373 (vol. 44).

Situations wanted advertisements from involuntarily unemployed mathematicians are accepted under certain conditions for free publication. Call toll-free 800-321-4AMS (321-4267) in the US and Canada or 401-455-4084 worldwide for further information.

Submission: Promotions Department, AMS, P.O. Box 6248, Providence, Rhode Island 02904; or via fax: 401-331-3842; or send email to `classads@ams.org`. AMS location for express delivery packages is 201 Charles Street, Providence, Rhode Island 02904. Advertisers will be billed upon publication.

through high-quality research programs and a strong faculty body consisting of distinguished scientists and visiting scholars.

Applicants are expected to have demonstrated exceptional research potential, including major contributions beyond or through the doctoral dissertation.

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Applications will be reviewed twice a year, May 20 and November 20, and selected applicants will be notified in a month after the review. In exceptional cases, applications can be reviewed other times based on the availability of positions. The starting date of the appointment is negotiable. Applications must include a complete vitae with a cover letter, a list of publications, a research plan, and three letters of recommendation (All documents should be in English). All should be sent by post or e-mail to:

Ms. Sojung Bae (email: mathkias@kias.re.kr)
School of Mathematics
Korea Institute for Advanced Study (KIAS)
85 Hoegiro (Cheongnyangni-dong 207-43), Dongdaemun-gu,
Seoul 02455, Republic of Korea

000009

PUBLICATIONS FOR SALE

DIFFERENTIAL EQUATIONS AND POLYNOMIALS, VOLUME 1

By Dr. Mehran Basti

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Infinity Publishing and Major Online Bookstores.

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DIFFERENTIAL EQUATIONS AND POLYNOMIALS, VOLUME 2

By Dr. Mehran Basti

Solving systems of differential equations (On a CD).

The method will be as powerful as Isaac Newton's gravitational formula.

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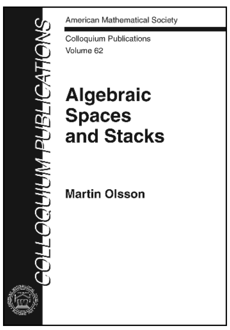
000007

New Publications Offered by the AMS

To subscribe to email notification of new AMS publications, please go to www.ams.org/bookstore-email.

Algebra and Algebraic Geometry



Algebraic Spaces and Stacks

Martin Olsson, *University of California, Berkeley, CA*

This book is an introduction to the theory of algebraic spaces and stacks intended for graduate students and researchers familiar with algebraic geometry at the level of a first-year graduate course. The first several chapters are devoted to background material including chapters on Grothendieck topologies, descent, and fibered categories. Following this, the theory of algebraic spaces and stacks is developed. The last three chapters discuss more advanced topics including the Keel-Mori theorem on the existence of coarse moduli spaces, gerbes and Brauer groups, and various moduli stacks of curves. Numerous exercises are included in each chapter ranging from routine verifications to more difficult problems, and a glossary of necessary category theory is included as an appendix.

It is splendid to have a self-contained treatment of stacks, written by a leading practitioner. Finally we have a reference where one can find careful statements and proofs of many of the foundational facts in this important subject. Researchers and students at all levels will be grateful to Olsson for writing this book.

—***William Fulton, University of Michigan***

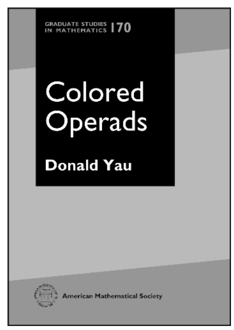
This is a carefully planned out book starting with foundations and ending with detailed proofs of key results in the theory of algebraic stacks.

—***Johan de Jong, Columbia University***

Contents: Introduction; Summary of background material; Grothendieck topologies and sites; Fibered categories; Descent and the stack condition; Algebraic spaces; Invariants and quotients; Quasi-coherent sheaves on algebraic spaces; Algebraic stacks: Definitions and basic properties; Quasi-coherent sheaves on algebraic stacks; Basic geometric properties and constructions for stacks; Coarse moduli spaces; Gerbes; Moduli of curves; Glossary of category theory; Bibliography; Index of notation; Index of terminology.

Colloquium Publications, Volume 62

April 2016, approximately 299 pages, Hardcover, ISBN: 978-1-4704-2798-6, LC 2015043394, 2010 *Mathematics Subject Classification:* 14A20; 14D23, 14D22, **AMS members US$79.20**, List US$99, Order code COLL/62



Colored Operads

Donald Yau, *The Ohio State University at Newark, OH*

The subject of this book is the theory of operads and colored operads, sometimes called symmetric multicategories. A (colored) operad is an abstract object which encodes operations with multiple inputs and one output and relations between such operations. The theory originated in the early 1970s in homotopy theory and quickly became very important in algebraic topology, algebra, algebraic geometry, and even theoretical physics (string theory). Topics covered include basic graph theory, basic category theory, colored operads, and algebras over colored operads. Free colored operads are discussed in complete detail and in full generality.

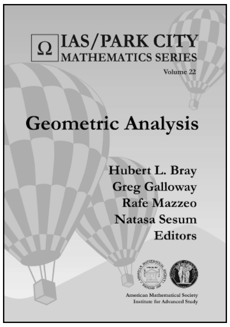
The intended audience of this book includes students and researchers in mathematics and other sciences where operads and colored operads are used. The prerequisite for this book is minimal. Every major concept is thoroughly motivated. There are many graphical illustrations and about 150 exercises. This book can be used in a graduate course and for independent study.

Contents: *Graphs and trees:* Directed graphs; Extra structures on graphs; Rooted trees; Collapsing an internal edge; Grafting of rooted trees; Grafting and extra structure; *Category theory:* Basic category theory; Symmetric monoidal categories; Colored symmetric sequences and objects; *Operads and algebras:* Motivation for colored operads; Colored operads; Operads in arity 1; Algebras over colored operads; Examples of algebras; Motivation for partial compositions; Colored pseudo-operads; *Free colored operads:* Motivation for free colored operads; General operadic composition; Free colored non-symmetric operads; Free colored operads; Further reading; Bibliography; List of main facts; Index.

Graduate Studies in Mathematics, Volume 170

April 2016, 428 pages, Hardcover, ISBN: 978-1-4704-2723-8, LC 2015036707, 2010 *Mathematics Subject Classification:* 18D50, 18D10, 18D20, 18A40, 05-01, 06F05, **AMS members US$71.20**, List US$89, Order code GSM/170

Analysis



Geometric Analysis

Hubert L. Bray, *Duke University, Durham, NC*, **Greg Galloway**, *University of Miami, Coral Gables, FL*, **Rafe Mazzeo**, *Stanford University, CA*, and **Natasa Sesum**, *Rutgers University, Piscataway, NJ*, Editors

This volume includes expanded versions of the lectures delivered in the Graduate Minicourse portion of the 2013 Park City Mathematics Institute session on Geometric Analysis. The papers give excellent high-level introductions, suitable for graduate students wishing to enter the field and experienced researchers alike, to a range of the most important areas of geometric analysis. These include: the general issue of geometric evolution, with more detailed lectures on Ricci flow and Kähler-Ricci flow, new progress on the analytic aspects of the Willmore equation as well as an introduction to the recent proof of the Willmore conjecture and new directions in min-max theory for geometric variational problems, the current state of the art regarding minimal surfaces in R^3, the role of critical metrics in Riemannian geometry, and the modern perspective on the study of eigenfunctions and eigenvalues for Laplace–Beltrami operators.

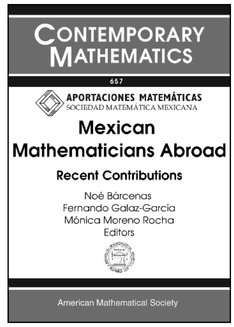
Titles in this series are co-published with the Institute for Advanced Study/Park City Mathematics Institute. Members of the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM) receive a 20% discount from list price.

Contents: G. Huisken, Heat diffusion in geometry; **P. Topping**, Applications of Hamilton's compactness theorem for Ricci flow; **B. Weinkove**, The Kähler-Ricci flow on compact Kähler manifolds; **S. Zelditch**, Park City lectures on eigenfunctions; **J. A. Viaclovsky**, Critical metrics for Riemannian curvature functionals; **F. C. Marques** and **A. Neves**, Min-max theory and a proof of the Willmore conjecture; **T. Riviére**, Weak immersions of surfaces with L^2-bounded second fundamental form; **B. White**, Introduction to minimal surface theory.

IAS/Park City Mathematics Series, Volume 22

April 2016, approximately 443 pages, Hardcover, ISBN: 978-1-4704-2313-1, LC 2015031562, 2010 *Mathematics Subject Classification:* 53-06, 35-06, 83-06, **AMS members US$79.20**, List US$99, Order code PCMS/22

General Interest



Mexican Mathematicians Abroad

Recent Contributions

Noé Bárcenas, *Centro de Ciencias Matemáticas, UNAM, Morelia, Mexico*, **Fernando Galaz-García**, *Karlsruher Institut Für Technologie, Germany*, and **Mónica Moreno Rocha**, *Centro de Investigación en Matemáticas, A.C., Guanajuato, Mexico*, Editors

This volume contains the proceedings of the First Workshop "Matemáticos Mexicanos Jóvenes en el Mundo", held from August 22–24, 2012, at Centro de Investigación en Matemáticas (CIMAT) in Guanajuato, Mexico.

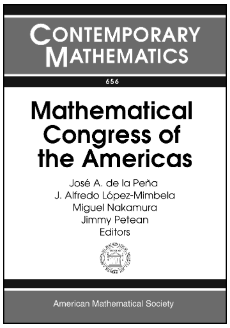
One of the main goals of this meeting was to present different research directions being pursued by young Mexican mathematicians based in other countries, such as Brazil, Canada, Colombia, Estonia, Germany, Spain and the United States, showcasing research lines currently underrepresented in Mexico.

Featured are survey and research articles in six areas: algebra, analysis, applied mathematics, geometry, probability and topology. Their topics range from current developments related to well-known open problems to novel interactions between pure mathematics and computer science. Most of the articles provide a panoramic view of the fields and problems the authors work on, making the book accessible to advanced graduate students and researchers in mathematics from different fields.

Contents: M. Abel and **R. M. Pérez-Tiscareño**, Locally pseudoconvex inductive limit of locally pseudoconvex Q-algebras; **O. Antolín Camarena**, A whirlwind tour of the world of $(\infty, 1)$-categories; **O. Arizmendi** and **C. Vargas**, Norm convergence in non-commutative central limit theorems: Combinatorial approach; **P. Dávalos**, Dynamical models for some toru homeomorphisms; **E. A. Duéñez-Guzmán** and **M. A. Ramírez-Ortegón**, A review of no free lunch theorems for search; **D. Labardini-Fragoso**, On triangulations, quivers with potentials and mutations; **J. Malagón-López**, Riemann-Roch without denominators for oriented cohomology theories; **L. Núñez-Betancourt**, **E. E. Witt**, and **W. Zhang**, A survey on the Lyubeznik numbers; **S. Ortega Castillo**, Cluster value problem in infinite-dimensional spaces; **R. Perales**, A survey on the convergence of manifolds with boundary; **C. Ramos-Cuevas**, Convexity is a local property in CAT(κ) spaces; **R. Ríos-Zertuche**, An introduction to the half-infinite wedge.

Contemporary Mathematics, Volume 657

February 2016, 237 pages, Softcover, ISBN: 978-1-4704-2192-2, LC 2015036424, 2010 *Mathematics Subject Classification:* 13D45, 13F60, 14C40, 18-01, 20C32, 37E45, 46G20, 46H05, 46L54, **AMS members US$86.40**, List US$108, Order code CONM/657



Mathematical Congress of the Americas

José A. de la Peña, **J. Alfredo López-Mimbela**, **Miguel Nakamura**, and **Jimmy Petean**, *CIMAT, Guanajuato, Mexico*, Editors

This volume contains the proceedings of the First Mathematical Congress of the Americas, held from August 5–9, 2013, in Guanajuato, México. With the participation of close to 1,000 researchers from more than 40 countries, the meeting set a benchmark for mathematics in the two continents.

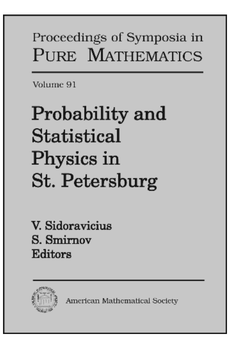
The papers, written by some of the plenary and invited speakers, as well as winners of MCA awards, cover new developments in classic topics such as Hopf fibrations, minimal surfaces, and Markov processes, and provide recent insights on combinatorics and geometry, isospectral spherical space forms, homogenization on manifolds, and Lagrangian cobordism, as well as applications to physics and biology.

Contents: **M. Clapp** and **A. Pistoia**, Symmetries, Hopf fibrations and supercritical elliptic problems; **F. C. Marques** and **A. Neves**, Min-max theory of minimal surfaces and applications; **G. Contreras**, Homogenization on manifolds; **O. Cornea**, Lagrangian cobordism: Rigidity and flexibility aspects; **A. Dickenstein**, Biochemical reaction networks: An invitation for algebraic geometers; **J. Denzler**, **H. Koch**, and **R. J. McCann**, Long-time asymptotic expansions for nonlinear diffusions in Euclidean space; **E. A. Lauret**, **R. J. Miatello**, and **J. P. Rossetti**, Non-strongly isospectral spherical space forms; **V. Rivero**, Entrance laws for positive self-similar Markov processes; **F. Rodriguez-Villegas**, Combinatorics and geometry; **M. Sambarino**, A (short) survey on dominated splittings; **E. V. Teixeira**, Geometric regularity estimates for elliptic equations.

Contemporary Mathematics, Volume 656

February 2016, 201 pages, Softcover, ISBN: 978-1-4704-2310-0, 2010 *Mathematics Subject Classification:* 00-02, 00A05, 00A99, 00B20, 00B25, **AMS members US$86.40**, List US$108, Order code CONM/656

Probability and Statistics



Probability and Statistical Physics in St. Petersburg

V. Sidoravicius, *Courant Institute, New York, NY, and New York University–Shanghai, China*, and **S. Smirnov**, *University of Geneva, Switzerland, and St. Petersburg State University, Russia*, Editors

This book brings a reader to the cutting edge of several important directions of the contemporary probability theory, which in many cases are strongly motivated by problems in statistical physics. The authors of these articles are leading experts in the field and the reader will get an exceptional panorama of the field from the point of view of scientists who played, and continue to play, a pivotal role in the development of the new methods and ideas, interlinking it with geometry, complex analysis, conformal field theory, etc., making modern probability one of the most vibrant areas in mathematics.

This item will also be of interest to those working in mathematical physics.

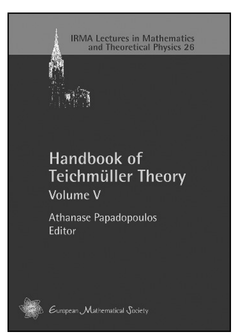
Contents: **Y. Bakhtin**, Ergodic theory of the Burgers equation; **V. Beffara** and **H. Duminil-Copin**, Critical point and duality in planar lattice models; **G. Ben Arous** and **A. Fribergh**, Biased random walks on random graphs; **A. Borodin** and **V. Gorin**, Lectures on integrable probability; **Y. Chang** and **Y. Le Jan**, Markov loops in discrete spaces; **A. Guionnet**, Random matrices and the Potts model on random graphs; **J. Komjáthy** and **Y. Peres**, Topics in Markov chains: Mixing and escape rate; **G. F. Lawler**, Random walk problems motivated by statistical physics; **G. Olshanski**, Markov dynamics on the dual object to the infinite-dimensional unitary group; **M. Sodin**, Lectures on random nodal portraits; **A. Vershik**, Smoothness and standardness in the theory of *AF*-algebras and in the problem on invariant measures; **O. Zeitouni**, Branching random walks and Gaussian fields.

Proceedings of Symposia in Pure Mathematics, Volume 91

April 2016, approximately 478 pages, Hardcover, ISBN: 978-1-4704-2248-6, LC 2015025728, 2010 *Mathematics Subject Classification:* 60K35, 82B43, 82C43, 60B20, 05C81, 82B41, 82C41, 60J25, **AMS members US$96**, List US$120, Order code PSPUM/91

New AMS-Distributed Publications

Analysis



Handbook of Teichmüller Theory: Volume V

Athanase Papadopoulos, *Université de Strasbourg, France*, Editor

This volume is the fifth in a series dedicated to Teichmüller theory in a broad sense, including the study of various deformation spaces and of mapping class group actions. It is divided into four parts:

Part A: The metric and the analytic theory

Part B: The group theory

Part C: Representation theory and generalized structures

Part D: Sources

The topics that are covered include identities for the hyperbolic geodesic length spectrum, Thurston's metric, the cohomology of moduli space and mapping class groups, the Johnson homomorphisms, Higgs bundles, dynamics on character varieties, and many others. Besides surveying important parts of the theory, several chapters contain conjectures and open problems. The last part contains two fundamental papers by Teichmüller, translated into English and accompanied by mathematical commentaries.

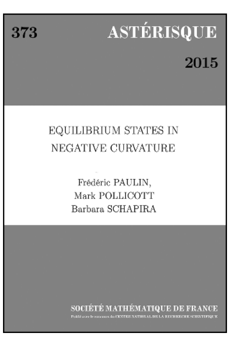
The papers, like those of the other volumes in this collection, are written by experts who have a broad view on the subject. Although the papers have an expository character (which fits with the purpose of the handbook), some of them also contain original and new material.

A publication of the European Mathematical Society. Distributed within the Americas by the American Mathematical Society.

Contents: **A. Papadopoulos**, Introduction to Teichmüller theory, old and new, V; *Part A. The metric and the analytic theory:* **M. Bridgeman** and **S. P. Tan**, Identities on hyperbolic manifolds; **W. Su**, Problems on the Thurston metric; *Part B. The group theory:* **Y. Kuno**, Meyer functions and the signature of fibered 4-manifolds; **N. Kawazumi** and **Y. Kuno**, The Goldman–Turaev Lie bialgebra and the Johnson homomorphisms; **T. Satoh**, A survey of the Johnson homomorphisms of the automorphism groups of free groups and related topics; *Part C. Representation theory and generalized structures:* **I. Kim**, Geometry and dynamics on character varieties; **L. Ji**, Compactifications and reduction theory of geometrically finite locally symmetric spaces; **L. Jeffrey**, Representations of fundamental groups of 2-manifolds; *Part D. Sources:* **O. Teichmüller**, Extremal quasiconformal mappings and quadratic differentials; **V. Alberge**, **A. Papadopoulos**, and **W. Su**, A commentary on Teichmüller's paper "Extremale quasikonforme Abbildungen und quadratische Differentiale"; **O. Teichmüller**, Determination of extremal quasiconformal mappings of closed oriented Riemann surfaces; **A. A'Campo-Neuen**, **N. A'Campo**, and **V. Alberge**, A commentary on Teichmüller's paper "Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen orientierten Riemannschen Flächen"; List of contributors; Index.

IRMA Lectures in Mathematics and Theoretical Physics, Volume 26

January 2016, 596 pages, Hardcover, ISBN: 978-3-03719-160-6, 2010 *Mathematics Subject Classification:* 30-00, 32-00, 57-00, 32G13, 32G15, 30F60; 11F06, 11F75, 14D20, 14H15, 14H60, 14H55, 14J60, 20F14, 20F28, 20F38, 20F65, 20F67, 20H10, 22E46, 30-03, 30C62, 30F20, 30F25, 30F10, 30F15, 30F30, 30F35, 30F40, 30F45, 32-03, 32S30, 53A35, 53B35, 53C35, 53C50, 53C80, 53D55, 53Z05, 57M07, 57M20, 57M27, 57M50, 57M60, 57N16, **AMS members US$86.40**, List US$108, Order code EMSILMTP/26



Equilibrium States in Negative Curvature

Frédéric Paulin, *Université Paris-Sud, France*, **Mark Pollicott**, *University of Warwick, Coventry, United Kingdom*, and **Barbara Schapira**, *Université Picardie Jules Verne, Amiens, France*

With their origin in thermodynamics and symbolic dynamics, Gibbs measures are crucial tools for studying the ergodic theory of the geodesic flow on negatively curved manifolds. The authors develop a framework (through Patterson–Sullivan densities) that allows them to get rid of compactness assumptions on the manifold, and prove many existence, uniqueness and finiteness results of Gibbs measures. They give many applications, to the variational principle, the counting and equidistribution of orbit points and periods, the unique ergodicity of the strong unstable foliation and the classification of Gibbs densities on some Riemannian covers.

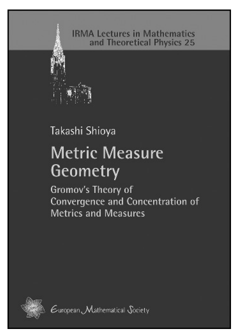
This item will also be of interest to those working in geometry and topology.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: Introduction; Background on negatively curved manifolds; A Patterson–Sullivan theory for Gibbs states; Critical exponent and Gurevich pressure; A Hopf–Tsuji–Sullivan–Roblin theorem for Gibbs states; Thermodynamic formalism and equilibrium states; The Liouville measure as a Gibbs measure; Finiteness and mixing of Gibbs states; Growth and equidistribution of orbits and periods; The ergodic theory of the strong unstable foliation; Gibbs states on Galois covers; List of symbols; Index; Bibliography.

Astérisque, Number 373

October 2015, 289 pages, Softcover, ISBN: 978-2-85629-818-3, 2010 *Mathematics Subject Classification:* 37D35, 53D25, 37D40, 37A25, 37C35, 53C12, **AMS members US$65.60**, List US$82, Order code AST/373



Metric Measure Geometry

Gromov's Theory of Convergence and Concentration of Metrics and Measures

Takashi Shioya, *Tohoku University, Mathematical Institute, Sendai, Japan*

This book studies a new theory of metric geometry on metric measure spaces. The theory was originally developed by M. Gromov in his book *Metric Structures for Riemannian and Non-Riemannian Spaces* and based on the idea of the concentration of measure phenomenon by Lévy and Milman. A central theme in this book is the study of the observable distance between metric measure spaces, defined by the difference between 1-Lipschitz functions on one space and those on the other. The topology on the set of metric measure spaces induced by the observable distance function is weaker than the measured Gromov–Hausdorff topology and allows the author to investigate a sequence of Riemannian manifolds with unbounded dimensions.

One of the main parts of this presentation is the discussion of a natural compactification of the completion of the space of metric measure spaces. The stability of the curvature-dimension condition is also discussed.

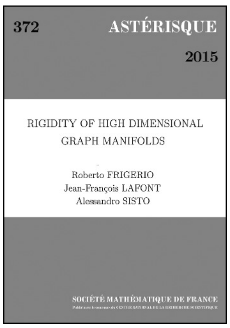
A publication of the European Mathematical Society. Distributed within the Americas by the American Mathematical Society.

Contents: Preliminaries from measure theory; The Lévy–Milman concentration phenomenon; Gromov–Hausdorff distance and distance matrix; Box distance; Observable distance and measurement; The space of pyramids; Asymptotic concentration; Dissipation; Curvature and concentration; Bibliography; Index.

IRMA Lectures in Mathematics and Theoretical Physics, Volume 25

January 2016, 194 pages, Hardcover, ISBN: 978-3-03719-158-3, 2010 *Mathematics Subject Classification:* 53C23; 28A33, 30Lxx, 35P15, 53C20, 54Exx, 58C40, 58J50, 60B10, **AMS members US$38.40**, List US$48, Order code EMSILMTP/25

Geometry and Topology



Rigidity of High Dimensional Graph Manifolds

Roberto Frigerio, *University of Pisa, Italy*, **Jean-François Lafont**, *Ohio State University, Columbus*, and **Alessandro Sistro**, *ETH Zurich, Switzerland*

This book is devoted to the definition and systematic study of graphed manifolds in large dimension. These are compact smooth manifolds supporting a decomposition into finitely many pieces, each of which is diffeomorphic to the product of a torus with a finite volume hyperbolic manifold with toric cusps. The pieces are glued by affine mappings of the boundary tori. The authors prove, in dimension larger or equal to 6, the Borel conjecture for the graphed manifolds and they establish the smooth rigidity. They analyze the structure of the groups which are quasi-isometric to the fundamental group of an irreducible graphed manifold.

This item will also be of interest to those working in algebra and algebraic geometry.

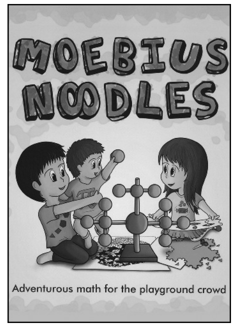
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: *Part I. Graph manifolds: topological and algebraic properties:* Quasi-isometries and quasi-actions; Generalized graph manifolds; Topological rigidity; Isomorphisms preserve pieces; Smooth rigidity; Algebraic properties; *Part II. Irreducible graph manifolds: coarse geometric properties:* Irreducible graph manifolds; Pieces of irreducible graph manifolds are quasi-preserved; Quasi isometry rigidity, I; Quasi isometry rigidity, II; *Part III. Concluding remarks:* Examples not supporting locally CAT(O) metrics; Directions for future research; Bibliography.

Astérisque, Number 372

October 2015, 189 pages, Softcover, ISBN: 978-2-85629-809-1, 2010 *Mathematics Subject Classification:* 53C24, 20F65, 53C23, 20E08, 20F67, 20F69, 19D35, **AMS members US$60**, List US$75, Order code AST/372

Math Education



Moebius Noodles

Adventurous Math for the Playground Crowd

Yelena McManaman and **Maria Droujkova**
Illustrations by Ever Salazar

This book is designed for parents and teachers who want to enjoy playful math with young children. It offers advanced math activities to fit the individual child's personality, interests, and needs and will open the door to a supportive online community that will answer questions and give ideas along the way.

Moebius Noodles will help readers take small, immediate steps toward the sense of mathematical power.

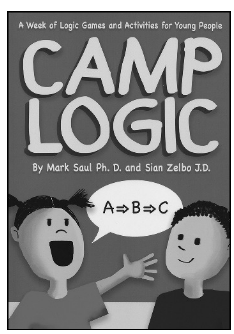
This item will also be of interest to those working in general interest.

A publication of Delta Stream Media, an imprint of Natural Math. Distributed in North America by the American Mathematical Society.

Contents: Why play this book; Questions and answers; *Symmetry:* Live mirror; Double doodle zoo; Mirror book; Special snowflake; Two-hand mirror drawing; *Number:* One-two-three and more; SuperAutoSimilarlyFractoalidocious; The big hunt for quantities; Real multiplication tables; *Function:* Function machine; Walk around in circles; New functions from old; Silly robot; *Grid:* Make your own grids; Grids and chimeras; The three bears and the middle way; Multiplication tower; Covariance monsters; Glossary.

Natural Math Series, Volume 3

April 2015, 88 pages, Softcover, ISBN: 978-0-9776939-5-5, **AMS members US$12**, List US$15, Order code NMATH/3



Camp Logic

A Week of Logic Games and Activities for Young People

Mark Saul and **Sian Zelbo**, *Courant Institute of Mathematical Sciences, New York University, New York*
Illustrations by Sian Zelbo

Most students encounter math through boring, rote memorization and drill and skill. Camp Logic reverses the trend by offering teachers fun, inquiry-based activities that get to the deeper elegance and joy of math with adaptations for different skill levels and learning environments. The work of Saul and Zelbo has redefined how math is taught in our programs.

—***Meghan Groome, Executive Director of Education and Public Programs at the New York Academy of Sciences***

Sian Zelbo and Mark Saul have created a user-friendly guide to help educators engage kids in finding patterns and using logic to solve puzzles. The program is a challenging yet fun approach to deepening math understanding.

—***Lisa Mielke, STEM Programs Manager for TASC, New York***

Mark and Sian have put together a delightful set of activities that are sure to captivate young minds (and old). Kids will enjoy games like Giotto for hours, all the while improving their math skills for years to come.

—***Ethan Berman, Founder, i2 Learning***

This book offers a deeper insight into what mathematics is, tapping every child's intuitive ideas of logic and natural enjoyment of games. Simple-looking games and puzzles quickly lead to deeper insights, which will eventually connect with significant formal mathematical ideas as the child grows.

This book is addressed to leaders of math circles or enrichment programs, but its activities can fit into regular math classes, homeschooling venues, or situations in which students are learning mathematics on their own. The mathematics contained in the activities can be enjoyed on many levels.

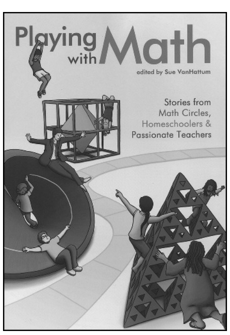
This item will also be of interest to those working in general interest.

A publication of Delta Stream Media, an imprint of Natural Math. Distributed in North America by the American Mathematical Society.

Contents: *Day One:* Animal puzzles (An introduction to logical reasoning with cryptarithms); The game of Giotto (Practice with pure logical reasoning); Lewis Carroll puzzles (Proof by contradiction); Cryptogram puzzle (More proof by contradiction); *Day Two:* Giotto puzzles (Analyzing the logical structure of Giotto); Watermelon language (Logic applied to number systems); Jittery soldiers (An introduction to invariants); The black and red problem (An introduction to parity); *Day Three:* Parity problem set; Discussion of black and red problem (A surprising connection to parity); Ginger's pigeons (Proofs with the pigeonhole principle); The mouse-and-cheese problem (Using the idea of an isomorphism to solve a problem); *Day Four:* Nim (An introduction to mathematical induction); Two-row nim and one-piece chess (Another example of an isomorphism); Leap frog activity (An activity for exploring invariants); *Day Five:* The boys and girls problem (Another example of the use of invariants); Hidden cards puzzle (Practice with logical deductions); Magic squares and 15 game (More practice with invariants and isomorphism).

Natural Math Series, Volume 2

December 2015, 134 pages, Softcover, ISBN: 978-0-9776939-6-2, **AMS members US$12**, List US$15, Order code NMATH/2



Playing with Math

Stories from Math Circles, Homeschoolers & Passionate Teachers

Sue VanHattum, Editor

Mathematics is a creative activity, like music. It requires some technique, and the technique has to be taught, but the main point is elsewhere—it is all about creativity, a sense of enjoyment, and higher purpose. This book goes a long way in that direction.

—***Ivar Ekeland, author of The Cat in Numberland***

…a marvelously useful and inspiring book. It is filled with stories by people who don't just love math, they share that love with others through innovative math activities. …is perfect for anyone eager to make math absorbing, entertaining, and fun.

—***Laura Grace Weldon, author of Free Range Learning***

The Internet is presently bursting with vibrant writing about mathematics learning; yet it can be difficult to navigate this wealth of resources. Sue VanHattum has carefully collected and arranged some of the best of this writing. Imagine having a cheerful, knowledgeable, caring, and patient native interpreter accompany you on a tour of a foreign land. That's Sue in the land of math. She and the authors collected here care deeply about welcoming everyone to the world of mathematics. Whether you play with math every day or are struggling to believe that one can play with math, this book will provide inspiration, ideas, and joy.

—***Christopher Danielson (talkingmathwithkids.com), author of Talking Math with Your Kids***

This book brings together the stories of over thirty authors who share their math enthusiasm with their communities, families, and students. After every chapter is a puzzle, game, or activity to encourage adults and children to play with math too. Thoughtful stories, puzzles, games, and activities will provide new insights.

This item will also be of interest to those working in general interest.

A publication of Delta Stream Media, an imprint of Natural Math. Distributed in North America by the American Mathematical Society.

Contents: *Math circles and more: Celebrating math:* Section introduction; The art of inquiry; Puzzle: Imbalance abundance; Rejoicing in confusion; Game: Parent bingo; Parents and kids together; Puzzle: Foxes and rabbits; On noticing and fairness; Puzzle: Is this for real?; Bionic algebra adventures; Story: Meet Alexandria Jones; Game: Pharaoh's pyramid; Puzzles: From ancient Egypt; The Oakland math circle; Game: Fantastic four, exploratorium staff; A culture of enthusiasm for math; Activity: Faces, edges, and vertices; Seized by a good idea; Puzzle: Math without words #1; A prison math circle; Puzzle: Math without words #2; Agents of math circles; Puzzle: Food for thought; The Julia Robinson mathematics festival; Puzzle: Saint Mary's Math Contest Sampler; Exploration: Candy conundrum; A young voice: Consider the circle; *Homeschoolers: Doing math:* Section introduction; Tying it all together; Game: Place value risk; Advice from Living Math forum; Puzzle: Deep arithmetic; Transitioning to living math; Game: Math card war; At the eye of the hurricane; Puzzle: Self-referential number square; One and a quarter pizzas; Game: Function machine; The math haters come around; Puzzle: Magic hexagon; Mapping the familiar; Game: Racetrack; Radically sensible ideas; Game: Dotsy; A young voice: An unschooler goes to college; *Passionate Teachers: Transforming Classroom Math:* Section introduction; Teach less, learn more; Game: Modular skirmish; Trust, Montessori style; Puzzle: Measuring with paper; Math in your feet; Game: Fizz buzz; Dinosaur math; Puzzle: Alien math; Better teaching through blogging; Activity: Candy launcher; Using math to describe gravity; Putting myself in my students' shoes; Puzzle: What number am I?; An argument against the real world; Puzzle: Octopus logic; Area of a circle; Exploration: Coloring cubes; Textbook-free; Activity: Guess my dice; Math is not linear; Puzzle: A little math magic; A young voice: My passion for math; *Community: Sharing Math:* Introduction and Internet resources; Math and the electronic commons; Creating math teachers at play; Math playground; Supporting girls; How to become invisible; Starting a math club or circle; Conclusion; *Resources:* Sue's book picks; Hints for puzzles; Meet the authors and artists; Acknowledgements; Where is the index?.

Natural Math Series, Volume 1

March 2015, 372 pages, Softcover, ISBN: 978-0-9776939-3-1, **AMS members US$15.20**, List US$19, Order code NMATH/1

Mathematics Calendar

This section contains new announcements of worldwide meetings and conferences of interest to the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. New announcements only are published in the print Mathematics Calendar featured in each *Notices* issue.

An announcement will be published in the *Notices* if it contains a call for papers and specifies the place, date, subject (when applicable), and the speakers. A second announcement will be published only if there are changes or necessary additional information. Asterisks (*) mark those announcements containing revised information.

In general, print announcements of meetings and conferences carry only the date, title of meeting, place of meeting, names of speakers (or sometimes a brief statement on the program), deadlines for abstracts or contributed papers, and source of further information. If there is any application deadline with respect to participation in the meeting, this fact should also be noted.

The complete listing of the Mathematics Calendar is available at: `www.ams.org/meetings/calendar/mathcal`

All submissions to the Mathematics Calendar should be done online via: `www.ams.org/cgi-bin/mathcal/mathcal-submit.pl`

Any questions or difficulties may be directed to `mathcal@ams.org`.

December 2015

21 – February 29, 2016 **New Call Intensive Research Programmes at the CRM**
Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona*
The Centre de Recerca Matemàtica (CRM) invites proposals for Intensive Research Programmes to be organized preferably after August 2017. CRM Research Programmes consist of periods ranging between 1 to 5 months (from September to July) of intensive research in any branch of mathematics and the mathematical sciences.
URL: `www.crm.cat/en/Host/SciEvents/IRP/Pages/CallApplication.aspx`

February 2016

8 – 11 **Workshop on Application on Statistical Tools in Research and Data Analysis**
Location: *Department of Applied Mathematics, Indian School of Mines, Dhanbad-826004*
Workshops on Application of Statistical Tools are often well packed with methods and techniques of data analyses, but with limited understanding of the logic underlying the various techniques, methods and research design as a whole. There exists a lack of logical and conceptual knowledge that would be helpful and support us in the research progression. Keeping this challenge in mind, the present workshop has been designed. The aim of this workshop is to provide the initial start towards the complex web of knowledge surrounding data analysis, when doing research.
URL: `ismdhanbad.ac.in/depart/math/courses/Workshop_2016.pdf`

March 2016

2 – 6 **Workshop on Function Spaces and High-dimensional Approximation**
Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain*
The workshop will promote the modern research connecting Fourier analysis, function spaces, and their links to modern developments in the high-dimensional approximation theory. The purpose of this meeting is to bring together the leading experts, and disseminate the latest progress in research, and in the interaction of these fields. The topics of the workshop include: · Function spaces and Embedding/Duality/Extension theorems · Smoothness of multivariate functions · Fourier transforms inequalities · Weighted inequalities · Hyperbolic cross approximation · Sparse approximation · Constructive methods of approximation
URL: `www.crm.cat/en/Activities/Curs_2015-2016/Pages/Function-Spaces.aspx`

9 – 11 **ANalysis and COntrol on NETworks: Trends and Perspectives**
Location: *Dipartimento di Ingegneria dell'Informazione Universita' di Padova via Gradenigo 6A, 35121 Padova, Italy*
Invited speakers B. Andreianov (Université de Tours) G. Bastin (Université Catholique de Louvain) A. Bressan (Penn State University) F. Camilli (Sapienza Università di Roma) G.-Q. G. Chen (University of Oxford) R. M. Colombo (Università di Brescia) P. Degond (Imperial College, London) M. Garavello (Università di Milano Bicocca) O. Glass (Université Paris-Dauphine) P. Goatin (INRIA Sophia Antipolis—Méditerranée) M. Gugat (F. -A.-Universität Erlangen-Nürnberg) K. Han (Imperial College, London) M. Herty (RWTH Aachen University) S. P. Hoogendoorn (Delft University of Technology) C. Imbert (CNRS, Ecole Normal Supérieure Paris) A. Klar (Technische Universität Kaiserslautern) C. Lattanzio (Università de L'Aquila) P. Marcati (GSSI, Università de L'Aquila) A. Maritan (Università di Padova) K. T. Nguyen (Penn State University) M. D. Rosini (Uniwersytet Marii Curie-Sklodowskiej Lublin) E. Trélat (Université Pierre et Marie Curie) E. Zuazua (Universidad Autònoma de Madrid)
URL: `https://events.math.unipd.it/anconet/node/1`

10 – 13 **Gone Fishing 2016, A workshop on Poisson Geometry**
Location: *University of Colorado at Boulder, USA*
This is the fifth meeting of the annual Gone Fishing event in North America, for the presentation and discussion of new ideas and results in Poisson geometry.
URL: `math.colorado.edu/gonefishing2016/`

11 – 12 **Shanks Workshop on Geometric Analysis**
Location: *Vanderbilt University, Nashville, Tennessee*
The Shanks Workshop on Geometric Analysis will take place March 11-12, 2016, at Vanderbilt University, Nashville, TN. Confirmed speakers: Michael Anderson (Stony Brook University), Boris Botvinnik (University of Oregon), Ken Knox (University of Tennessee at Knoxville), Michael Lock (University of Texas Austin), Jie Qing (University of California Santa Cruz), Bianca Santoro (CUNY-City College of NY). Financial support is available for travel and local expenses, with graduate students and recent PhDs especially encouraged to apply. Preference for financial support will be given to those registered by February 1. More information can be found at: https://my.vanderbilt.edu/shanksgeomanalysis/
URL: `my.vanderbilt.edu/shanksgeomanalysis/`

14 – 18 **IMA Annual Program Workshop: Computational Methods for Control of Infinite-dimensional Systems**
Location: *IMA Annual Program Workshop: Computational Methods for Control of Infinite-dimensional Systems*
There are many challenges and research opportunities associated with developing and deploying computational methodologies for problems of control for systems modeled by partial differential equations and delay equations. The state of these systems lies in an infinite-dimensional space, but finite-dimensional approximations must be used. Fundamental issues in applied and computational mathematics are essential to the development of practical computational algorithms. The focus will be on applications, physics-based modeling, numerical methods, sensor/actuator location and optimal control. Although computation and optimization are the key themes that tie the areas together, topics in infinite-dimensional systems theory will be discussed since these are the foundation for all the topics.
URL: `www.ima.umn.edu/2015-2016/W3.14-18.16/`

18 – 22 **Nirenberg Lectures in Geometric Analysis at the CRM: Gunther Uhlmann (University of Washington)**
Location: *Centre de recherches mathématiques Université de Montréal Pavillon André-Aisenstadt 2920, Chemin de la tour, 5th floor Montréal (Québec) H3T 1J4*
Gunther Uhlmann (University of Washington) will give three lectures: Harry Potter's Cloak via Transformation Optics, Journey to the Center of the Earth, and Seeing Through Space-Time.
URL: `www.crm.umontreal.ca/Nirenberg2016`

April 2016

3 – 9 **Talbot Workshop 2016: Equivariant stable homotopy theory and the Kervaire invariant**
Location: *Herriman, UT*
The 2016 Talbot workshop, aimed toward graduate students and other young researchers, will be about Hill-Hopkins-Ravenel's proof of the Kervaire invariant one problem. It will be mentored by Mike Hill and Douglas Ravenel. The workshop will start with an introduction to equivariant stable homotopy theory, and then will delve into the proof in detail.
URL: `math.mit.edu/conferences/talbot`

4 – 8 **RENORMALIZATION IN DYNAMICS - PISA 2016 Organized by Centro di Ricerca Matemàtica Ennio De Giorgi with additional financial support from FAN project and INdAM-GNAMPA**
Location: *Centro di Ricerca Matemàtica "Ennio De Giorgi", Collegio Puteano, Piazza dei Cavalieri 3, 56126 PISA, Italy*
The aim of this conference is to start collaboration between the "Fractals and Numeration (FAN)" project and the Dynamical systems team of the Scuola Normale Superiore and the University of Pisa. The conference will focus on renormalization schemes, which often enable to understand the dynamics of a parabolic or elliptic system in a given family through the study of an auxiliary hyperbolic system.
URL: `crm.sns.it/event/360/`

20 – 23 **Boundary Value Problems, Functional Equations and Applications 3**
Location: *Faculty of Mathematics and Natural Sciences, University of Rzeszow, Rzeszow, Poland*
The mathematical communities of Boundary value problems and Functional equations have been separately developed for many years and significant results have been obtained. The goal of the "Boundary Value Problems, Functional Equations and Applications" workshops is to develop the theory of these domains in order to find joint points and to apply them to applied problems. It is planned to present existing joint works, to present modern methods in Boundary value problems and Functional equations and to discuss their various applications. The conference is included in the series of conferences associated with the International Society for Analysis, its Applications and Computation.
URL: `www.bfa.ur.edu.pl`

May 2016

1 – 2 **International Conference on Mathematics and Mechanics**
Location: *Venice, Italy, conference hall*
It is the third ICMM conference. ICMM 2014 took place in Vienna, Austria and ICMM 2015 took place in Paris, France. ICMM 2016, is to bring together innovative academics and industrial experts in the field of Mathematics and Mechanics to a common forum. The aim of the conference is to promote research in the field of Mathematics and Mechanics. Another goal is to facilitate exchange of new ideas in these fields and to create a dialogue between scientists and practitioners.
URL: `mathematics.conference-site.com/`

9 – 11 **Advanced Course by Jill Pipher**
Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain*
These advanced courses are devoted to different topics in connection with High-dimensional approximation, Harmonic Analysis, and closed areas, such as PDEs. The courses will focus on the problems which have attracted a lot of attention in the recent years.
URL: `www.crm.cat/en/Activities/Curs_2015-2016/Pages/AdvancedPipher.aspx`

13 – 15 **Midwest Several Complex Variables Conference, 2016**
Location: *University of Toledo, Toledo, OH*
MWSCV 2016 continues the long tradition of Midwest SCV meetings. This time there will be a Mini-Course by Siqi Fu of Rutgers Camden on "Spectral Theory of Complex Laplacians and Applications".
URL: `sites.google.com/a/umich.edu/mw-scv-16/`

16 – 18 **46th Annual John H. Barrett Memorial Lectures: Modeling and Analysis of Nonlinear PDE Models in Spatial Ecology**
Location: *Department of Mathematics, The University of Tennessee, Knoxville, TN*
Plenary Speakers (each gives two 50 minute talks): Chris Cosner, University of Miami, Coral Gables, FL; Mark Lewis, University of Alberta, Edmonton, Alberta, Canada; Yuan Lou, The Ohio University, Columbus, OH. Invited Speakers (each gives one 50 minute talk): Anna R. Ghazaryan, Miami University, Oxford, OH; Judith Miller, Georgetown University, DC; Nancy Rodriguez, University of North Carolina, Chapel Hill, NC; Wenxian Shen, Auburn University, AL; Michael Winkler, University of Paderborn, Germany; Jin Yu, University of Nebraska, NE. Registration is available at the conference website. Slots for contributed talks, and poster session will be considered and arranged for participants who submit the titles and abstracts of the talks. This conference is being sponsored by The University of Tennessee Mathematics Department, National Institute for Mathematical and Biological Synthesis (NIMBioS), College of Arts & Sciences, and the Office of Research & Engagement.
URL: `www.math.utk.edu/barrett/`

16 – 19 **Experimental Chaos and Complexity Conference 2016 Travel grants available for new investigators, underserved populations**
Location: *Banff, Alberta, Canada*
The 14th Experimental Chaos and Complexity Conference (ECC 2016) brings together an international interdisciplinary group involving physicists, engineers, mathematicians, chemists, biologists, and neuroscientists focused on various aspects of Experimental Chaos and Complexity. This meeting will focus on experimental approaches in physics, engineering, neuroscience, chemistry, and biology, linked together by modern non-linear dynamics. Travel grants supported by the US NSF will provide partial to full support including airfare (US carriers as specified in NSF rules), registration, housing, depending upon the number of applications. For more information, or to submit applications, please email Harold M. Hastings, Bard College at Simon's Rock, `hhastings@simons-rock.edu`.
URL: `wcm.ucalgary.ca/ecc2016/`

24 – 27 **Algebraic Groups, Quantum Groups and Geometry**
Location: *University of Virginia, Charlottesville, VA, USA*
The workshop "Algebraic Groups, Quantum Groups and Geometry" in the Southeastern Lie Theory series will be held at the University of Virginia, May 24-27, 2016. The conference is supported by the National Science Foundation and the Institute of Mathematics Science (UVA). May 26th will be a special day in recognition of the work of Brian Parshall.
URL: `people.virginia.edu/~btw4e/AQG/index.html`

26 – 29 **International Conference on Applications of Mathematics to Nonlinear Sciences (AMNS-2016)**
Location: *Nepal Academy of Tourism and Hotel Management, Kathmandu, Nepal*
The Association of Nepalese Mathematicians in America (ANMA), Nepal Mathematical Society (NMS) and mathematics departments of Tribhuvan University and Kathmandu University are jointly organizing the International Conference on Applications of Mathematics to Nonlinear Sciences (AMNS-2016) in Kathmandu, Nepal, on May 26-29, 2016. The conference provides a forum to a diverse group of scientists in applications of mathematics to natural and health sciences, engineering and finance. Specific areas include analysis, topology, mathematics education, statistics, big data, optimization, operations research, quantitative finance, mathematical biology, biomedical science, biophysics, and public health. The conference intends to bring together researchers from a variety of disciplines which impact nonlinear analysis and applications in bio- and physical sciences from the south-east Asian countries and around the globe.
URL: `anmaweb.org/AMNS-2016/`

27 – 29 **92. Arbeitstagung Allgemeine Algebra**
Location: *Prague, Czech Republic*
The AAA series of workshops in the field of general algebra has taken place twice a year for more than 40 years, usually at universities in Central Europe. Traditional topics include universal algebra, lattice theory, ordered algebraic structures, and applications in logic and elsewhere.
URL: `aaa.karlin.mff.cuni.cz/`

29 – 31 **Canadian Society for History and Philosophy of Mathematics Annual Meeting**
Location: *University of Calgary*
The CSHPM will be holding its 2016 Annual Meeting at the University of Calgary in conjunction with the 2016 Congress of the Humanities and Social Sciences. Members are invited to present papers on any subject relating to the history of mathematics, its use in the teaching of mathematics, the philosophy of mathematics, or a related topic. Talks in either English or French are welcome. Graduate students who present are eligible for the CSHPM Student Award. The Call for Papers with submission information is available at: `www.cshpm.org/meeting/2016CSHPMSCHPM_Call_for_Papers.pdf`
URL: `www.cshpm.org/meeting/`

30 – June 3 **US-Mexico Conference in Representation Theory, Categorification, and Noncommutative Algebra**
Location: *University of Southern California, Los Angeles, CA*
This is the second meeting of an annual series of conference alternating between the US and Mexico. The program covers a broad spectrum of topics in Representation Theory and Noncommutative Algebra. Organizers: Andrea Appel (UCS), Christof Geiss (UNAM), Aaron Lauda (USC), and Milen Yakimov (LSU). Speakers: Vyjayanthi Chari, Jose-Antonio De la Pena, Eric Friedlander, Sachin Gautam, Gustavo Jasso, Vladislav Kharchenko, Alexander Kleshchev, Daniel Labardini-Fragoso, Bernard Leclerc, Octavio Mendoza, Susan Montgomery, Jose Pablo Pelaez, Nicolai Reshetikhin, Raphael Rouquier, Rita Jimenez Rolland, Vera Serganova, Joshua Sussan, Valerio Toledano Laredo, Edith Corina Saenz Valadez, Ernesto Vallejo, Monica Vazirani, Weiqiang Wang, Chelsea Walton, Lauren Williams, Birge Huisgen-Zimmermann.
URL: `www-bcf.usc.edu/~andreaap/conference2016.html`

30 – June 4 **Advanced Course on Constructive Approximation and Harmonic Analysis**
Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain*

These advanced courses are devoted to different topics in connection with High-dimensional approximation, Harmonic Analysis, and closed areas, such as PDE's. The courses will focus on the problems which have attracted a lot of attention in the recent years.

URL: www.crm.cat/en/Activities/Curs_2015-2016/Pages/Constructive-Approximation.aspx

30 – June 11 **Seminaire de Mathematiques Superieures 2016: Dynamics of Biological Systems**

Location: *UNIVERSITY OF ALBERTA FOR PIMS*

The purpose of this summer school is to focus on the interplay of dynamical and biological systems, developing the rich connection between science and mathematics that has been so successful to date. Our focus will be on understanding the mathematical structure of dynamical systems that come from biological problems, and then relating the mathematical structures back to the biology to provide scientific insight.

URL: www.msri.org/summer_schools/775

June 2016

2 – 8 **Algebraic Cycles and Moduli**

Location: *Centre de recherches mathématiques Université de Montréal Pavillon André-Aisenstadt 2920, Chemin de la tour, 5th floor Montréal (Québec) H3T 1J4 CANADA*

This workshop will focus on recent advances in moduli theory and algebraic cycles, with a particular emphasis on Hodge-theoretic aspects and new connections between these two subjects. One such point of contact is the representation theory of the Mumford-Tate group, which gives a powerful toolbox for classifying variations and degenerations of Hodge structure, and its interplay with recent activity on compactifications of moduli stemming from geometric invariant theory, the minimal model program, and mirror symmetry. The normal functions arising from cycles and the interesting loci in moduli supporting them, as well as the Hodge Conjecture, provide further sources of interactions between the two subjects.

URL: www.crm.umontreal.ca/2016/Cycles16/venueCRM_e.php

5 – 25 **Mathematics Research Communities (Research conferences for early-career mathematicians.)**

Location: *Snowbird, Utah*

A program of the AMS funded principally by the National Science Foundation, Mathematics Research Communities foster the formation of self-sustaining cohorts of early-career mathematical scientists by supporting summer conferences, collaboration grants, and special sessions at the Joint Mathematics Meetings focused around research topics of common interest. In 2016 the summer conferences are: June 5–11, (a) Lie Group Representations, Discretization, and Gelfand Pairs and (2) Character Varieties-Experiments and New Frontiers; June 12–18, Algebraic Statistics; and June 19–25, Mathematics in Physiology and Medicine. The 2016 program is aimed at mathematicians whose career stage is between two years pre-PhD and three years post-PhD, with interest in one of these areas, and whose background makes them ready to engage in hands-on collaborative research under the mentorship of experts in the respective fields. Applications are open through March 1, 2016, on the program web site.

URL: www.ams.org/programs/mrc-16

6 – 10 **Topology Students Workshop**

Location: *Georgia Institute of Technology, Atlanta, GA*

A combination professional development workshop and topology conference for graduate students. Sessions on writing/giving lectures, mathematical etiquette, career options, writing grant proposals, etc.

URL: tsw.gatech.edu

6 – 10 **Conference on Harmonic Analysis and Approximation Theory (HAAT 2016)**

Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain*

The main goal of the conference HAAT2016, besides presenting the recent developments in Constructive Approximation and Harmonic Analysis, is promoting their integration and research exchange. In particular, the conference promotes the idea of applying the research tools from one research area for problems in the other area. As a consequence, such an integration will possibly result in solving many applied problems in other areas of science.

URL: www.crm.cat/en/Activities/Curs_2015-2016/Pages/HAAT2016.aspx

7 – 8 **It is the second ICMCS conference. First one took place on November 2015. ICMCS 2016, is to bring together innovative academics and industrial experts in the field of Mathematics and Computer Science**

Location: *Vienna, Austria, conference hall*

It is the second ICMCS conference. First one took place on November 2015. ICMCS 2016, is to bring together innovative academics and industrial experts in the field of Mathematics and Computer Science to a common forum. The aim of the conference is to promote research in the field of Mathematics and Computer Science. Another goal is to facilitate exchange of new ideas in these fields and to create a dialogue between scientists and practitioners. Keynote speaker - Prof. Enoch Opeyemi, Federal University Oye-Ekiti, Nigeria Speech: Proof of Riemann Hypothesis

URL: computer.conference-site.com/register.html

13 – 17 **Recent Advances in Commutative Rings and Module Theory**

Location: *Bressanone/Brixen, Alto Adige/South Tirol, Italy*

The goal of this Conference is to present recent progress and new trends in the area of commutative ring theory and module theory. It also aims to gather together mathematicians sharing similar algebraic interests for a fruitful interaction. The main subjects include, but not limited to: Spaces of valuation domains and applications; Multiplicative Ideal Theory, lattices, star and semistar operations; Factorization and divisibility properties in the ideal theory of commutative rings; Rings of integer-valued polynomials; Topological and algebraic entropy over commutative rings; Almost t-injective and fully inert modules over integral domains. The Conference will be held in Bressanone/Brixen, a very pleasant small town, about 40 kilometers North of Bolzano/Bozen, at "Casa della Gioventà" (University of Padova at Bressanone).

Organizers: Francesca Tartarone (Università degli Studi, Roma Tre) and Paolo Zanardo (Università di Padova)
URL: www.rings-modules-bressanone2016.blogspot.it

16 – 18 **International scientific conference Actual Problems in Theory of Partial Differential Equations, dedicated to the centenary of Andrey V. Bitsadze**
Location: *Moscow State University by Lomonosov, Russia, Moscow*
Faculty of Computational Mathematics and Cybernetics of the M.V. Lomonosov Moscow State University and the V.A. Steklov Mathematical Institute of Russian Academy of Sciences hold an international scientific conference 'Actual problems of the theory of partial differential equations', dedicated to the centenary of A.V. Bitsadze.
URL: bitsadze2016.cs.msu.ru/en

20 – 25 **The Fifth International School-Seminar "NONLINEAR ANALYSIS AND EXTREMAL PROBLEMS"—2016**
Location: *Institute for System Dynamics and Control Theory SB RAS, Irkutsk, Russia*
The aim of the school-seminar is to introduce young researchers to some topics of current research in the fields of: • nonlinear analysis and its applications • dynamical systems • evolution equations and partial differential equations • calculus of variations and optimal control. The main part of the school-seminar will consist of series of lectures by leading scientists (P. Krejci (Czech Republic), A. Cellina (Italy), A. Dontchev (USA), G. Akagi (Japan)) in the above fields, and the rest of the time will be devoted to short talks of other participants. The working languages of the school-seminar are Russian and English. The city of Irkutsk, the venue of the conference, is one of the oldest cities in Siberia which has many interesting tourist attractions. It is located in an immediate vicinity of the famous lake Baikal. A one day trip to the lake will be also organized.
URL: www.idstu.irk.ru/en/nla-2016

20 – July 1 **Mixed Integer Nonlinear Programming: Theory, algorithms and applications**
Location: *IMUS, The Institute of Mathematics of The University of Seville*
This school is oriented to the presentation of theory, algorithms and applications for the solution of mixed integer nonlinear problems (MINLP).
URL: www.msri.org/summer_schools/773

22 – 27 **8th Conference of the Euro-American Consortium for Promoting the Application of Mathematics in Technical and Natural Sciences (AMiTaNS'16)**
Location: *Black-Sea resort of Albena, Bulgaria*
The conference will be scheduled in plenary and keynote lectures followed by special and contributed sessions. The accents of the conference will be on Mathematical Physics, Solitons and Transport Processes, Numerical Methods, Scientific Computing, Continuum Mechanics, Applied Analysis, Applied Physics, Biomathematics, which can be complemented by some specific topics in contributed special sessions.
URL: 2016.eac4amitans.eu

27 – July 1 **The 14th International Conference on Permutation Patterns (PP 2016)**
Location: *Howard University, Washington, DC*
The scope of this meeting includes all topics related to patterns in permutations, words, and other combinatorial objects. The conference will include invited and contributed talks, as well as poster sessions. There will be no parallel sessions. The invited speakers will be Anders Claesson (University of Strathclyde; starting 2016, University of Iceland) and Ira Gessel (Brandeis University).
URL: permutationpatterns2016.wordpress.com

27 – July 2 **Summer School on Algebra, Statistics and Combinatorics**
Location: *Aalto University, Helsinki, Finland*
Our goal is to provide an opportunity for graduate students and postdocs to learn current developments on the interaction of Algebra, Statistics and Combinatorics. Special lectures will be given by Petter Brändén, Alexander Engström, Pauliina Ilmonen, Steffen Lauritzen and Bernd Sturmfels. Mini courses are held by Kaie Kubjas, Patrik Norén, Louis Theran and Piotr Zwiernik. Application deadlines are on February 29 (funding) and on May 31 (no funding).
URL: asci.aalto.fi/en/project_funding/big_data/summer_school/

July 2016

4 – 7 **Analysis and Probability. Conference in honour of Jean-Pierre Kahane**
Location: *Orsay and Paris, France*
This conference in analysis and probability is organized in honour of Jean-Pierre Kahane who will celebrate his 90th birthday in 2016.
URL: www.mathconf.org/kahane90

4 – 8 **Complex Boundary and Interface Problems: Theoretical models, Applications and Mathematical Challenges**
Location: *Centre de recherches mathématiques Université de Montréal Pavillon André-Aisenstadt 2920, Chemin de la tour, 5th floor Montréal (Québec) H3T 1J4*
Organizers: Jean-Christophe Nave (McGill), Robert Owens (Montréal), Pascal Poullet (Antilles), Hongkai Zhao (UC Irvine)
URL: www.crm.umontreal.ca/Computational2016

4 – 9 **Conference on Ulam's Type Stability**
Location: *Cluj-Napoca, Romania*
The conference is organized by Department of Mathematics of Technical University of Cluj-Napoca in cooperation with Department of Mathematics of Babeş-Bolyai University, Department of Mathematics of Pedagogical University of Cracow, and Faculty of Applied Mathematics of AGH University of Science and Technology. The participants are invited to give talks on stability of difference, differential, functional and integral equations; stability of inequalities and other mathematical objects; hyperstability and superstability; various (direct, fixed point, invariant mean, etc.) methods for proving Ulam's type stability results; generalized (in the sense of Aoki and Rassias, Bourgin and Găvruţa) stability; stability on restricted domains and in various (metric, Banach, non-Archimedean, fuzzy, quasi-Banach, etc.) spaces,

relations between Ulam's type stability and fixed point results, and related topics. Moreover, some special invited plenary lectures are planned.
URL: `cuts.up.krakow.pl`

4 – 14 **2016 CRM Summer School: Spectral Theory and Applications**
Location: *Université Laval Quebec City.*
The goal of the 2016 CRM Summer School in Quebec City is to prepare students for research involving spectral theory. The school will give an overview of a selection of topics from spectral theory, interpreted in a broad sense. It will cover topics from pure and applied mathematics, each of which will be presented in a 5-hour mini-course by a leading expert. These lectures will be complemented by supervised computer labs and exercise sessions. At the end of the school, invited speakers will give specialized talks. This rich subject intertwines several sub-disciplines of mathematics, and it will be especially beneficial to students. The subject is also very timely, as spectral theory is witnessing major progresses both in its mathematical sub-disciplines and in its applications to technology and science in general.
URL: `www.crm.umontreal.ca/2016/Quebec16/index_e.php`

5 – 8 **International Workshop on Operator Theory and Operator Algebras - WOAT 2016**
Location: *Instituto Superior Técnico, University of Lisbon, Portugal*
The International Workshop on Operator Theory and Operator Algebras, WOAT 2016, will be held at Instituto Superior Técnico, University of Lisbon, Portugal, from July 5 to July 8, 2016. WOAT 2016 will consist of talks presented by invited speakers and by participants that wish to make a contribution. This workshop continues a series of conferences organized in Lisbon since 2006 that aim to stimulate communication between researchers in Operator Theory and Operator Algebras. WOAT 2016 will also include two special sessions on Matrix Theory and Applications. A small number of grants is available to students and young researchers. Deadline for abstract submission: May 13, 2016.
URL: `https://woat2016.math.tecnico.ulisboa.pt/`

5 – 9 **Conference on Differential Geometry**
Location: *UQAM (downtown Montréal) Pavillon Sherbrooke 200, Sherbrooke St West Room: SH-3420*
Honouring Claude LeBrun's mathematical contributions, this conference aims to foster interaction among various topics of Differential Geometry, Geometric Analysis, and Mathematical Physics, centered around hot areas of current research. Specifically, the conference will explore on the following subjects: Special structures in geometry and physics; Complex methods in conformal geometry and twistor theory; Extremal Kähler metrics. The primary aim of the conference is to gather together leading experts in the above topics, to discuss recent advances and new directions of research in these vibrant areas, and to expose graduate students and young mathematicians to these exciting developments.
URL:
`www.crm.umontreal.ca/2016/LeBrunFest16/index_e.php`

11 – 17 **Foliations 2016**
Location: *The Conference Centre of the Polish Academy of Sciences, Bedlewo, Poland*
We are pleased to announce that a conference FOLIATIONS 2016 will held in Bedlewo (Poland) during the period July 11–17, 2016. Our aim is to exchange scientific information about most recent results on foliations and related topics. Theory of foliations can be considered as geometric theory of integrable systems of partial differential systems (on manifolds). It has several aspects: topological, geometrical, dynamical etc. All the aspects are supposed to be covered by the lecturers. Some of the problems are related to those for nonintegrable systems (contact and symplectic structures, confoliations, Engel and Goursat structures and so on), therefore some talks will deal with non-integrable systems. Also, several properties of foliations are reflected by their holonomy groups and pseudogroups, some of the talks will be devoted to group (and pseudogroup) actions on manifolds.
URL: `foliations2016.math.uni.lodz.pl/`

11 – 22 **An Introduction to Character Theory and the McKay Conjecture**
Location: *Mathematical Sciences Research Institute, Berkeley CA*
Character Theory of Finite Groups provides one of the most powerful tools to study groups. In this course we will give a gentle introduction to basic results in the Character Theory, as well as some of the main conjectures in Group Representation Theory, with particular emphasis on the McKay Conjecture.
URL: `www.msri.org/summer_schools/767`

25 – 29 **International Conference "Patterns of Dynamics"**
Location: *Free University of Berlin, Berlin, Germany*
The conference will showcase recent advances of dynamical systems theory and their interplay with a wide range of applications in the sciences and engineering. The underlying mathematical theories can help extract structures from experimental observations (real-world dynamics), and conversely, shed light on the formation, dynamics, and control of spatio-temporal patterns in applications. The meeting will bring together speakers who engage with applications, build and develop mathematical techniques, and use mathematical approaches for prediction and control. The conference will also honor the long-standing contributions of Bernold Fiedler to these topics. The scientific program will consist of 45-minute keynote and 30-minute invited lectures, as well as 20-minute contributed presentations.
URL:
`conference.mi.fu-berlin.de/patterns-of-dynamics/`

31 – August 6 **Computational Methods in Applied Mathematics (CMAM-7), 100 years of the Galerkin Method**
Location: *University of Jyväskylä, Finland*
Celebrating 100 years of Petrov-Galerkin method, the CMAM-7 conference will be held at the University of Jyväskylä, Finland, in July 31–August 6, 2016. The Conference is organized under the aegis of the de Gruyter journal Computational Methods in Applied Mathematics (CMAM) and will be focused on various aspects of mathematical modeling and numerical methods for problems arising in natural sciences and engineering. The CMAM-7 conference is organized by the University of

Jyväskylä, Finland, and the Steklov Institute of Mathematics, Russian Academy of Sciences, St. Petersburg, Russia. The list of confirmed invited speakers includes Carsten Carstensen, Roland Glowinski, Boris Khoromskij, Yuri Kuznetsov, Raytcho Lazarov, Markus Melenk, Roderick Melnik, Stefan Sauter, Rolf Stenberg, Ragnar Winther, Jun Zou. Details can be found at `www.jyu.fi/cmam7` and registration starts November 1, 2015; a call for minisymposia will follow.
URL: `www.mit.jyu.fi/scoma/cmam2016/`

August 2016

1 – 4 **22nd Conference on Applications of Computer Algebra**
Location: *Kassel University, Kassel, Germany*
The ACA conference series is devoted to promoting all manner of computer algebra applications and encouraging the interaction of developers of computer algebra systems and packages with researchers and users (including scientists, engineers, educators, and mathematicians). Topics include, but are not limited to, computer algebra in the sciences, engineering, communication, medicine, pure and applied mathematics, education and computer science. The meeting will be run in the standard ACA format where individuals are invited to organize special sessions.
URL: `www.mathematik.uni-kassel.de/ACA2016/`

2 – 4 **International Conference and Workshop on Mathematical Analysis 2016 (ICWOMA 2016)**
Location: *Langkawi, Malaysia*
The main goal of this conference is to bring together experts and young talented scientists from all over the world to discuss the modern and recent aspects of the mathematical analysis. It is also to ensure exchange of ideas in various applications of Mathematics in Engineering, Physics, Economics, Biology, etc.
URL: `einspem.upm.edu.my/icwoma2016/`

22 – 26 **Master class on "Sums of self-adjoint operators—Kasparov products and applications"**
Location: *University of Copenhagen, Denmark*
This master class will introduce the necessary technical tools for constructing Kasparov products in the unbounded picture of KK as well as discussing the many open problems related to sums of self-adjoint operators, the standard model in physics, topological insulators and operator spaces.
URL: `www.math.ku.dk/english/research/conferences/2016/ncg_sums/`

22 – 26 **Putting the Theory Back in Density Functional Theory: A Summer School**
Location: *Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, CA*
The purpose of this school is to teach the theory behind DFT. Lectures will be pedagogical and range from fundamentals (Hohenberg-Kohn theorem) to the latest approximations, and will help connect DFT to other areas of mathematics and theory.
URL: `www.ipam.ucla.edu/programs/summer-schools/putting-the-theory-back-in-density-functional-theory/`

26 – 28 **Workshop on Nonlinear PDEs in Applied Mathematics**
Location: *Izmir Institute of Technology, Izmir, Turkey*
This program's goal is to bring together leading mathematicians, researchers and students to exchange and share their scientific experiences and research results about different aspects of Nonlinear PDEs. The workshop also aims to provide a fruitful environment for researchers to present and discuss the most recent discoveries, methods, problems and challenges in the field of Nonlinear PDEs.
URL: `pde.iyte.edu.tr`

28 – September 4 **XIX Geometrical Seminar**
Location: *Student's Resting-House "Ratko Mitrović" Kraljeve vode bb, 31315 Zlatibor, Serbia*
XIX Geometrical seminar is organized by Faculty of Science, University of Kragujevac, Kragujevac, Serbia Faculty of Mathematics, University of Belgrade, Belgrade, Serbia in collaboration with: Faculty of Science and Mathematics, University of Niš, Niš, Serbia Mathematical Institute of the Serbian Academy of Sciences and Arts (SANU), Belgrade, Serbia XIX Geometrical seminar is supported by Ministry of Education, Science and Technological Development of the Republic Serbia Conference topics Differential Geometry Topology Lie Groups Mathematical Physics Discrete Geometry Integrable Systems Visualization
URL: `tesla.pmf.ni.ac.rs/people/geometrijskiseminarxix/index.php`

September 2016

7 – 9 **5th IMA Numerical Linear Algebra and Optimization**
Location: *University of Birmingham, UK*
The success of modern codes for large-scale optimization is heavily dependent on the use of effective tools of numerical linear algebra. On the other hand, many problems in numerical linear algebra lead to linear, nonlinear or semidefinite optimization problems. The purpose of the conference is to bring together researchers from both communities and to find and communicate points and topics of common interest. Mini-symposium proposals and contributed talks are invited on all aspects of numerical linear algebra and optimization. Mini-symposium proposals should be submitted to conferences@ima.org.uk by 31 March 2016. A mini-symposium is limited to at most two sessions on a single topic (maximum eight speakers). Organizers will be advised of acceptance by 11 April 2016. Contributed talks and mini-symposia talks will be accepted on the basis of a one page extended abstract which should be submitted by 30 April 2016 by email to `conferences@ima.org.uk`.
URL: `tinyurl.com/IMANLAO2016`

19 – 23 **18th International Workshop on Computer Algebra in Scientific Computing**
Location: *University of Bucharest, Bucharest, Romania*
The ongoing development of computer algebra systems, including their integration and adaptation to modern software environments, puts them to the forefront in scientific computing and enables the practical solution of many complex applied problems in the domains of natural sciences and

engineering. The workshop covers all basic areas of scientific computing (see web page for a more detailed list).
URL: `www.casc.cs.uni-bonn.de/2016/`

October 2016

14 – 16 **International Conference on Statistical Distributions and Applications (ICOSDA 2016)**
Location: *Crowne Plaza, Niagara Falls, Canada*
This international conference is being organized to provide a platform for researchers and practitioners to share and discuss recent advancements on statistical distributions and their applications, and to provide opportunities for collaborative work.
URL: `people.cst.cmich.edu/lee1c/icosda2016/`

November 2016

14 – 18 **Mal'tsev Meeting**
Location: *Sobolev Institute of Mathematics, Novosibirsk, Russia*
"Mal'tsev Meeting" is a series of annual conferences on algebra, mathematical logic, and their applications in computer science. In 2016, the meeting will be dedicated to the 70th birthdays of Professors Nikolai S. Romanovskii and Valerii A. Churkin. The organisers invite specialists in group theory, ring theory, universal algebra and model theory, computability theory, nonclassical logics, and theoretical computer science to contribute and to participate.
URL: `math.nsc.ru/conference/malmeet/16/`

December 2016

5 – 9 **Synergies between Machine Learning and Physical Models**
Location: *Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, CA*
This workshop will broadly address the reaches and limitations of ML as applied to the modeling of physical systems and highlight examples where physical models can be successfully combined or even derived from ML algorithms. The application deadline is Monday, October 10, 2016.
URL: `www.ipam.ucla.edu/mpsws4`

14 – 17 **International Conference of The Indian Mathematics Consortium (TIMC) in cooperation with American Mathematical Society (AMS)**
Location: *DST-Centre for Interdisciplinary Mathematical Sciences, Banaras Hindu University, Varanasi-221005, India*
This is the first major event being organized by TIMC in co-operation with an organization from outside India. Eight plenary talks by distinguished mathematicians and about 15 Symposia on themes of current interest in mathematical research are being planned for the meeting. These will be supplemented by various other activities.
URL: `www.bhu.ac.in/seminar/timcams2016/`

September 2017

19 – 22 **11th International Conference on Parametric Optimization and Related Topics (ParaoptXI)**
Location: *Charles University, Prague, Czech Republic*
Parametric optimization is a part of mathematical programming and has emerged as an exciting research area in theory, numerics and applications. ParaoptXI welcomes papers as well as proposals for special sessions on any area in parametric optimization or related topics.
URL: `paraoptxi.fsv.cuni.cz/`

MEETINGS & CONFERENCES OF THE AMS

MARCH TABLE OF CONTENTS

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at: **www.ams.org/meetings/**.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to page 88 in the January 2016 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LaTeX. Visit **www.ams.org/cgi-bin/abstracts/abstract.pl**. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

MEETINGS IN THIS ISSUE

Conferences in Cooperation with the AMS

Indian Mathematics Consortium, December 14–17, 2016, Danaras Aindu University, Varanasi, India

See **www.ams.org/meetings/** for the most up-to-date information on these conferences.

ASSOCIATE SECRETARIES OF THE AMS

Central Section: Georgia Benkart, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; e-mail: steve.weintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403, e-mail: brian@math.uga.edu; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: lapidus@math.ucr.edu; telephone: 951-827-5910.

Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See www.ams.org/meetings/.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL .

Athens, Georgia

University of Georgia

March 5–6, 2016

Saturday – Sunday

Meeting #1117

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: January 2016
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 37, Issue 2

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Michele Benzi, Department of Mathematics and Computer Science, Emory University, *Numerical Analysis of Quantum Graphs*.

Erik Demaine, MIT Computer Science and Artificial Intelligence Laboratory, *Fun with Fonts: Mathematical Typography* (Einstein Public Lecture in Mathematics).

Frank G. Garvan, University of Florida, *Dyson's Conjectures and Predictions in the Work of Ramanujan*.

William Graham, University of Georgia, *A generalization of the Springer resolution*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Active Learning in Undergraduate Mathematics, **Darryl Chamberlain, Jr**, **Aubrey Kemp**, **Leslie Meadows**, **Harrison Stalvey**, and **Draga Vidakovic**, Georgia State University.

Algebraic Structures in Knot Theory, **Sam Nelson**, Claremont McKenna College, and **Mohamed Elhamdadi**, University of South Florida.

Algebraic Structures in Mathematical Physics: Lie Algebras, Vertex Algebras, Quantum Algebras, **Iana I. Anguelova**, College of Charleston, and **Bojko Bakalov**, North Carolina State University.

Algebraic and Combinatorial Methods in Mathematical Biology, **Elena Dimitrova** and **Svetlana Poznanovic**, Clemson University.

Bioinformatics and Molecular Biology: Dynamic Models, Structural Analysis, and Computational Methods, **Christine Heitsch**, **Chi-Jen Wang**, and **Haomin Zhou**, Georgia Institute of Technology.

Combinatorial and Computational Algebra, **Huy Tai Ha**, Tulane University, **Kuei-Nan Lin**, Penn State Greater Allegheny, and **Augustine O'Keefe**, Connecticut College.

Commutative Algebra, **Jon F. Carlson**, University of Georgia, and **Andrew Kustin**, University of South Carolina.

Discrete and Applied Algebraic Geometry, **Cynthia Vinzant**, North Carolina State University, and **Josephine Yu**, Georgia Institute of Technology.

Elliptic Curves, **Abbey Bourdon** and **Pete L. Clark**, University of Georgia.

Experimental Mathematics, **Frank Garvan**, University of Florida, and **Andrew Sills**, Georgia Southern University.

Financial Mathematics, **Arash Fahim** and **Alec Kercheval**, Florida State University.

Harmonic Analysis and Applications, **Irina Holmes**, Georgia Institute of Technology, and **Brett D. Wick**, Washington University.

Interactions Between Algebraic and Tropical Geometry, **Matthew Ballard**, University of South Carolina, **Noah Giansiracusa**, University of Georgia, and **Jesse Kass**, University of South Carolina.

Invariant Measures of Dynamical Systems, **Miaohua Jiang** and **Chris Johnson**, Wake Forest University, and **Martin Schmoll**, Clemson University.

Lie Theory, Representation Theory, and Geometry, **Shrawan Kumar**, University of North Carolina, and **Daniel K. Nakano** and **Paul Sobaje**, University of Georgia.

Low-dimensional Topology and Geometry, **David Gay** and **Gordana Matic**, University of Georgia.

Mathematical Physics and Spectral Theory, **Stephen Clark**, Missouri University of Science and Technology, and **Roger Nichols**, The University of Tennessee at Chattanooga.

Mathematics and Music, **Mariana Montiel**, Georgia State University, and **Robert Peck**, Louisiana State University.

Moduli Spaces and Vector Bundles, **Patricio Gallardo** and **Anna Kazanova**, University of Georgia.

New Developments in Discrete and Intuitive Geometry (Dedicated to the 75th birthday of Wlodzimierz Kuperberg), **Andras Bezdek**, Auburn University, **Oleg Musin**, University of Texas at Brownsville, and **Gabor Fejes Toth**, Renyi Institute of Mathematics, Hungary (AMS-AAAS).

Numerical Methods and Scientific Computing, **Michele Benzi**, Emory University, and **Edmond Chow**, Georgia Institute of Technology.

PDE Analysis in Fluid Flows, **Geng Chen**, **Ronghua Pan**, and **Yao Yao**, Georgia Institute of Technology.

Probabilistic and Analytic Tools in Convexity, **Joseph Fu**, University of Georgia, **Galyna Livshyts**, Georgia Institute of Technology, and **Elisabeth Werner**, Case Western Reserve University.

Sharp Estimates and Bellman Functions in Harmonic Analysis, **Kabe Moen**, University of Alabama, **Leonid Slavin**, University of Cincinnati, and **Alex Stokolos**, Georgia Southern University.

Symplectic and Contact Geometry, **Yi Lin** and **Stefan Müller**, Georgia Southern University, **Michael Usher**, University of Georgia, and **François Ziegler**, Georgia Southern University.

The Combinatorics of Symmetric Functions, **Sarah K. Mason**, Wake Forest University, and **Elizabeth Niese**, Marshall University.

Theory and Applications of Graphs, **Colton Magnant** and **Hua Wang**, Georgia Southern University.

Topics in Graph Theory, **Guantao Chen**, Georgia State University, and **Songling Shan**, Vanderbilt University.

Topology and Dynamical Systems, **Alexander Blokh**, University of Alabama at Birmingham, **Krystyna Kuperberg**, Auburn University, and **John Mayer** and **Lex Oversteegen**, University of Alabama at Birmingham.

Stony Brook, New York

State University of New York at Stony Brook

March 19–20, 2016

Saturday – Sunday

Meeting #1118

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: January 2016
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 37, Issue 2

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Simon Donaldson, Stony Brook University, *Survey of progress and problems on manifolds with G_2 holonomy.*

Dmitry Kleinbock, Brandeis University, *Homogeneous Dynamics and Intrinsic Approximation.*

Irena Lasiecka, University of Memphis, *Mathematical theory of PDE-dynamics arising in fluid/flow-structure interactions.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Analysis, Probability and Mathematical Physics on Fractals, **Joe P. Chen** and **Luke Rogers**, University of Connecticut, **Robert Strichartz**, Cornell University, and **Alexander Teplyaev**, University of Connecticut.

Commutative Ring Theory, **Alan Loper**, Ohio State University, and **Nick Werner**, State University of New York at Old Westbury.

Complex Geometric Analysis, **Xiuxiong Chen**, Stony Brook University, **Weiyong He**, University of Oregon, and **Ioana Suvaina**, Vanderbilt University.

Evolution of Partial Differential Equations and their Control, **George Avalos**, University of Nebraska, and **Irena Lasiecka** and **Roberto Triggiani**, University of Memphis.

G_2 Geometry, **Sergey Grigorian**, University of Texas, Rio Grande Valley, **Sema Salur**, University of Rochester, and **Albert J. Todd**, University of South Alabama.

Geometric Measure Theory and Its Applications, **Matthew Badger**, University of Connecticut, and **Christopher J. Bishop** and **Raanan Schul**, Stony Brook University.

Graph Vulnerability Parameters and their Role in Network Analysis, **Michael Yatauro**, Pennsylvania State University-Brandywine.

Holomorphic Dynamics, **Artem Dudko** and **Raluca Tanase**, Stony Brook University.

Homogeneous Dynamics and Related Topics, **Dmitry Kleinbock**, Brandeis University, and **Han Li**, Wesleyan University.

Invariants of Closed Curves on Surfaces, **Ara Basmajian**, Hunter College and Graduate Center, City University of New York, and **Moira Chas**, Stony Brook University.

Mathematical General Relativity, **Lan-Hsuan Huang**, University of Connecticut, **Marcus Khuri**, Stony Brook University, and **Christina Sormani**, Lehman College and City University of New York Graduate Center.

Mathematicians in Mathematics Education, **Lisa Berger**, Stony Brook University, and **Melkana Brakalova**, Fordham University.

PDE Methods in Geometric Flows, **Mihai Bailesteanu**, Central Connecticut State University, and **Andrew Cooper**, North Carolina State University.

Teichmüller Theory and Related Topics, **Sudeb Mitra** and **Dragomir Saric**, Queens College of the City University of New York and City University of New York Graduate Center.

Topology and Combinatorics of Arrangements (in honor of Mike Falk), **Daniel C. Cohen**, Louisiana State University, and **Alexander I. Suciu**, Northeastern University.

Vertex Algebra and Related Algebraic and Geometric Structures, **Katrina Barron**, University of Notre Dame, **Antun Milas**, State University of New York at Albany, and **Jinwei Yang**, University of Notre Dame.

Salt Lake City, Utah

University of Utah

April 9–10, 2016

Saturday – Sunday

Meeting #1119

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: January 2016
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 37, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 16, 2016

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Daniel Bump, Stanford University, *From Whittaker Functions to Quantum Groups.*

James McKernan, University of California, San Diego, *Classification of algebraic varieties.*

Ravi Vakil, Stanford University, *Cutting and pasting in algebraic geometry* (Erdős Memorial Lecture).

Stephanie van Willigenburg, University of British Columbia, *An introduction to quasisymmetric Schur functions.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Combinatorics (Code: SS 5A), **Susanna Fishel**, Arizona State University, **Edward Richmond**, Oklahoma State University, and **Stephanie van Willigenburg**, University of British Columbia.

Algebraic Geometry (association with the Erdős Lecture by Ravi Vakil) (Code: SS 1A), **Ravi Vakil**, Stanford University, and **Christopher Hacon** and **Karl Schwede**, University of Utah.

Automorphic Forms, Combinatorics and Representation Theory (Code: SS 6A), **Anna Puskás**, University of Alberta, **Daniel Bump**, Stanford University, **Paul Gunnells**, University of Massachusetts Amherst, and **Solomon Friedberg**, Boston College.

CR Geometry and Partial Differential Equations in Complex Analysis (Code: SS 4A), **Yuan Yuan**, Syracuse University, and **Yuan Zhang**, Indiana University-Purdue University Fort Wayne.

Combinatorial and Computational Commutative Algebra and Algebraic Geometry (Code: SS 10A), **Hirotachi Abo**, University of Idaho, **Zach Teitler**, Boise State University, **Jim Wolper**, Idaho State University, and **Alex Woo**, University of Idaho.

Commutative Algebra (Code: SS 7A), **Adam Boocher** and **Linquan Ma**, University of Utah.

Descriptive Set Theory and its Applications (Code: SS 9A), **Christian Rosendal**, University of Illinois at Chicago, and **Alexander Kechris**, California Institute of Technology.

Ergodic Theory and Dynamical Systems (Code: SS 14A), **Jon Chaika** and **Yiannis Konstantoulas**, University of Utah.

Extremal Problems in Graph Theory (Code: SS 8A), **Andre Kundgen** and **Mike Picollelli**, California State University San Marcos.

Fusion Categories and Topological Phases of Matter (Code: SS 11A), **Paul Bruillard**, Pacific Northwest National Laboratory, and **Julia Plavnik**, Texas A&M University.

Infinite Dimensional and Stochastic Dynamical Systems (Code: SS 15A), **Peter W. Bates**, Michigan State University, and **Kening Lu**, Brigham Young University.

Inverse Problems (Code: SS 2A), **Hanna Makaruk**, Los Alamos National Laboratory (LANL), and **Robert Owczarek**, University of New Mexico, Albuquerque and UNM, Los Alamos.

Representations of Reductive p-adic Groups (Code: SS 3A), **Shiang Tang** and **Gordan Savin**, University of Utah.

Structure and Emergent Properties of Biological Networks (Code: SS 16A), **Fred Adler**, **Katrina Johnson**, **Anna Miller**, and **Laura Strube**, University of Utah.

Topics in Probability (Code: SS 13A), **Tom Alberts** and **Arjun Krishnan**, University of Utah.

Topics in Stochastic Partial Differential Equations (Code: SS 12A), **Jingyu Huang** and **Davar Khosnevisan**, University of Utah.

Fargo, North Dakota

North Dakota State University

April 16–17, 2016

Saturday – Sunday

Meeting #1120

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: February 2016
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 37, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 23, 2016

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Rodrigo Banuelos, Purdue University, *Title to be announced.*

Laura Matusevich, Texas A&M University, *Title to be announced.*

Jeff Viaclovsky, University of Wisconsin-Madison, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic and Geometric Combinatorics (Code: SS 16A), **Kevin Dilks** and **Jessica Striker**, North Dakota State University.

Applications of Microlocal Analysis: Eigenfunctions and Dispersive PDE (Code: SS 20A), **Hans Christianson** and **Jason Metcalfe**, University of North Carolina.

Combinatorial Ideals and Applications (Code: SS 10A), **Laura Matusevich** and **Christopher O'Neill**, Texas A&M University.

Commutative Algebra and Its Interactions with Combinatorics and Algebraic Geometry (Code: SS 4A), **Susan Cooper**, North Dakota State University, and **Adam Van Tuyl**, McMaster University.

Commutative Ring Theory (Code: SS 6A), **Catalin Ciuperca** and **Sean Sather-Wagstaff**, North Dakota State University.

Contemporary Issues in Mathematics Education (Code: SS 8A), **Abraham Ayebo**, North Dakota State University.

Convexity and Harmonic Analysis (Code: SS 2A), **Maria Alfonseca-Cubero**, North Dakota State University, and **Dmitry Ryabogin**, Kent State University.

Discrete Probability (Code: SS 9A), **Jonathon Peterson**, Purdue University, and **Arnab Sen**, University of Minnesota.

Dynamics, Inverse Semigroups, and Operator Algebras (Code: SS 15A), **Benton Duncan**, North Dakota State University, and **David Pitts**, University of Nebraska-Lincoln.

Ergodic Theory and Dynamical Systems (Code: SS 1A), **Dogan Comez**, North Dakota State University, and **Mrinal Kanti Roychowdhury**, University of Texas Rio Grand Valley.

Extremal Graph Theory (Code: SS 13A), **Michael Ferrara** and **Stephen Hartke**, University of Colorado Denver.

Frames, Harmonic Analysis, and Operator Theory (Code: SS 7A), **Gabriel Picioroaga**, University of South Dakota, and **Eric Weber**, Iowa State University.

Frames, Wavelets and Gabor Systems (Code: SS 11A), **Yeonhyang Kim** and **Sivaram K. Narayan**, Central Michigan University.

Integrable Dynamical Systems and Special Functions (Code: SS 5A), **Oksana Bihun**, University of Colorado, Colorado Springs.

Interactions with Algebraic Geometry (Code: SS 19A), **Julie Rana** and **Kaisa Taipale**, University of Minnesota.

Low Dimensional and Symplectic Topology (Code: SS 12A), **Anar Akhmedov**, University of Minnesota, and **Josef G. Dorfmeister**, North Dakota State University.

Mathematical Finance (Code: SS 3A), **Indranil SenGupta**, North Dakota State University.

Matrix and Operator Theory (Code: SS 14A), **Shaun Fallat** and **Douglas Farenick**, University of Regina.

Probabilistic and Extremal Combinatorics (Code: SS 17A), **Jonathan Cutler**, and **Jamie Radcliffe**, University of Nebraska-Lincoln.

Probability and Complex Analysis Inspired by Schramm and Loewner (Code: SS 21A), **Michael Kozdron**, University of Regina.

Topological and Smooth Dynamics (Code: SS 18A), **Azer Akhmedov** and **Michael Cohen**, North Dakota State University.

Brunswick, Maine

Bowdoin College

September 24–25, 2016

Saturday – Sunday

Meeting #1121

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: June 2016
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 37, Issue 3

Deadlines

For organizers: February 24, 2016
For abstracts: July 19, 2016

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Tim Austin, New York University, *Title to be announced*.

Moon Duchin, Tufts University, *Title to be announced*.

Thomas Lam, University of Michigan, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Noncommutative Ring Theory and Noncommutative Algebra (Code: SS 1A), **Jason Gaddis**, Wake Forest University, and **Manuel Reyes**, Bowdoin College.

Denver, Colorado

University of Denver

October 8–9, 2016

Saturday – Sunday

Meeting #1122

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: August 2016
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 37, Issue 3

Deadlines

For organizers: March 8, 2016
For abstracts: August 16, 2016

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Henry Cohn, Microsoft Research, New England, *Title to be announced*.

Ronny Hadani, University of Texas, Austin, *Title to be announced*.

Chelsea Walton, Temple University, Philadelphia, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Above and Beyond Fluid Flow studies: In celebration of the 60th birthday of Prof. William Layton (Code: SS 12A), **Traian Iliescu**, Virginia Polytechnic Institute and State University, **Alexander Labovsky**, Michigan Technological University, **Monika Neda**, University of Nevada, Las Vegas, and **Leo Rebholz**, Clemson University.

Algebraic Logic (Code: SS 1A), **Nick Galatos**, University of Denver, and **Peter Jipsen**, Chapman University.

Analysis on Graphs and Spectral Graph Theory (Code: SS 2A), **Paul Horn** and **Mei Yin**, University of Denver.

Foundations of Numerical Algebraic Geometry (Code: SS 14A), **Abraham Martin del Campo**, CIMAT, Guanajuato, Mexico, and **Frank Sottile**, Texas A&M University.

Nonassociative Algebra (Code: SS 3A), **Izabella Stuhl**, University of Debrecen and University of Denver, and **Petr Vojtěchovský**, University of Denver.

Noncommutative Geometry and Fundamental Applications (Code: SS 4A), **Frederic Latremoliere**, University of Denver.

Nonlinear and Stochastic Partial Differential Equations (Code: SS 13A), **Michele Coti Zelati**, University of Maryland, **Nathan Glatt-Holtz**, Virginia Polytechnic Institute and State University, and **Geordie Richards**, University of Rochester.

Operator Algebras and Applications (Code: SS 5A), **Alvaro Arias**, University of Denver.

Quantum Algebra (Code: SS 11A), **Chelsea Walton**, Temple University, **Ellen Kirkman**, Wake Forest University, and **James Zhang**, University of Washington, Seattle.

Recent Trends in Semigroup Theory (Code: SS 6A), **Michael Kinyon**, University of Denver, and **Ben Steinberg**, City College of New York.

Set Theory of the Continuum (Code: SS 7A), **Natasha Dobrinen** and **Daniel Hathaway**, University of Denver.

Unimodularity in Randomly Generated Graphs (Code: SS 8A), **Florian Sobieczky**, University of Denver.

Vertex Algebras and Geometry (Code: SS 9A), **Andrew Linshaw**, University of Denver, and **Thomas Creutzig** and **Nicolas Guay**, University of Alberta.

Zero Dimensional Dynamics (Code: SS 10A), **Nic Ormes** and **Ronnie Pavlov**, University of Denver.

Minneapolis, Minnesota

University of St. Thomas

October 28–30, 2016
Friday – Sunday

Meeting #1123
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: August 2016
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 37, Issue 4

Deadlines
For organizers: March 29, 2016
For abstracts: August 30, 2016

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Thomas Nevins, University of Illinois Urbana-Champaign, *Title to be announced*.

Charles Rezk, University of Illinois Urbana-Champaign, *Title to be announced*.

Christof Sparber, University of Illinois at Chicago, *Title to be announced*.

Samuel Stechmann, University of Wisconsin-Madison, *Title to be announced*.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Chip-Firing and Divisors on Graphs and Complexes (Code: SS 3A), **Caroline Klivans**, Brown University, and **Gregg Musiker** and **Victor Reiner**, University of Minnesota.

Enumerative Combinatorics (Code: SS 4A), **Eric Egge**, Carleton College, and **Joel Brewster Lewis**, University of Minnesota.

Geometric Flows, Integrable Systems and Moving Frames (Code: SS 2A), **Joseph Benson**, St. Olaf College, **Gloria Mari-Beffa**, University of Wisconsin-Madison, **Peter Olver**, University of Minnesota, and **Rob Thompson**, Carleton College.

Modeling and Predicting the Atmosphere, Oceans, and Climate (Code: SS 1A), **Sam Stechmann**, University of Wisconsin-Madison.

Raleigh, North Carolina

North Carolina State University at Raleigh

November 12–13, 2016
Saturday – Sunday

Meeting #1124
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: September 2016
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 37, Issue 4

Deadlines
For organizers: April 12, 2016
For abstracts: September 13, 2016

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Ricardo Cortez, Tulane University, *Title to be announced*.

Jason Metcalfe, University of North Carolina at Chapel Hill, *Title to be announced*.

Agnes Szanto, North Carolina State University, *Title to be announced*.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Difference Equations and Applications (Code: SS 2A), **Michael A. Radin**, Rochester Institute of Technology, and **Youssef Raffoul**, University of Dayton.

Homological Methods in Commutative Algebra (Code: SS 1A), **Alina Iacob** and **Saeed Nasseh**, Georgia Southern University.

Mathematical String Theory (Code: SS 3A), **Paul Aspiniall**, Duke University, **Ilarion Melnikov**, James Madison University, and **Eric Sharpe**, Virginia Tech.

Metric and Topological Oriented Fixed Point Theorems (Code: SS 5A), **Clement Boateng Ampadu**, Boston, MA, **Sartaj Ali**, National College of Business Administration and Economics, Lahore, Pakistan, **Xiaorong Liu**, University of Colorado at Boulder, and **Xavier Alexius Udo-Utun**, University of Uyo, Uyo, Nigeria.

Varieties, Their Fibrations and Automorphisms in Mathematical Physics and Arithmetic Geometry (Code: SS 4A), **Jimmy Dillies** and **Enka Lakuriqi**, Georgia Southern University, and **Tony Shaska**, Oakland University.

Atlanta, Georgia

Hyatt Regency Atlanta and Marriott Atlanta Marquis

January 4–7, 2017
Wednesday – Saturday

Meeting #1125
Joint Mathematics Meetings, including the 123rd Annual Meeting of the AMS, 100th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: October 2016
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 38, Issue 1

Deadlines
For organizers: April 1, 2016
For abstracts: To be announced

Charleston, South Carolina

College of Charleston

March 10–12, 2017
Friday – Sunday

Meeting #1126
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: November 10, 2016
For abstracts: To be announced

Bloomington, Indiana

Indiana University

April 1–2, 2017
Saturday – Sunday

Meeting #1127
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Pullman, Washington

Washington State University

April 22–23, 2017
Saturday – Sunday

Meeting #1128
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

New York, New York

Hunter College, City University of New York

May 6–7, 2017
Saturday – Sunday

Meeting #1129
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: September 14, 2016
For abstracts: March 21, 2017

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `www.ams.org/cgi-bin/abstracts/abstract.pl`.

Commutative Algebra (Code: SS 1A), **Laura Ghezzi**, New York City College of Technology-CUNY, and **Jooyoun Hong**, Southern Connecticut State University.

Recent Advances in Function Spaces, Operators and Nonlinear Differential Operators (Code: SS 2A), **David Cruz-Uribe**, University of Alabama, **Jan Lang**, The Ohio State University, and **Osvaldo Mendez**, University of Texas at El Paso.

Montréal, Quebec Canada

McGill University

July 24–28, 2017
Monday – Friday

Meeting #1130
The second Mathematical Congress of the Americas (MCA 2017) is being hosted by the Canadian Mathematical Society (CMS) in collaboration with the Pacific Institute for the Mathematical Sciences (PIMS), the Fields Institute (FIELDS), Le Centre de Recherches Mathématiques (CRM), and the Atlantic Association for Research in the Mathematical Sciences (AARMS).
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: July 31, 2016
For abstracts: To be announced

Buffalo, New York

State University of New York at Buffalo

September 16–17, 2017
Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: February 14, 2017
For abstracts: To be announced

Orlando, Florida

University of Central Florida, Orlando

September 23–24, 2017
Saturday – Sunday
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: February 23, 2017
For abstracts: July 25, 2017

Riverside, California

University of California, Riverside

November 4–5, 2017
Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 10–13, 2018
Wednesday – Saturday
Joint Mathematics Meetings, including the 124th Annual Meeting of the AMS, 101st Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2017
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: April 1, 2017
For abstracts: To be announced

Portland, Oregon

Portland State University

April 14–15, 2018
Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Baltimore, Maryland

Baltimore Convention Center, Hilton Baltimore, and Baltimore Marriott Inner Harbor Hotel

January 16–19, 2019
Wednesday – Saturday
Joint Mathematics Meetings, including the 125th Annual Meeting of the AMS, 102nd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM)and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2018
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: April 2, 2018
For abstracts: To be announced

Denver, Colorado

Colorado Convention Center

January 15–18, 2020
Wednesday – Saturday
Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: November 1, 2019
Issue of *Abstracts*: To be announced

Deadlines
For organizers: April 1, 2019
For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center

January 6–9, 2021
Wednesday – Saturday
Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: October 2020
Program first available on AMS website: November 1, 2020
Issue of *Abstracts*: To be announced

Deadlines
For organizers: April 1, 2020
For abstracts: To be announced

THE BACK PAGE

"Mathematics is a true art, the art of avoiding brute-force calculations by the development of concepts and techniques that permit us to travel more lightly. ... if computers had been available in...the Fourth Century, mathematics today would be a pale image of itself."

—*F. F. Bonsall, quoted in the* Bulletin of the Spanish Royal Math. Society, *November 2, 2014.*

Artwork by Michael Berg.

1900s PhDs with the Most Descendants:

Oswald Veblen, PhD 1903
10,371 descendants
Erhard Schmidt, PhD 1905
9,731 descendants

Source: Mathematics Genealogy Project; as of 1/22/16.

My TA

Artwork by Sam White.

QUESTIONABLE MATHEMATICS

The Reuters news agency claimed on March 19, 2002, that "The Antarctic Peninsular has warmed by 36 degrees Fahrenheit over the past half century, far faster than elsewhere on the ice-bound continent or the rest of the world." (Apparently they substituted the correct 2.5°C into the temperature conversion formula.)

Reported by John A. Shonder

What crazy things happen to you? Readers are invited to submit original short amusing stories, math jokes, cartoons, and other material to: `noti-backpage@ams.og`.

American Mathematical Society

100 years from now you can still be advancing mathematics.

When it's time to think about your estate plans, consider making a provision for the American Mathematical Society to extend your dedication to mathematics well into the future.

Professional staff at the AMS can share ideas on wills, trusts, life insurance plans, and more to help you achieve your charitable goals.

For more information:
Robin Marek at 401.455.4089 or rxm@ams.org
Or visit **www.ams.org/support**

Photo by Goen South

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