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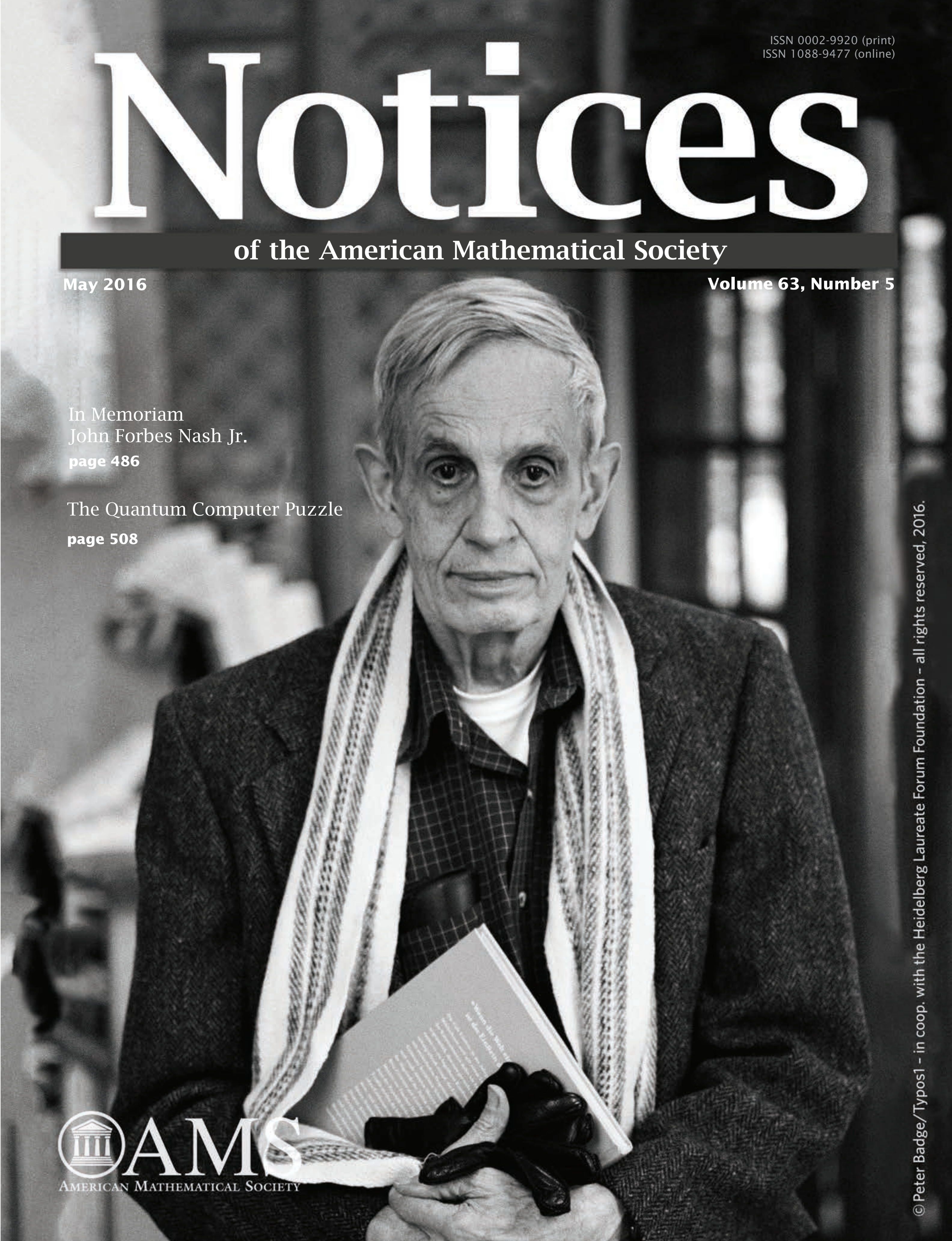

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2017

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for Outstanding Research in Mathematics by an Undergraduate Student

The prize is awarded each year to an undergraduate student (or students having submitted joint work) for outstanding research in mathematics. Any student who is an undergraduate in a college or university in the United States or its possessions, or Canada or Mexico, is eligible to be considered for this prize.

The prize recipient's research need not be confined to a single paper; it may be contained in several papers. However, the paper (or papers) to be considered for the prize must be submitted while the student is an undergraduate; they cannot be submitted after the student's graduation. The research paper (or papers) may be submitted for consideration by the student or a nominator. All submissions for the prize must include at least one letter of support from a person, usually a faculty member, familiar with the student's research. Publication of research is not required.

The recipients of the prize are to be selected by a standing joint committee of the AMS, MAA, and SIAM. The decisions of this committee are final. Nominations for the 2017 Morgan Prize are due no later than June 30, 2016. Those eligible for the 2017 prize must have been undergraduates in December 2015.

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AMERICAN MATHEMATICAL SOCIETY

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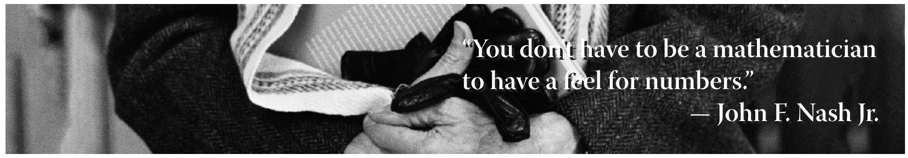

May 2016

FEATURES

We honor John F. Nash Jr. on this the first anniversary of his dramatic last week of life, when he received the 2015 Abel Prize from King Harald of Norway.

You can post general comments and suggestions for this and future issues at www.ams.org/notices.
—Frank Morgan, Editor-in-Chief

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IN EVERY ISSUE

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Interview with Abel Laureate John F. Nash Jr.

This article originally appeared in the September 2015 issue of the Newsletter of the European Mathematical Society www.ems-ph.org/journals/newsletter/pdf/2015-09-97.pdf*, pp. 26–31, and is reprinted here with permission of the EMS. (An analogous interview with Louis Nirenberg appeared in the February issue of the Notices.)*

Martin Raussen and Christian Skau

Courtesy of Eirik Baardsen.

John F. Nash Jr.

This interview took place in Oslo on May 18, 2015, the day before the prize ceremony and only five days before the tragic accident that led to the death of John Nash and his wife Alicia. Nash's untimely death made it impossible to follow the usual procedure for Abel interviews where interviewees are asked to proof-read and to edit first drafts. All possible misunderstandings are thus the sole responsibility of the interviewers.

Martin Raussen is professor of mathematics of Aalborg University, Denmark. His email address is raussen@math.aau.dk.

Christian Skau is professor of mathematics at the Norwegian University of Science and Technology, Trondheim, Norway. His email address is csk@math.ntnu.no.

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The Prize

Raussen and Skau: *Professor Nash, we would like to congratulate you as the Abel laureate in mathematics for 2015, a prize you share with Louis Nirenberg. What was your reaction when you first learned that you had won the Abel Prize?*

Professor Nash: I did not learn about it like I did with the Nobel Prize. I got a telephone call late on the day before the announcement, which was confusing. However, I wasn't entirely surprised. I had been thinking about the Abel Prize. It is an interesting example of a newer category of prizes that are quite large and yet not entirely predictable. I was given sort of a pre-notification. I was told on the telephone that the Abel Prize would be announced on the morning the next day. Just so I was prepared.

Raussen and Skau: *But it came unexpected?*

Professor Nash: It was unexpected, yes. I didn't even know when the Abel Prize decisions were announced. I had been reading about them in the newspapers but not following closely. I could see that there were quite respectable persons being selected.

Youth and Education

Raussen and Skau: *When did you realize that you had an exceptional talent for mathematics? Were there people that encouraged you to pursue mathematics in your formative years?*

Professor Nash: Well, my mother had been a school teacher, but she taught English and Latin. My father was an electrical engineer. He was also a schoolteacher immediately before World War I.

While at the grade school I was attending, I would typically do arithmetic—addition and multiplication—with multi-digit numbers instead of what was given at the school, namely multiplying two-digit numbers. So I got to work with four- and five-digit numbers. I just got pleasure in trying those out and finding the correct procedure. But the fact that I could figure this out was a sign, of course, of mathematical talent.

Then there were other signs also. I had the book by E. T. Bell, "Men of Mathematics", at an early age. I could read that. I guess Abel is mentioned in that book?

Raussen and Skau: *Yes, he is. In 1948, when you were twenty years of age, you were admitted as a graduate student in mathematics at Princeton University, an elite institution that hand-picked their students. How did you like the atmosphere at Princeton? Was it very competitive?*

Professor Nash: It was stimulating. Of course it was competitive also—a quiet competition of graduate students. They were not competing directly with each other like tennis players. They were all chasing the possibility of some special appreciation. Nobody said anything about that but it was sort of implicitly understood.

Games and Game Theory

Raussen and Skau: *You were interested in game theory from an early stage. In fact, you invented an ingenious game of a topological nature that was widely played, by both faculty members and students, in the Common Room at Fine Hall, the mathematics building at Princeton. The game was called "Nash" at Princeton but today it is commonly known as "Hex". Actually, a Danish inventor and designer Piet Hein independently discovered this game.*

Why were you interested in games and game theory?

Professor Nash: Well, I studied economics at my previous institution, the Carnegie Institute of Technology in Pittsburgh (today Carnegie Mellon University). I observed people who were studying the linkage between games and mathematical programming at Princeton. I had some ideas: some related to economics, some related to games like you play as speculators at the stock market—which is really a game. I can't pin it down exactly but it turned out that von Neumann [1903–1957] and Morgenstern [1902–1977] at Princeton had a proof of the solution to a two-person game that was a special case of a general theorem for the equilibrium of n-person games, which is what I found. I associated it with the natural idea of equilibrium and of the topological idea of the Brouwer fixed-point theorem, which is good material.

Exactly when and why I started, or when von Neumann and Morgenstern thought of that, that is something I am uncertain of. Later on, I found out about the Kakutani fixed-point theorem, a generalisation of Brouwer's theorem. I did not realise that von Neumann had inspired it and that he had influenced Kakutani [1911–2004]. Kakutani was a student at Princeton, so von Neumann wasn't surprised with the idea that a topological argument could yield equilibrium in general. I developed a theory to study a few other aspects of games at this time.

Raussen and Skau: *You are a little ahead of us now. A lot of people outside the mathematical community know that you won the Nobel Memorial Prize in Economic Sciences in 1994.*

Professor Nash: That was much later.

Raussen and Skau: *Yes. Due to the film "A Beautiful Mind", in which you were played by Russell Crowe, it became known to a very wide audience that you received the Nobel Prize in economics. But not everyone is aware that the Nobel Prize idea was contained in your PhD thesis, which was submitted at Princeton in 1950, when you were twenty-one-years-old. The title of the thesis was "Noncooperative games."*

Did you have any idea how revolutionary this would turn out to be? That it was going to have impact, not only in economics but also in fields as diverse as political science and evolutionary biology?

Professor Nash: It is hard to say. It is true that it can be used wherever there is some sort of equilibrium and there are competing or interacting parties. The idea of evolutionists is naturally parallel to some of this. I am getting off on a scientific track here.

Raussen and Skau: *But you realized that your thesis was good?*

Professor Nash: Yes. I had a longer version of it but it was reduced by my thesis advisor. I also had material for cooperative games but that was published separately.

Raussen and Skau: *Did you find the topic yourself when you wrote your thesis or did your thesis advisor help to find it?*

Professor Nash: Well, I had more or less found the topic myself and then the thesis advisor was selected by the nature of my topic.

Raussen and Skau: *Albert Tucker [1905–1995] was your thesis advisor, right?*

Professor Nash: Yes. He had been collaborating with von Neumann and Morgenstern.

Princeton

Raussen and Skau: *We would like to ask you about your study and work habits. You rarely attended lectures at Princeton. Why?*

Professor Nash: It is true. Princeton was quite liberal. They had introduced, not long before I arrived, the concept of an N-grade. So, for example, a professor giving a course would give a standard grade of N, which means "no grade". But this changed the style of working. I think that Harvard was not operating on that basis at that time. I don't know if they have operated like that since. Princeton has continued to work with the N-grade, so that the number of people actually taking the courses (formally taking courses where grades are given) is less in Princeton than might be the case at other schools.

Raussen and Skau: *Is it true that you took the attitude that learning too much second-hand would stifle creativity and originality?*

Professor Nash: Well, it seems to make sense. But what is second-hand?

Raussen and Skau: *Yes, what does second-hand mean?*

Professor Nash: Second-hand means, for example, that you do not learn from Abel but from someone who is a student of abelian integrals.

Raussen and Skau: *In fact, Abel wrote in his mathematical diary that one should study the masters and not their pupils.*

Professor Nash: Yes, that's somewhat the idea. Yes, that's very parallel.

Raussen and Skau: *While at Princeton you contacted Albert Einstein and von Neumann, on separate occasions. They were at the Institute for Advanced Study in Princeton, which is located close to the campus of Princeton University. It was very audacious for a young student to contact such famous people, was it not?*

Professor Nash: Well, it could be done. It fits into the idea of intellectual functions. Concerning von Neumann, I had achieved my proof of the equilibrium theorem for game theory using the Brouwer fixed-point theorem, while von Neumann and Morgenstern used other things in their book. But when I got to von Neumann, and I was at the blackboard, he asked: "Did you use the fixed-point theorem?" "Yes," I said. "I used Brouwer's fixed-point theorem."

I had already, for some time, realized that there was a proof version using Kakutani's fixed-point theorem, which is convenient in applications in economics since the mapping is not required to be quite continuous. It has certain continuity properties, so-called generalized continuity properties, and there is a fixed-point theorem in that case as well. I did not realize that Kakutani proved that after being inspired by von Neumann, who was using a fixed-point theorem approach to an economic problem with interacting parties in an economy (however, he was not using it in game theory).

Raussen and Skau: *What was von Neumann's reaction when you talked with him?*

Professor Nash: Well, as I told you, I was in his office and he just mentioned some general things. I can imagine now what he may have thought, since he knew the Kakutani fixed-point theorem and I did not mention that (which I could have done). He said some general things, like: "Of course, this works." He did not say too much about how wonderful it was.

Raussen and Skau: *When you met Einstein and talked with him, explaining some of your ideas in physics, how did Einstein react?*

Professor Nash: He had one of his student assistants there with him. I was not quite expecting that. I talked about my idea, which related to photons losing energy on long travels through the Universe and as a result getting a red-shift. Other people have had this idea. I saw much later that someone in Germany wrote a paper about it but I can't give you a direct reference. If this phenomenon existed then the popular opinion at the time of the expanding Universe would be undermined because what would appear to be an effect of the expansion of the Universe (sort of a Doppler red-shift) could not be validly interpreted in that way because there could be a red-shift of another origin. I developed a mathematical theory about this later on. I will present this here as a possible interpretation, in my Abel lecture tomorrow.

There is an interesting equation that could describe different types of space-times. There are some singularities that could be related to ideas about dark matter and dark energy. People who really promote it are promoting the idea that most of the mass in the Universe derives from dark energy. But maybe there is none. There could be alternative theories.

Raussen and Skau: *John Milnor, who was awarded the Abel Prize in 2011, entered Princeton as a freshman the same year as you became a graduate student. He made the observation that you were very much aware of unsolved problems, often cross-examining people about these. Were you on the lookout for famous open problems while at Princeton?*

Professor Nash: Well, I was. I have been in general. Milnor may have noticed at that time that I was looking at some particular problems to study.

Milnor made various spectacular discoveries himself. For example, the nonstandard differentiable structures on the seven-sphere. He also proved that any knot has a certain amount of curvature although this was not really a new theorem, since someone else had—unknown to Milnor—proved that.

Courtesy of the Institute for Advanced Study. Photo by Serge J.-F. Levy.

John F. Nash Jr. at the Common Room, Institute for Advanced Study.

A Series of Famous Results

Raussen and Skau: *While you wrote your thesis on game theory at Princeton University, you were already working on problems of a very different nature, of a rather geometric flavor. And you continued this work while you were on the staff at MIT in Boston, where you worked from 1951 to 1959. You came up with a range of really stunning results. In fact, the results that you obtained in this period are the main motivation for awarding you the Abel Prize this year. Before we get closer to your results from this period, we would like to give some perspective by quoting Mikhail Gromov, who received the Abel Prize in 2009. He told us, in the interview we had with him six years ago, that your methods showed "incredible originality". And moreover: "What Nash has done in geometry is from my point of view incomparably greater than what he has done in economics, by many orders of magnitude." Do you agree with Gromov's assessment?*

Professor Nash: It's simply a question of taste, I say. It was quite a struggle. There was something I did in algebraic geometry, which is related to differential geometry with some subtleties in it. I made a breakthrough there. One could actually gain control of the geometric shape of an algebraic variety.

Raussen and Skau: *That will be the subject of our next question. You submitted a paper on real algebraic manifolds when you started at MIT, in October 1951. We would like to quote Michael Artin at MIT, who later made use of your result. He commented: "Just to conceive such a theorem was remarkable."*

Could you tell us a little of what you dealt with and what you proved in that paper, and how you got started?

Professor Nash: I was really influenced by space-time and Einstein, and the idea of distributions of stars, and I thought: "Suppose some pattern of distributions of stars could be selected; could it be that there would be a manifold, something curving around and coming in on itself that would be in some equilibrium position with those distributions of stars?" This is the idea I was considering.

Ultimately, I developed some mathematical ideas so that the distribution of points (interesting points) could be chosen, and then there would be some manifold that would go around in a desired geometrical and topological way. So I did that and developed some additional general theory for doing that at the same time, and that was published.

Later on, people began working on making the representation more precise because I think what I proved may have allowed some geometrically less beautiful things in the manifold that is represented, and it might come close to other things. It might not be strictly finite. There might be some part of it lying out at infinity.

Ultimately, someone else, A. H. Wallace [1926–2008], appeared to have fixed it, but he hadn't—he had a flaw. But later it was fixed by a mathematician in Italy, in Trento, named Alberto Tognoli [1937–2008].

Raussen and Skau: *We would like to ask you about another result, concerning the realisation of Riemannian manifolds. Riemannian manifolds are, loosely speaking, abstract smooth structures on which distances and angles are only locally defined in a quite abstract manner. You showed that these abstract entities can be realised very concretely as sub-manifolds in sufficiently high-dimensional Euclidean spaces.*

Professor Nash: Yes, if the metric was given, as you say, in an abstract manner but was considered as sufficient to define a metric structure then that could also be achieved by an embedding, the metric being induced by the embedding. There I got on a side-track. I first proved it for manifolds with a lower level of smoothness, the C^1 case. Some other people have followed up on that. I published a paper on that. Then there was a Dutch mathematician, Nicolaas Kuiper [1920–1994], who managed to reduce the dimension of the embedding space by one.

Raussen and Skau: *Apart from the results you obtained, many people have told us that the methods you applied were ingenious. Let us, for example, quote Gromov and John Conway.*

Gromov said, when he first read about your result: "I thought it was nonsense, it couldn't be true. But it was true, it was incredible." And later on: "He completely changed the perspective on partial differential equations."

And Conway said: "What he did was one of the most important pieces of mathematical analysis in the twentieth century." Well, that is quite something!

Professor Nash: Yes.

Raussen and Skau: *Is it true, as rumours have it, that you started to work on the embedding problem as a result of a bet?*

Professor Nash: There was something like a bet. There was a discussion in the Common Room, which is the meeting place for faculty at MIT. I discussed the idea of an embedding with one of the senior faculty members in geometry, Professor Warren Ambrose [1914–1995]. I got from him the idea of the realization of the metric by an embedding. At the time, this was a completely open problem; there was nothing there beforehand.

I began to work on it. Then I got shifted onto the C^1 case. It turned out that one could do it in this case with very few excess dimensions of the embedding space compared with the manifold. I did it with two but then Kuiper did it with only one. But he did not do it smoothly, which seemed to be the right thing—since you are given something smooth, it should have a smooth answer.

But a few years later, I made the generalisation to smooth. I published it in a paper with four parts. There is an error, I can confess now. Some forty years after the paper was published, the logician Robert M. Solovay from the University of California sent me a communication pointing out the error. I thought: "How could it be?" I started to look at it and finally I realized the error in that if you want to do a smooth embedding and you have an infinite manifold, you divide it up into portions and you have embeddings for a certain amount of metric on each portion. So you are dividing it up into a number of things: smaller, finite manifolds. But what I had done was a failure in logic. I had proved that—how can I express it?—that points local enough to any point where it was spread out and differentiated perfectly if you take points close enough to one point; but for two different points it could happen that they were mapped onto the same point. So the mapping, strictly speaking, wasn't properly embedded; there was a chance it had self-intersections.

Raussen and Skau: *But the proof was fixed? The mistake was fixed?*

Professor Nash: Well, it was many years from the publication that I learned about it. It may have been known without being officially noticed, or it may have been noticed but people may have kept the knowledge of it secret.

Raussen and Skau: *May we interject the following to highlight how surprising your result was? One of your colleagues at MIT, Gian-Carlo Rota [1932–1999], professor of mathematics and also philosophy at MIT, said: "One of the great experts on the subject told me that if one of his graduate students had proposed such an outlandish idea, he would throw him out of his office."*

Professor Nash: That's not a proper liberal, progressive attitude.

Courtesy of Erik Baardsen.

Nash interviewed by Christian Skau and Martin Raussen.

Partial Differential Equations

Raussen and Skau: *But nevertheless it seems that the result you proved was perceived as something that was out of the scope of the techniques that one had at the time.*

Professor Nash: Yes, the techniques led to new methods to study PDEs in general.

Raussen and Skau: *Let us continue with work of yours purely within the theory of PDEs. If we are not mistaken, this came about as a result of a conversation you had with Louis Nirenberg, with whom you are sharing this year's Abel Prize, at the Courant Institute in New York in 1956. He told you about a major unsolved problem within non-linear partial differential equations.*

Professor Nash: He told me about this problem, yes. There was some work that had been done previously by a professor in California, C. B. Morrey [1907–1984], in two dimensions. The continuity property of the solution of a partial differential equation was found to be intrinsic in two dimensions by Morrey. The question was what happened beyond two dimensions. That was what I got to work on, and De Giorgi [1928–1996], an Italian mathematician, got to work on it also.

Raussen and Skau: *But you didn't know of each other's work at that time?*

Professor Nash: No, I didn't know of De Giorgi's work on this, but he did solve it first.

Raussen and Skau: *Only in the elliptic case though.*

Professor Nash: Yes, well, it was really the elliptic case originally but I sort of generalized it to include parabolic equations, which turned out to be very favorable. With parabolic equations, the method of getting an argument relating to an entropy concept came up. I don't know; I am not trying to argue about precedents but a similar entropy method was used by Professor Hamilton in New York and then by Perelman. They use an entropy which they can control in order to control various improvements that they need.

Raussen and Skau: *And that was what finally led to the proof of the Poincaré Conjecture?*

Professor Nash: Their use of entropy is quite essential. Hamilton used it first and then Perelman took it up from there. Of course, it's hard to foresee success. It's a funny thing that Perelman hasn't accepted any prizes. He rejected the Fields Prize and also the Clay Millennium Prize, which comes with a cash award of one million dollars.

Raussen and Skau: *Coming back to the time when you and De Giorgi worked more or less on the same problem. When you first found out that De Giorgi had solved the problem before you, were you very disappointed?*

Professor Nash: Of course I was disappointed but one tends to find some other way to think about it. Like water building up and the lake flowing over, and then the outflow stream backing up, so it comes out another way.

Raussen and Skau: *Some people have been speculating that you might have received the Fields Medal if there had not been the coincidence with the work of De Giorgi.*

Professor Nash: Yes, that seems likely; that seems a natural thing. De Giorgi did not get the Fields Medal either, though he did get some other recognition. But this is not mathematics, thinking about how some sort of selecting body may function. It is better to be thought about by people who are sure they are not in the category of possible targets of selection.

Raussen and Skau: *When you made your major and really stunning discoveries in the 1950s, did you have anybody that you could discuss with, who would act as some sort of sounding board for you?*

Professor Nash: For the proofs? Well, for the proof in game theory there is not so much to discuss. Von Neumann knew that there could be such a proof as soon as the issue was raised.

Raussen and Skau: *What about the geometric results and also your other results? Did you have anyone you could discuss the proofs with?*

Professor Nash: Well, there were people who were interested in geometry in general, like Professor Ambrose. But they were not so much help with the details of the proof.

Raussen and Skau: *What about Spencer [1912–2001] at Princeton? Did you discuss with him?*

Professor Nash: He was at Princeton and he was on my General Exam committee. He seemed to appreciate me. He worked in complex analysis.

Raussen and Skau: *Were there any particular mathematicians that you met either at Princeton or MIT that you really admired, that you held in high esteem?*

Professor Nash: Well, of course, there is Professor Levinson [1912–1975] at MIT. I admired him. I talked with Norman Steenrod [1910–1971] at Princeton and I knew Solomon Lefschetz [1884–1972], who was Department Chairman at Princeton. He was a good mathematician. I did not have such a good rapport with the algebra professor at Princeton, Emil Artin [1898–1962].

The Riemann Hypothesis

Raussen and Skau: *Let us move forward to a turning point in your life. You decided to attack arguably the most famous of all open problems in mathematics, the Riemann*

Hypothesis, which is still wide open. It is one of the Clay Millennium Prize problems that we talked about. Could you tell us how you experienced mental exhaustion as a result of your endeavor?

Professor Nash: Well, I think it is sort of a rumor or a myth that I actually made a frontal attack on the hypothesis. I was cautious. I am a little cautious about my efforts when I try to attack some problem because the problem can attack back, so to say. Concerning the Riemann Hypothesis, I don't think of myself as an actual student but maybe some casual—whatever—where I could see some beautiful and interesting new aspect.

I am a little cautious … when I try to attack some problem because the problem can attack back.

Professor Selberg [1917–2007], a Norwegian mathematician who was at the Institute for Advanced Study, proved back in the time of World War II that there was at least some finite measure of these zeros that were actually on the critical line. They come as different types of zeros; it's like a double zero that appears as a single zero. Selberg proved that a very small fraction of zeros were on the critical line. That was some years before he came to the Institute. He did some good work at that time.

And then, later on, in 1974, Professor Levinson at MIT, where I had been, proved that a good fraction—around 1/3—of the zeros were actually on the critical line. At that time he was suffering from brain cancer, which he died from. Such things can happen; your brain can be under attack and yet you can do some good reasoning for a while.

A Very Special Mathematician?

Raussen and Skau: *Mathematicians who know you describe your attitude toward working on mathematical problems as very different from that of most other people. Can you tell us a little about your approach? What are your sources of inspiration?*

Professor Nash: Well, I can't argue that at the present time I am working in such and such a way, which is different from a more standard way. In other words, I try to think of what I can do with my mind and my experiences and connections. What might be favourable for me to try? So I don't think of trying anything of the latest popular nonsense.

Raussen and Skau: *You have said in an interview (you may correct us) something like: "I wouldn't have had good scientific ideas if I had thought more normally." You had a different way of looking at things.*

Professor Nash: Well, it's easy to think that. I think that is true for me just as a mathematician. It wouldn't be worth it to think like a good student doing a thesis. Most mathematical theses are pretty routine. It's a lot of work but sort of set up by the thesis advisor; you work until you have enough and then the thesis is recognized.

Interests and Hobbies

Raussen and Skau: *Can we finally ask you a question that we have asked all the previous Abel Prize laureates? What are your main interests or hobbies outside of mathematics?*

Professor Nash: Well, there are various things. Of course, I do watch the financial markets. This is not entirely outside of the proper range of the economics Nobel Prize but there is a lot there you can do if you think about things. Concerning the great depression, the crisis that came soon after Obama was elected, you can make one decision or another decision which will have quite different consequences. The economy started on a recovery in 2009, I think.

Raussen and Skau: *It is known that when you were a student at Princeton you were biking around campus whistling Bach's "Little Fugue". Do you like classical music?*

Professor Nash: Yes, I do like Bach.

Raussen and Skau: *Other favorite composers than Bach?*

Professor Nash: Well, there are lot of classical composers that can be quite pleasing to listen to, for instance when you hear a good piece by Mozart. They are so much better than composers like Pachelbel and others.

Raussen and Skau: *We would like to thank you very much for a very interesting interview. Apart from the two of us, this is on behalf of the Danish, Norwegian and European Mathematical Societies.*

Afterword: After the end of the interview proper, there was an informal chat about John Nash's main current interests. He mentioned again his reflections about cosmology. Concerning publications, Nash told us about a book entitled "Open Problems in Mathematics" that he was editing with the young Greek mathematician Michael Th. Rassias, who was conducting postdoctoral research at Princeton University during that academic year.

Photo by Danielle Alio, Princeton University, Office of Communications.

John F. Nash Jr. and wife Alicia.

John Forbes Nash Jr. (1928–2015)

Camillo De Lellis, Coordinating Editor

John Forbes Nash Jr. was born in Bluefield, West Virginia, on June 13, 1928 and was named after his father, who was an electrical engineer. His mother, Margaret Virginia (née Martin), was a school teacher before her marriage, teaching English and sometimes Latin. After attending the standard schools in Bluefield, Nash entered the Carnegie Institute of Technology in Pittsburgh (now Carnegie Mellon University) with a George Westinghouse Scholarship. He spent one semester as a student of chemical engineering, switched momentarily to chemistry and finally decided to major in mathematics. After graduating in 1948 with a BS and a MS at the same time, Nash was offered a scholarship to enter as a graduate student at either Harvard or Princeton. He decided for Princeton, where in 1950 he earned a PhD degree with his celebrated work on noncooperative games, which won him the Nobel Prize in Economics thirty-four years later.

Courtesy of Martha Nash Legg and John D. Stier.

A picture of Nash taken the day of his graduation in Princeton.

In the summer of 1950 he worked at the RAND (Research and Development) Corporation, and although he went back to Princeton during the autumn of the same year, he remained a consultant and occasionally worked at RAND for the subsequent four years, as a leading expert on the Cold War conflict. He was fired from RAND in 1954 after being arrested for indecent exposure in Santa Monica, although the charges were dropped.

In 1951 he joined the mathematics faculty of MIT as a C.L.E. Moore Instructor, where he remained until his resignation in the spring of 1959. In 1951 he wrote his

Courtesy of John D. Stier.

John and Alicia Nash on the day of their wedding.

groundbreaking paper "Real algebraic manifolds", cf. [39], much of which was indeed conceived at the end of his graduate studies: According to his autobiographical notes, cf. [44], Nash was prepared for the possibility that the game theory work would not be regarded as acceptable as a thesis at the Princeton mathematics department. Around this time Nash met Eleanor Stier, with whom he had his first son, John David Stier, in 1953.

After his work on real algebraic manifolds he began his deep studies on the existence of isometric embeddings of Riemannian manifolds, a fundamental and classical open problem, which Nash solved completely in his two subsequent revolutionary papers [40] and [41]. During the academic year 1956–1957 he received an Alfred P. Sloan grant and decided to spend the year as a temporary member of the Institute for Advanced Study in Princeton. It is during this period that he got interested in another classical question, the continuity of solutions to uniformly elliptic and parabolic second order equations, which would have lead to a solution of the 19th Hilbert problem. Nash published his solution [42] and learned slightly after that a different independent proof, in the case of elliptic equations, had just been given by De Giorgi [14].

During his academica sabbatical at the Institute for Advanced Study Nash married Alicia Lopez-Harrison de

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Lardé and shortly after, in 1958, he earned a tenured position at MIT. In the last months of 1958 and the early months of 1959 the first signs of mental disorder had become evident, while his wife was pregnant with their child, John Charles. This was the start of a long miserable period of mental illness, during which Nash still managed to produce some remarkable pieces of mathematics, such as [45], [43], [46] (published a couple of decades later) and the idea of the "Nash blow-up."

Nash and de Lardé divorced in 1962. However, after his final hospital discharge in 1970, Nash lived in the house of his former wife and the couple eventually remarried in 2003. After a long period Nash gradually recovered from his paranoid delusions, was allowed by Princeton to audit classes and finally to teach again.

After he received the Nobel Memorial Prize in Economic Sciences in 1994, jointly with John Harsanyi and Reinhard Selten, Nash's dramatic life attracted the attention of the media and was the subject of Sylvia Nasar's bestseller *A Beautiful Mind*, which inspired the 2001 movie with the same title. During this period Nash became an icon of genius in popular culture.

In 1978 he was awarded the John von Neumann Theory Prize for his discovery of the Nash Equilibria. In 1999 he received a Leroy P. Steele Prize for Seminal Contribution to Research from the American Mathematical Society and finally in 2015 he was one of the two recipients of the Abel Prize, the other one being Louis Nirenberg. On May 23, 2015, on their way back home after spending one week in Oslo on the occasion of the Abel prize ceremony, John and Alicia Nash were killed in a taxi accident on the New Jersey Turnpike.

John Milnor

About John Nash

John Forbes Nash was an amazing person, highly original, and determined to make a name for himself by attacking the most difficult and important mathematical problems.

His most widely influential work is surely the 1950 Princeton Thesis, in which he introduced what we now call a ***Nash equilibrium***. I have heard that this was described by von Neumann as "just another fixed point theorem". Whether or not this is a true quotation, this evaluation is certainly valid from the point of view of pure mathematics. However, when mathematics is applied to the real world, the important question is not whether it represents the most cutting edge mathematical techniques, but whether it tells us something meaningful about reality. The theory of two-person zero-sum games had been firmly established by the work of Zermelo, von Neumann and Morgenstern; but before Nash's work the theory of any more general form of conflict between two or more parties was a wasteland of complicated mathematics with no apparent relation to reality. Nash's ideas transformed the subject, and over the years, they have become basic and central in fields as diverse as economic theory and evolutionary biology. (See the exposition by Nachbar and Weinstein below.)

In 1952, Nash created a relation between differential and algebraic manifolds by showing that every smooth compact manifold is diffeomorphic to an essentially isolated smooth subset of some real algebraic variety. (See the exposition by Henry King below. One important application was given by Michael Artin and Barry Mazur [4] thirteen years later: For any smooth compact manifold M, they used Nash's result in proving that any smooth mapping from M to itself can be smoothly approximated by one for which the number of isolated periodic points of period n grows at most exponentially with n. For related results by V. Kaloshin, see [27].)

An amazing person, highly original, and determined to make a name for himself.

Nash had not forgotten about application of mathematical ideas to real world problems. A 1954 RAND Corporation memorandum described his ideas for the architecture and programing of a parallel processing computer. This was well before any such machine existed. In 1955, he wrote a letter to the National Security Agency which proposed an encypherment procedure, and explained his ideas about computational complexity and cryptography. Long before such ideas were generally known, he realized that a key criterion for secure cryptography is that the computation time for determining the key, given other information about the system, should increase exponentially with the key length. He conjectured that this criterion should be satisfied, but very hard to prove, for many possible encryption schemes. (This is perhaps an early relative of the P versus NP problem, which was posed by Stephen Cook sixteen year later, see [12].) More explicitly, Nash stated that "I cannot prove [this conjecture], nor do I expect it to be proven." His message was filed and presumably forgotten by the NSA, but declassified and released in 2012.

Returning to the study of smooth manifolds, the following classical statement could easily have been proved by Gauss, if he had considered such questions: *A compact surface which is smoothly embedded in 3-dimensional Euclidean space must have points of positive Gaussian curvature.* More precisely, the proof requires that the embedding should be twice continuously differentiable. A reasonable person would assume that C^2-differentiability is just a technicality, but Nash was never a reasonable person. His 1954 paper, as sharpened one year later by Nicholaas Kuiper, shows in particular that *every* compact surface with a smooth Riemannian metric can be C^1-isometrically embedded in Euclidean 3-space. Such exotic C^1-embeddings are very hard to visualize, and it is only in the last year or so that a determined French team

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Courtesy of John D. Stier.

Nash and his first son John David Stier in a picture taken at Princeton in the mid-1970s.

has managed to provide computer visualizations, and even 3-D printed models, for a flat torus C^1-isometrically embedded in 3-space (see [7]).

For $k > 1$, the problem of C^k-isometric embedding of a smooth manifold in a suitable Euclidean space is less dramatic, but far more important for most applications. Its effective solution by Nash in 1956 required the invention of new and important methods in the study of partial differential equations. One step in the proof was extracted by Jürgen Moser ten years later [32], [33] and used to study periodic orbits in celestial mechanics. The resulting Nash-Moser Inverse Function Theorem is a basic tool; but is not easy to explain. (Richard Hamilton in 1982 took more than 150 pages to explain it, see [23].)

A reasonable person would assume that C^2-differentiability is just a technicality, but Nash was never a reasonable person.

Further information on Nash's Embedding Theory can be found in the article by De Lellis and Székelyhidi below. This was just the beginning of Nash's work in partial differential equations. For his 1957–1958 study of parabolic and elliptic equations, see the article by Villani below. It is hard for a nonspecialist to understand the details of this work, but it is surely notable for its originality and depth.

During these years, Nash was bouncing back and forth between the Courant Institute in New York and the Institute for Advanced Study in Princeton. He was full of ideas on every subject. At Courant he was talking about partial differential equations and fluid mechanics, for example, with Louis Nirenberg and Peter Lax. In Princeton he was talking with number theorists such as Atle Selberg about ideas towards the Riemann Hypothesis, and arguing with physicists such as Robert Oppenheimer about the foundations of quantum mechanics.

Nash's work was drastically interrupted by a breakdown in early 1959. (Many years later, he blamed his collapse on efforts to resolve the contradictions in quantum mechanics.) Whatever the cause, the next thirty years were quite miserable for Nash and for his friends, although he did manage to write a few more papers. It was a wonderful relief when he began to recover in the early 1990s. It was also wonderful that he lived to see his life's work validated, both by a Nobel Prize in Economics in 1994, and by an Abel Prize in Mathematics this May, just a few days before his untimely death.

Comments and Further References

One convenient source is *The Essential John Nash*, edited by Harold Kuhn and Sylvia Nasar, Princeton University Press, 2002. This includes biographical and autobiographical material, as well as the complete texts of a number of papers, including the following:

- Real Algebraic Manifolds [39].
- Parallel Control [an otherwise unpublished RAND Corporation memorandum from 1954].
- The Imbedding Problem for Riemannian Manifolds [41], plus an erratum.
- Continuity of Solutions of Parabolic and Elliptic Equations [42].

For Nash's letter to the NSA, see `https://www.nsa.gov/public_info/_files/nash_letters/nash_letters1.pdf`. For a discussion of Nash's cryptosystem by Ron Rivest and Adi Shamir, see `www.iacr.org/conferences/eurocrypt2012/Rump/nash.pdf`.

For video illustrating a flat torus in 3-space, see `hevea.imag.fr/Site/Hevea_images-eng.html`.

John Nachbar and Jonathan Weinstein

Nash Equilibrium

Game theory is a mathematical framework for analyzing conflict and cooperation. It was originally motivated by recreational games and gambling, but has subsequently seen application to a wide range of disciplines, including the social sciences, computer science, and evolutionary

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biology. Within game theory, the single most important tool has proven to be Nash equilibrium. Our objective here is to explain why John Nash's introduction of Nash equilibrium (Nash called it an "equilibrium point") in [37] and [38] caused a radical shift in game theory's research program.

We start with some terminology. A *finite strategic-form game* (henceforth simply *game*), is a triple $G = (N, S, u)$ where $N = \{1, \dots, n\}$ is a finite set of players, $S = \prod_{i \in N} S_i$, where S_i denotes the finite set of strategies available to player i, and $u = (u_1, \dots, u_n)$ where $u_i : S \to \mathbb{R}$ describes the utility achieved by player i at each *strategy profile* $s \in S$. A *mixed strategy* σ_i is a probability distribution over S_i. Players attempt to maximize their utilities, or, if facing randomness, the expected value of their utilities; we extend our notation by letting $u_i(\sigma_1, \dots, \sigma_n)$ be the expectation of u_i with respect to the independent distribution over strategy profiles induced by $(\sigma_1, \dots, \sigma_n)$.

Two-player *zero-sum* games (two-player games for which $u_1(s) + u_2(s) = 0$ for all $s \in S$) are games of pure conflict. The central result for such games was first established by von Neumann ([55]):

Theorem 1 (Minimax Theorem)**.** *For every two-player zero-sum game, there is a number V such that:*

$$V = \max_{\sigma_1} \min_{\sigma_2} u_1(\sigma_1, \sigma_2) = \min_{\sigma_2} \max_{\sigma_1} u_1(\sigma_1, \sigma_2).$$

Player 1 can thus guarantee an average utility of at least V, called the *security value* of the game, while Player 2 can guarantee that Player 1 achieves at most V, or equivalently (since the game is zero-sum) that Player 2 achieves at least $-V$. This provides a strong basis for the prediction that players will achieve average utilities of V and $-V$. Any other outcome involves some player achieving less than he or she could have guaranteed. In standard formalizations of Rock-Paper-Scissors, for example, $V = 0$, which players can guarantee by randomizing equally over "rock", "paper", and "scissors".

In game theory, the single most important tool has proven to be Nash equilibrium.

At the time Nash began working on game theory, the *de facto* bible in the discipline was [56] by von Neumann and Morgenstern (hereafter VN-M). VN-M made the following proposal for how to extend the Minimax Theorem to general games, games that may combine elements of both cooperation and conflict. Given a general n-player game, construct an $(n+1)$-player zero-sum game by adding a dummy player. For each *coalition* (nonempty set of players), construct a two-player zero-sum game in which the two players are the coalition and its complement; implicitly, each coalition is assumed to cooperate perfectly within itself. The *value* of the coalition is the value V from the Minimax Theorem in the induced two-player zero-sum game. VN-M thus converted a general n-player game in strategic form into an $(n+1)$-player game in *coalition form*. For games in coalition form, VN-M proposed a solution concept now called a *stable set*, consisting of a set of payoff profiles with certain properties. Finally, VN-M proposed that for a general n-player game in strategic form, the *solution* is the set of utility profiles that correspond to elements of the stable set for the associated $(n+1)$-player game in coalition form, with the additional restriction that the solution maximize the total utility to the nondummy players.

Courtesy of John D. Stier.

Nash and his second son John Charles.

The VN-M solution is difficult to compute for games of four or more players. When there are only two players, however, the VN-M solution is simply the set of all utility profiles such that (1) each player gets at least his security value (which is defined even in a nonzero-sum game) and (2) the sum of player utilities is maximal. We refer to such utility profiles as *efficient*.

Consider, in particular, a game of the Prisoner's Dilemma form.

	C	D
C	$4,4$	$0,5$
D	$5,0$	$1,1$

Here, player 1 is the row player and player 2 is column. If they play the strategy profile (C, D), for example, then player 1 gets 0 and player 2 gets 5. The VN-M solution for this game is the set of utility profiles such that the utilities sum to 8 and each player gets at least 1.

As an alternative to the VN-M solution, Nash ([37]) proposed what is now called a *Nash equilibrium* (NE): a NE is a strategy profile (possibly involving mixed strategies) such that each player maximizes his or her own expected utility given the profile of (mixed) strategies of the other players. The focus of NE is thus on individual, rather than collective, optimization.

The zero-sum game Rock-Paper-Scissors has a unique NE in which each player randomizes equally over "rock", "paper", and "scissors". This NE yields an expected utility profile of (0,0), which is the VN-M solution.

On the other hand, in the Prisoner's Dilemma, the unique NE is (D, D). The induced utility profile $(1, 1)$ is inefficient, hence is not an element of the VN-M solution. The Prisoner's Dilemma is the canonical example of a game in which individual incentives lead players away from collective optimality. The VN-M solution, in contrast, assumes this inefficiency away.

Nash proved:

Theorem 2 (Existence of Nash Equilibrium)**.** *For every finite game, there is a Nash Equilibrium profile* $(\sigma_1^*, \dots, \sigma_n^*)$.

As noted in [37], Theorem 2 is an almost immediate consequence of [26], which extended Brouwer's fixed point theorem to correspondences for the express purpose of aiding proofs in economics and game theory. ([38] provided an alternate proof directly from Brouwer.) In contrast, it was unknown at that time whether every finite game had a VN-M solution; [30] later provided an example of a game with no VN-M solution.

That Theorem 2 is a generalization of the Minimax Theorem can be seen by noting that Theorem 1 is equivalent to:

Theorem 3 (Minimax Theorem, Equilibrium Version)**.** *For every two-player zero-sum game, there is a pair* (σ_1^*, σ_2^*) *such that*

$$u_1(\sigma_1^*, \sigma_2^*) = \max_{\sigma_1} u_1(\sigma_1, \sigma_2^*)$$

and

$$u_2(\sigma_1^*, \sigma_2^*) = \max_{\sigma_2} u_2(\sigma_1^*, \sigma_2).$$

Thus, both the VN-M solution and NE generalize the Minimax Theorem, but along very different paths. To characterize the difference between the approaches, Nash ([38]) coined the terms *cooperative* game theory (for games in coalition form, solved by concepts such as the stable set) and *noncooperative* game theory (for games in strategic form, solved by NE and related concepts). This choice of language can be deceptive. In particular, noncooperative game theory does not rule out cooperation.

For example, a standard explanation for cooperation in the Prisoner's Dilemma is that the players interact repeatedly. But if this is the case, then the actual game isn't the Prisoner's Dilemma as written above but a more complicated game called a repeated game. If, in this repeated game, players are sufficiently patient, then there are NE that are cooperative: the players play (C, C) in every period, and this cooperation is enforced by the threat of retaliation in future periods if either player ever deviates and plays D.

As this example illustrates, noncooperative game theory requires that the analyst specify the strategic options for the players correctly: if the game is played repeatedly, or if players can negotiate, form coalitions, or make binding agreements, then all of that should be represented in the strategic form. By highlighting both individual optimization and the importance of the fine details of the strategic environment, non-cooperative game theory allows us to investigate when, or to what degree, cooperation can be sustained. Such questions could not even be posed within the research program advocated by VN-M.

Photo credit: Princeton University.

A press conference in Princeton on occasion of Nash winning the 1994 Nobel Prize. At left facing the camera is Princeton mathematician and game theorist Harold Kuhn.

Noncooperative game theory has become the dominant branch of game theory, and research on noncooperative game theory began with Nash's formulation of NE, [37] and [38]. It was appropriate, therefore, that the 1994 Sveriges Riksbank Prize in Economic Science in Memory of Alfred Nobel (the Nobel Prize in Economics), which Nash shared with two other prominent game theorists, cited Nash not only for Nash equilibrium, but also for launching noncooperative game theory as a whole.

Additional Reading

For more on game theory generally, see [17] and [48]. For motivation for, and interpretation of, NE, see [8] (introspective reasoning), [36] (learning), and [49] (evolution). For a gloss on whether NE is predictively accurate, and why testing this is not straightforward, see [29]. For connections between cooperative and noncooperative game theory (often called the Nash program), see [52]. Finally, see [35] for a more thorough history of NE. In particular, [35] discusses at length an issue that we omitted: the relationship between Nash's work and that of Cournot ([13]).

Henry C. King

Nash's Work on Algebraic Structures

I first learned of Nash's work on algebraic structures from Dick Palais who shaped my understanding of the subject. I never met Nash, but am grateful to him for the many enjoyable mathematical excursions his work made possible.

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An m-dimensional differentiable submanifold M of $\mathbb{R}^n$ is locally given as the zeroes of $n-m$ differentiable functions f_i with linearly independent gradients. By asking that each f_i be polynomial (or a generalization now called a Nash function[1]) we get an algebraic structure on M. In [39], Nash showed that any compact differentiable manifold M has a unique algebraic structure. The meat of this result is showing existence, in particular that M has a representation as a submanifold V_0 of $\mathbb{R}^n$ locally given by polynomials f_i as above. This is what Nash calls a proper representation: There is a real algebraic set $V \subset \mathbb{R}^n$; i.e., V is the set of solutions of a collection of polynomial equations in n variables, and V_0 is a union of connected components of V. If $V = V_0$ it is called a pure representation. There is also a plain old representation (where the f_i are Nash functions), an example being the image of a polynomial embedding of a proper representation.

Here are some examples if M is the circle. The algebraic set X in $\mathbb{R}^2$ given by $x^2+y^2=1$ is a pure representation of the circle. Let Y be the cubic $y^2 = x^3 - x$. The portion Y_0 of Y with $-1 \le x \le 0$ is a proper representation of the circle[2]. The cubic Y contains other points Y_1 with $x \ge 1$ but these are in a different connected component of Y. Now consider the image $p(Y)$ under the map

$$p(x,y) = (x - x^2(x+1)^2/2,\, y).$$

Elimination theory tells us that this image is an algebraic set Z, as long as we include any real images of complex solutions[3] of $y^2 = x^3 - x$. Then $p(Y_0)$ is a representation of the circle but is not proper since $p(Y_1)$ intersects $p(Y_0)$ at $(-1,0) = p(1,0) = p(-1,0)$.

Nash finds an algebraic representation of M by writing $M \subset \mathbb{R}^n$ as the zeroes of some differentiable functions, approximating these functions by polynomials, and concluding that the zeroes of the polynomials have connected components which are a slightly perturbed copy of M. Unfortunately, to make this work Nash must add some auxiliary variables and the proper representation ends up in $\mathbb{R}^{n+m}$.

Let $y(x)$ denote the closest point in M to x; then M is the zeroes of $x - y(x)$. Approximate $x - y(x)$ (and its derivatives) near M by some polynomial $u(x)$. We would not expect the zeroes of u to approximate M; after all, $u(x) = 0$ is n equations in n unknowns so we expect its solutions to have dimension 0. Let $K(x)$ be the matrix of orthogonal projection to the $n-m$ plane normal to M at $y(x)$. If we could approximate $K(x)$ by a polynomial $P(x)$ so that $P(x)$ had rank $n-m$ near M we would be in business; $\{x \mid P(x)u(x) = 0\}$ would have connected components which are a perturbed copy of M. To see this, restrict to the plane normal to M at a point p, $K(p)P(x)u(x)$ approximates the identity and thus has a unique zero near p. But $K(p)P(x)u(x) = 0$ implies $P(x)u(x) = 0$ near p, so we have a one-to-one correspondence between M and the components of $\{x \mid P(x)u(x) = 0\}$ near M. If we approximate $K(x)$ by a polynomial $L(x)$ we would not expect L to have rank $n-m$. But let $\alpha(t) = t^m + \delta_1 t^{m-1} + \cdots + \delta_m = (t-r_1)(t-r_2)\cdots(t-r_m)$ where the r_i are the eigenvalues of $L(x)$ close to 0. Then $P(x) = \alpha(L(x))$ has rank $n-m$ and $P(x) \approx K^m(x) = K(x)$. The coefficients δ_i are polynomially related to x, set to 0 the remainder of the quotient of the characteristic polynomial of $L(x)$ by α. So at the expense of adding the auxiliary variables δ_i, we can perturb M to a proper algebraic representation.

Nash's paper mentions the following questions, among others.

(1) Can every compact differentiable submanifold M of $\mathbb{R}^n$ be approximated by a proper algebraic representation in $\mathbb{R}^n$? He tried proving this without success.
(2) Can every compact differentiable submanifold M of $\mathbb{R}^n$ be approximated by a pure algebraic representation in $\mathbb{R}^n$? He speculated that this is plausible.
(3) Does every compact differentiable manifold M have a pure algebraic representation in some $\mathbb{R}^n$? He thought this was probably true.

In [57], Wallace claimed to prove conjecture 1. Unfortunately, there was a serious error (he neglected to include the real images of complex solutions in his projections). However he did prove conjecture 3 in the case where M is the boundary of a compact differentiable manifold W. Glue two copies of W together along M. By Nash, we may assume this is a component V_0 of an algebraic subset V of some $\mathbb{R}^n$. Let f be a differentiable function which is positive on one copy of W, negative on the other copy of W, zero on M, and positive on $V - V_0$. Approximate f by a polynomial p and then $V \cap p^{-1}(0)$ is a pure representation of M.

In [53], Tognoli proved conjecture 3 by greatly improving on this idea of Wallace. By work of Thom and Milnor, we know that any compact differentiable manifold M is cobordant to a nonsingular real algebraic set S; i.e., there is a compact differentiable manifold W whose boundary is $M \cup S$ where S is a pure representation of some manifold. Glue two copies of W together along their boundaries. Tognoli then does a careful version of Nash to make the result a component V_0 of a real algebraic set V so that $S \subset V$ is still a nonsingular algebraic set. Let f be a differentiable function which is positive on one copy of W, negative on the other copy of W, zero on M and S, and positive on $V - V_0$. Approximate f by a polynomial p, being careful to ensure that p still vanishes on S, and then $V \cap p^{-1}(0) = M' \cup S$ is an algebraic set with M' diffeomorphic to M. It turns out that M' is by itself an algebraic set and the conjecture is proven.

This method of Tognoli ends up being very useful and gives us a general rule of thumb: If a differentiable situation is cobordant to a real algebraic situation, then it can be perturbed to be real algebraic.

[1] *Nash functions are only needed for uniqueness; we shall ignore them here.*

[2] *The map* $(x,y) \mapsto (2x+1,\, 2y/\sqrt{1-x})$ *gives a Nash diffeomorphism from* Y_0 *to* X.

[3] *We'll have to include* $(2-2\sqrt{3}, \pm\sqrt[4]{12}) = p((-\sqrt{3} \pm \sqrt{-5})/2, \pm\sqrt[4]{12})$.

In [54], Tognoli claimed to prove conjecture 2 but the proof had serious errors detailed in [2]. This inspired Akbulut and me to prove Conjecture 1 [2] and to use the above rule of thumb [1] to reduce conjecture 2 to a cobordism statement: A compact differentiable submanifold M of $\mathbb{R}^n$ can be approximated by a pure representation if and only if there is a compact differentiable submanifold W of $\mathbb{R}^n \times [0,1]$ whose boundary is $M \times 0 \cup S \times 1$ where $S \subset \mathbb{R}^n$ is a pure representation of some manifold. The proof in [2] consists of being careful with the images of complex solutions. Nash's proof gives a nonsingular component V_0 of a real algebraic set V and a polynomial embedding $p\colon V_0 \to \mathbb{R}^n$ so that $p(V_0)$ is a perturbation of M. We alter one coordinate of p to make sure that $p(V - V_0)$ is far from $p(V_0)$ and also any real images of nonreal solutions of the polynomial equations of V lie far from $p(V_0)$. Then $p(V_0)$ is a proper representation approximating M.

Photo credit: Robert P. Matthews, Princeton University.

Camillo De Lellis and László Székelyhidi Jr.

Nash's Work on Isometric Embeddings

Nash wrote three papers on isometric embeddings of Riemannian manifolds in Euclidean space, which are landmark papers not only for the mathematical problem they solved, but more importantly because of the impact they had on other fields, encompassing applications that go well beyond differential geometry. In these papers Nash studied the following problem:

> Given a smooth compact n-dimensional Riemannian manifold M with metric g, can we find an embedding of M into some Euclidean space $\mathbb{R}^N$ which preserves the metric structure?

This was a fundamental issue, aimed at linking the notion of submanifolds of $\mathbb{R}^N$, and hence of classical surfaces, to the abstract concept arising from the pioneering work of Riemann and his contemporaries.

In the statement of the problem there are two complementary requirements on the map $u : M \to \mathbb{R}^N$:

(i) it should be a topological embedding, that is, continuous and injective;

(ii) it should be continuously differentiable and preserve the length of curves; in other words the length of any rectifiable curve $\gamma \subset M$ should agree with the length of its image $u(\gamma) \subset \mathbb{R}^N$:

$$\ell(u \circ \gamma) = \ell(\gamma) \quad \text{for all rectifiable } \gamma \subset M. \tag{1}$$

In local coordinates the condition (ii) amounts to the following system of partial differential equations

$$\sum_{k=1}^{N} \partial_i u_k \partial_j u_k = g_{ij}, \quad i,j = 1 \dots n \tag{2}$$

consisting of $s_n := n(n+1)/2$ equations in N unknowns.

An important relaxation of the concept above is that of *short embedding*. A C^1 embedding $u : M \to \mathbb{R}^N$ is called *short* if it reduces (rather than preserving) the length of all curves, i.e. if (1) holds with $\leq$ replacing the equality sign. In coordinates this means that $(\partial_i u \cdot \partial_j u) \leq (g_{ij})$ in the sense of quadratic forms.

Nash realized that given a smooth embedding $u : M \to \mathbb{R}^N$, which is not necessarily isometric but it is short, one may try to solve (2) via *local perturbations* which are small in C^0, because being an embedding is a stable property with respect to a large class of such perturbations (since (2) alone guarantees that the differential of u has maximal rank, i.e. that u is an immersion). Let us assume for simplicity that $g \in C^\infty$. The three main theorems concerning the solvability of the system of partial differential equations (2) are the following:

(A) If $N \geq n+1$, then any short C^1 embedding can be uniformly approximated by isometric embeddings of class C^1 (Nash [40] proved the statement for $N \geq n+2$, Kuiper [28] extended it to $N = n+1$) .

(B) If $N \geq s_n + \max\{2n, 5\}$, then any short C^1 embedding can be uniformly approximated by isometric embeddings of class C^∞ (Nash [41] proved the *existence of isometric embeddings* for $N \geq 3s_n + 4n$; the approximation statement above was first shown by Gromov and Rokhlin for $N \geq s_n + 4n + 5$ [20]; subsequently the threshold was lowered by Gromov [19] to $N \geq s_n + 2n + 3$ and by Günther [21] to $N \geq s_n + \max\{2n, 5\}$, see also [22]).

(C) If g is real analytic and $N \geq s_n + 2n + 3$, then any short C^1 embedding can be uniformly approximated by analytic isometric embeddings (Nash [43] extended his C^∞ existence theorem to the analytic case, whereas the approximation statement was shown first by Gromov for $N \geq s_n + 3n + 5$ [18] and lowered to the threshold above [19]).

Nash's papers on isometric embeddings of Riemannian manifolds in Euclidean space are landmark papers.

Corresponding theorems can also be proved for noncompact manifolds M, but they are more subtle (for instance the noncompact case of (C) was left in [43] as an open problem; we refer the reader to [18], [19] for more details).

For M compact, any C^1 embedding of M into $\mathbb{R}^N$ can be made short after multiplying it by a sufficiently small

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The award ceremony of the Abel prize, with King Harald V of Norway and Louis Nirenberg. Courtesy of The Norwegian Academy of Science and Letters.

constant. Thus, (A), (B) and (C) are not merely existence theorems: they show that the set of solutions is huge (essentially C^0-dense). Naively, this type of flexibility could be expected for high codimension as in (B) and (C), since then there are many more unknowns than equations in (2). Statement (A) on the other hand is rather striking, not just because the problem is formally over-determined in dimension $n \geq 3$, but also when compared to the classical rigidity result concerning the Weyl problem: if (S^2, g) is a compact Riemannian surface with positive Gauss curvature and $u \in C^2$ is an isometric immersion into $\mathbb{R}^3$, then u is uniquely determined up to a rigid motion [11, 24]. Notice on the other hand that if u is required merely to be Lipschitz, then condition (ii) still makes sense in the form (1) and it is not difficult to construct a large class of non-equivalent isometric embeddings of any (orientable) surface in $\mathbb{R}^3$: just think of crumpling paper!

The results (A) and (B)-(C) rely on two, rather different, iterative constructions, devised by Nash to solve the underlying set of equations (2). In order to explain the basic idea, let us write (2) in short-hand notation as

$$du \cdot du = g. \tag{3}$$

Assuming that we have an approximation u_k, i.e. such that $\varepsilon_k := \|du_k \cdot du_k - g\|_{C^0}$ is small, we wish to add a perturbation w_k so that $u_{k+1} := u_k + w_k$ is a better approximation. The quadratic structure of the problem yields the following equation for w_k:

$$[dw_k \cdot du_k + du_k \cdot dw_k] + dw_k \cdot dw_k = g - du_k \cdot du_k.$$

A basic geometric insight in both constructions is that, assuming u_k is a short embedding, the perturbation w_k should *increase* lengths and thus it makes sense to choose w_k normal to the image $u_k(M)$. This amounts to the differential condition $du_k \cdot w_k = 0$, from which one easily deduces $du_k \cdot dw_k = -d^2u_k \cdot w_k$.

For the construction in (B)-(C) the idea is now to follow the Newton scheme: assuming that w_k and dw_k are comparable and small, $dw_k \cdot dw_k$ is much smaller than the linear term $[dw_k \cdot du_k + du_k \cdot dw_k]$, hence a good approximation can be obtained by solving for w_k the *linearization*

$$[dw_k \cdot du_k + du_k \cdot dw_k] = g - du_k \cdot du_k.$$

This can be reduced to an algebraic system for w_k by using $du_k \cdot w_k = 0$ and $du_k \cdot dw_k = -d^2u_k \cdot w_k$. The central analytic difficulty in carrying out the iteration is that, by solving the corresponding algebraic system, estimates on w_k will depend on estimates of d^2u_k - the mathematical literature refers to this phenomenon as *loss of derivative* and Nash dealt with this by introducing an additional regularization step.

The latter obviously perturbs the estimates on how small $u_{k+1} - u_k$ is. However, Nash's key realization is that Newton-type iterations converge so fast that such loss in the regularization step does not prevent the convergence of the scheme. Regularizations are obviously easier in the C^∞ category, where for instance standard convolutions with compactly supported mollifiers are available. It is thus not surprising that the real analytic case requires a subtler argument and this is the reason why Nash dealt with it much later in the subsequent paper [43].

Nash's scheme has numerous applications in a wide range of problems in partial differential equations where a purely functional-analytic implicit function theorem fails. The first author to put Nash's ideas in the framework of an abstract implicit function theorem was J. Schwartz, cf. [51]. However the method became known as the Nash-Moser iteration shortly after Moser succeeded in developing a general framework going beyond an implicit function theorem, which he applied to a variety of problems in his fundamental papers [32], [33], in particular to the celebrated KAM theory. Several subsequent authors generalized these ideas and a thorough mathematical theory has been developed by Hamilton [23], who defined the categories of "tame Fréchet spaces" and "tame nonlinear maps."

It is rather interesting to notice that in fact neither the results in (B) nor those in (C) ultimately really needthe Nash-Moser hard implicit function theorem. In fact in case (B) Günther has shown that the perturbation w_k can be generated inverting a suitable elliptic operator and thus appealing to standard contraction arguments in Banach spaces. Case (C) can instead be reduced to the local solvability of (2) in the real analytic case (already known in the thirties, cf. [25], [9]); such reduction uses another idea of Nash on approximate decompositions of the metric g (compare to the decomposition in primitive metrics explained below).

Contrary to the iteration outlined above to handle the results in (B) and (C), in the construction used for (A) w_k and dw_k have different orders of magnitude. More precisely, if w_k is a highly oscillatory perturbation of the type

$$w_k(x) \sim \mathrm{Re}\left(\frac{a_k(x)}{\lambda_k} e^{i\lambda_k x\cdot\xi_k}\right), \tag{4}$$

then the linear term is $O(\lambda_k^{-1})$ whereas the quadratic term is $O(1)$. For the sake of our discussion, assume for the moment the following:

(*) w_k can be chosen with oscillatory structure (4) in such a way that $dw_k \cdot dw_k \sim g - du_k \cdot du_k$.

Then the amplitude of the perturbation will be $\|a_k\|_{C^0} \sim \|g - du_k \cdot du_k\|_{C^0}^{1/2}$ whereas the new error will be $\varepsilon_{k+1} = O(\lambda_k^{-1})$. Since

$$\|du_{k+1} - du_k\|_{C^0} = \|dw_k\|_{C^0} \sim \|a_k\|_{C^0},$$

the C^1 convergence of the sequence u_k is guaranteed when $\sum_k \sqrt{\varepsilon_k} < \infty$, which is easily achieved by choosing a sequence λ_k which blows up sufficiently rapidly. Furthermore, $\|u_{k+1} - u_k\|_{C^0} = O(\lambda_k^{-1})$, so that topological properties of the map u_k (e.g. being an embedding) will be easily preserved. On the other hand it is equally clear that in this way $\|u_k\|_{C^2} \to \infty$, so that the final embedding will be C^1 but not C^2.

Nirenberg did not hesitate to use the word 'genius'.

It should be added that in fact it is not possible to achieve (*) as stated above: it is easy to check that a single oscillatory perturbation of the type (4) adds a rank-1 tensor to $du_k \cdot du_k$, modulo terms of order $O(\lambda_k^{-1})$. Nash overcame this difficulty by decomposing $g - du_k \cdot du_k$ as a sum of finitely many (symmetric and positive semidefinite) rank-1 tensors, which nowadays are called *primitive metrics*: the actual iterative step from u_k to u_{k+1} consists then in the (serial) addition of finitely many oscillatory perturbations of type (4).

Nash's iteration served as a prototype for a technique developed by Gromov, called convex integration, which unraveled the connection between the Nash-Kuiper theorem and several other counterintuitive constructions in geometry, cf. [19]. In recent decades this technique has been applied to show similar phenomena (called h-principle statements) in many other geometric contexts. More recently, Müller and Šverak [34] discovered that a suitable modification of Gromov's ideas provides a further link between the geometric instances of the h-principle and several theorems with the same flavor proved in the 1980s and in the 1990s in partial differential equations. This point of view can be used to explain the existence of solutions to the Euler equations that do not preserve the kinetic energy, cf. [15]. Although the latter phenomenon was discovered only rather recently in the mathematical literature by Scheffer [50], in the theory of turbulence it was predicted already in 1949 by a famous paper of Onsager, cf. [47]. Mil

Even nowadays the Nash-Kuiper theorem defies the intuition of most scholars. In spite of the fact that Nash's iteration is constructive and indeed rather explicit, its numerical implementation has been attempted only in the last few years. After overcoming several hard computational problems, a team of French mathematicians have been able to produce its first computer-generated illustrations, cf. [7].

Photo: Eirik Furu Baardsen.

Nash at the Abel Lectures. Courtesy of the University of Oslo.

Cedric Villani

On Nash's Regularity Theory for Parabolic Equations in Divergence Form

In the fall of 1958 the *American Journal of Mathematics* published what may possibly be, to this date, the most famous article in its long history: *Continuity of solutions of elliptic and parabolic equations*, by John Nash. At twenty-four pages, this is a quite short paper by modern standards in partial differential equations; but it was solving a major open problem in the field, and was immediately considered by experts (Carleson, Nirenberg, Hörmander, to name just a few) as an extraordinary achievement. Nirenberg did not hesitate to use the word "genius" to comment on the paper; as for me, let me say that I remember very well the emotion and marvel which I felt at studying it, nearly forty years after its writing.

Here is one form of the main result in Nash's manuscript.

Theorem 4. *Let $a_{ij} = a_{ij}(x,t)$ be a $n\times n$ symmetric matrix depending on $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$. Assume that (a_{ij}) is uniformly elliptic, that is*

$$\forall \xi \in \mathbb{R}^n, \qquad \lambda|\xi|^2 \le \sum_{ij} a_{ij}\,\xi_i\,\xi_j \le \Lambda|\xi|^2, \tag{1}$$

for some positive constants λ and Λ. Let $f = f(x,t) \ge 0$ solve the divergence form linear parabolic equation

$$\frac{\partial f}{\partial t} = \sum_j \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial f}{\partial x_j}\right)$$

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in $\mathbb{R}^n \times \mathbb{R}_+$. Then f is automatically continuous, and even C^α (Hölder-continuous) for some exponent $\alpha > 0$, when $t > 0$. The exponent α, as well as a bound on the C^α norm, can be made explicit in terms of λ, Λ, n and the (time-independent) $L^1(\mathbb{R}^n)$ norm of f.

The two key features in the assumptions of this theorem are that

(a) *no regularity assumption* of any kind is made on the diffusion matrix: the coefficients a_{ij} should just be measurable, and this is in contrast with the older classical regularity theories for parabolic equations, which required at least Hölder continuity of the coefficients;

(b) Equation (1) is in divergence form; actually, equations in nondivergence form would later be the object of a quite different theory pioneered by Krylov and Safonov.

The fact that the equation is of parabolic nature, on the other hand, is not so rigid: elliptic equations can be considered just the same, as a particular, stationary, case. Also, this theorem can be localized by classical means and considered in the geometric setting of a Riemannian manifold.

The absence of regularity assumptions on the diffusion matrix makes it possible to use this theorem to study nonlinear diffusion equations with a nonlinear dependence between the diffusion matrix and the solution itself. In this spirit, Nash hoped that these new estimates would be useful in fluid mechanics. Still, the first notable use of this theorem was the solution of Hilbert's nineteenth problem on the analyticity of minimizers of functionals with analytic integrand. Namely, consider a nonnegative minimizer for $\int L(\nabla v(x))\,dx$, with a uniformly convex analytic L: is v analytic too? Classical calculus of variations shows that $C^{1,\alpha}$ solutions are analytic; then Nash's estimate completes the proof by establishing the Hölder-continuity of ∇v. Indeed, if v is a minimizer, then for any index k, $u = \partial_k v$ solves the divergence form linear elliptic Euler–Lagrange equation $\sum_{ij} \partial_i (a_{ij} \partial_j u) = 0$, where $a_{ij} = \partial^2_{ij} L(\nabla v)$ is uniformly elliptic (this requires a few clever manipulations of mixed derivatives). Note that in this case, when we apply the theorem, absolutely nothing is known on the regularity of a_{ij}, which directly depends on the unknown function v.

Still, it is not only its contents, and this foray into Hilbert's problem, that would make this paper unique, but also the amazing set of circumstances and human passion surrounding it.

First, although a complete outsider in the field, Nash had managed to solve in just a few months the problem, which had been submitted to him by Nirenberg.

Then it was discovered by accident that De Giorgi—future icon, but completely unknown at the time—had just published an alternative solution [14], in the form of an even shorter article in a journal that was obscure (at least in comparison with the AJM). For decades to come, the coincidence of the solutions of Nash and De Giorgi would be regarded by all analysts as the example *par excellence* of simultaneous discovery.

As for his own paper, Nash, amazingly, withdrew it immediately upon its acceptance by *Acta Mathematica*, where the referee was none other than Hörmander; and he resubmitted it to the AJM, in an unsuccessful hope of winning the 1959 Bôcher Prize. Just a few months later, Nash's health would deteriorate to a point that would (among other much more tragical consequences) stop his scientific career for many years, leaving him only a couple of later opportunities for additional contributions.

In spite of all this, when I read the detailed account by Nasar [44, Chapters 30–31] or when I had the opportunity to discuss with a prime witness like Nirenberg, what most fascinated me was the genesis of the paper. (How I would have loved that the movie *A Beautiful Mind* pay proper tribute to this truly inspiring adventure, rather than choosing to forget the science and focus on the illness with such heavy pathos.)

In order to get to his goal, Nash had not developed his own tools, but rather orchestrated fragmented efforts from his best fellow analysts, combining his own intuition with the skills of specialists. A typical example is Nash's interpolation inequality

$$(2)\qquad \int_{\mathbb{R}^n} f^2\,dx \le C(n) \left(\int_{\mathbb{R}^n} |\nabla_x f|^2\,dx\right)^{1-\theta} \left(\int_{\mathbb{R}^n} f\,dx\right)^{2\theta}, \qquad \theta = \frac{2}{n+2}.$$

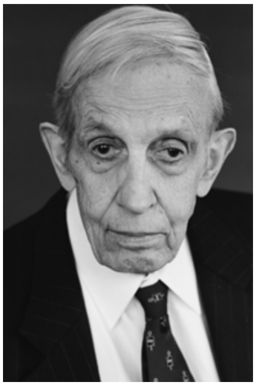
As Nash acknowledged in the manuscript, this inequality was actually proven, on his request, by Stein; but it was Nash who understood the crucial role that it could play in the regularity theory of diffusion processes, and which has been later explored in great generality.

> *One should praise Nash's informal style, intended to convey not only the proof, but also the ideas underlying it.*

Another example is the jaw-dropping use of Boltzmann's entropy, $S = -\int f \log f$, completely out of context. Entropy became famous as a notion of disorder or information, mainly in statistical physics; but it certainly had nothing to do with a regularity issue. Still, Nash brilliantly used the entropy to measure the spreading of a distribution, and related this spreading to the smoothing. Again, the tool was borrowed from somebody else: I learnt from Carleson that it was him who initiated Nash to the notion of entropy. This was the start of a long tradition of using nonlinear integral functionals of the solution as an approach to regularity bounds.

The next thing that one should praise is Nash's informal style, all intended to convey not only the proof, but also the ideas underlying it – or "powerful."

But then, it is also the construction of the proof which is a work of art. Nash uses a rather visual strategy, inspired by physics: think of the solution as the spreading of some quantity of heat, But then, it is also the construction of the proof which



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and be interested in the contribution of an initial point source of heat; displacement of "sources of heat" will imply strict positivity, which in turn will imply overlapping of nearby contributions, which in turn will imply the continuity. He also uses fine tactics, in particular to find dynamical relations between appropriate "summary" quantities. As a typical start: Nash shows how the L^2 norm of the solution has to decrease immediately, which implies an unconditional bound on the maximum temperature, which in turn implies a lower bound on the entropy. Then he shows that entropy goes with spreading (high entropy implies spreading; but through diffusion, spreading increases entropy). These ideas have been quite influential, and can be found again, for instance, in the beautiful work [10] by Carlen and Loss on the 2-dimensional incompressible Navier–Stokes equation.

Various authors rewrote, simplified and pushed further the De Giorgi–Nash theory. The two most important contributors were Moser [31] and Aronson [3]. Moser introduced the versatile Moser iteration, based on the study of the time-evolution of successive powers, which simplifies the proof and avoids the explicit use of the entropy. (Entropy is a way to consider the regime $p \to 1$ in the L^p norm; a dual approach is to consider the regime $p \to \infty$ as Moser.) Moser further proved what can be called the Moser–Harnack inequality: positive solutions of an elliptic divergence equation satisfy an estimate of the form

$$\sup_{B(x,r)} f \le C \inf_{B(x,2r)} f,$$

where C only depends on r, n and the ellipticity bounds. As for Aronson, he established a Gaussian-type bound on the associated heat kernel: $p_t(x,y)$ is bounded from above and below by functions of the form

$$\frac{K}{t^{n/2}}\, e^{-B|x-y|^2/t}.$$

These three results—the Hölder continuity, the Moser–Harnack inequality, and the Gaussian type bounds—are all connected and in some sense equivalent. Fine expositions of this can be found in Bass [5] (Chapter 7), [6], and Fabes & Stroock [16]. They have also been extended to nonsmooth geometries. Actually, these techniques have been so successful that some elements of proof now look so familiar even when we are not aware of it!

To conclude this exposition, following Fabes & Stroock, here is a brief sketch of the proof of Aronson's upper bound, using Nash's original strategy. By density, we may pretend that f is smooth, so it is really about an a priori estimate. First fix $q \in (1,\infty)$ and consider the time-evolution of the power q of the solution: the divergence assumption leads to a neat dissipation formula,

$$\frac{d}{dt}\int f^q = -q(q-1)\int \langle a\nabla f, \nabla f\rangle f^{q-2} \le -Kq(q-1)\int |\nabla f|^2 f^{q-2}.$$

Using the chain-rule, we deduce

$$\frac{d}{dt}\int f^q \le -K\left(\frac{q-1}{q}\right)\int |\nabla f^{q/2}|^2.$$

Now, the Nash inequality (2) tells us that the integral on the right-hand side controls a higher power of the integral on the left-hand side: more precisely, if, say, $q \ge 2$,

$$\frac{d}{dt}\int f^q \le -K\frac{(\int f^q)^{1+\beta}}{\int f^{q/2}},$$

for some $\beta = \beta(n) > 0$. This relates the evolution of the L^q norm and the evolution of the $L^{q/2}$ norm; it implies a bound for $\|f\|_{L^q}$ in terms of t and $\|f\|_{L^{q/2}}$, which can be made explicit after some work. Iterating this bound up to infinity, we may obtain an estimate on $\|f\|_{L^p}$ as $p \to \infty$, and eventually to $\|f\|_{L^\infty}$: writing $f_0 = f(0,\cdot)$ we have

$$\|f\|_{L^\infty} \le \frac{C}{t^{n/4}}\|f_0\|_{L^2}.$$

Combining this with the dual inequality

$$\|f\|_{L^2} \le \frac{C}{t^{n/4}}\|f_0\|_{L^1}$$

(which can also be proven from Nash's inequality), we obtain

$$\|f\|_{L^\infty} \le \frac{C}{t^{n/2}}\|f_0\|_{L^1}.$$

This is the sharp L^∞ estimate in short time. Now do all the analysis again with f replaced by $f\,e^{-\alpha\cdot x}$, for some $\alpha \in \mathbb{R}^n$. Error terms will arise in the differential equations, leading to

$$\frac{d}{dt}\|f\|_{L^q} \le -\frac{K}{q}\|f\|_{L^q}^{1+\beta q/2}\|f\|_{L^{q/2}}^{-\beta q/2} + \frac{|\alpha|^2 q}{2\lambda}\|f\|_{L^q}.$$

Iteration and the study of these ordinary differential inequalities will lead to a similar bound on $f\,e^{-\alpha\cdot x}$ as on f; after some optimization this will imply the Gaussian bound.

As can be seen, the method is elementary, but beautifully arranged, and obviously flexible. Whether in the original version, or in the modern rewritings, Nash's proof is a gem; or, to use the expression of Newton, a beautiful pebble.

References

1. S. Akbulut and H. King, *Algebraicity of immersions in* $\mathbf{R}^n$, Topology **31** (1992), no. 4, 701–712. MR 1191374 (94d:57055)
2. ———, *On approximating submanifolds by algebraic sets and a solution to the Nash conjecture*, Invent. Math. **107** (1992), no. 1, 87–98. MR 1135465 (93d:57051)
3. D. G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc. **73** (1967), 890–896. MR 0217444 (36 #534)
4. M. Artin and B. Mazur, *On periodic points*, Ann. of Math. (2) **81** (1965), 82–99. MR 0176482 (31 #754)

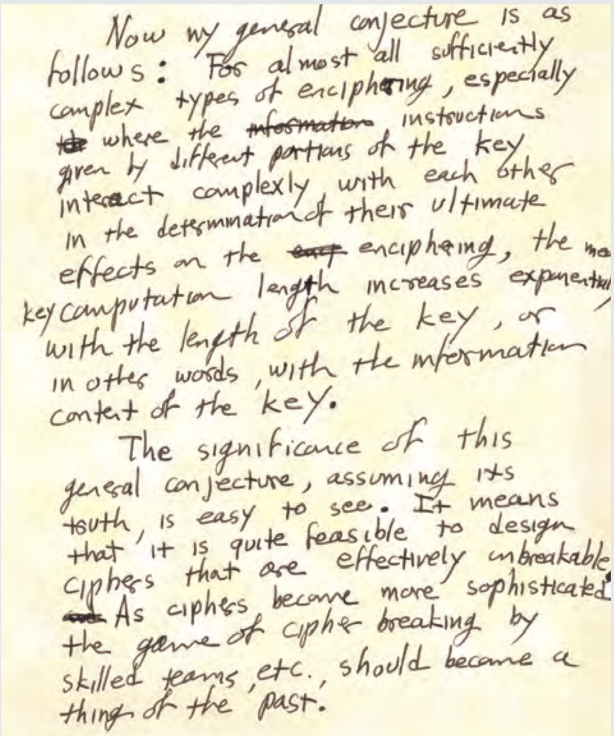
5. R. F. Bass, *Diffusions and elliptic operators*, Probability and its Applications (New York), Springer-Verlag, New York, 1998. MR 1483890 (99h:60136)
6. ______, *On Aronson's upper bounds for heat kernels*, Bull. London Math. Soc. **34** (2002), no. 4, 415–419. MR 1897420 (2003c:35054)
7. V. Borrelli, S. Jabrane, F. Lazarus, and B. Thibert, *Flat tori in three-dimensional space and convex integration*, Proc. Natl. Acad. Sci. USA **109** (2012), no. 19, 7218–7223. MR 2935570
8. A. Brandenburger, *Epistemic game theory: An overview*, The New Palgrave Dictionary of Economics, ed. Steven N. Durlauf & Lawrence E. Blume. Second Edition., Palgrave Mcmillan, 2008.
9. C. Burstin, *Ein Beitrag zum Problem der Einbettung der Riemannschen Räume in euklidischen Räumen*, Rec. Math. Moscou **38** (1931), no. 3-4, 74–85 (German).
10. E. A. Carlen and M. Loss, *Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2-D Navier-Stokes equation*, Duke Math. J. **81** (1995), no. 1, 135–157 (1996), A celebration of John F. Nash, Jr. MR 1381974 (96m:35199)
11. S. Cohn-Vossen, *Zwei Sätze über die Starrheit der Eiflächen*, Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. **1927** (1927), 125–134 (German).
12. S. A. Cook, *The complexity of theorem-proving procedures*, Proceedings of the Third Annual ACM Symposium on Theory of Computing (New York, NY, USA), STOC '71, ACM, 1971, pp. 151–158.
13. A. Cournot, *Researches into the mathematical principles of the theory of wealth*, Kelley, 1838, Translation from the French by Nathaniel T. Bacon. Translation publication date: 1960.
14. E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43. MR 0093649 (20 #172)
15. C. De Lellis and L. Székelyhidi, Jr., *Dissipative continuous Euler flows*, Invent. Math. **193** (2013), no. 2, 377–407. MR 3090182
16. E. B. Fabes and D. W. Stroock, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Rational Mech. Anal. **96** (1986), no. 4, 327–338. MR 855753 (88b:35037)
17. D. Fudenberg and J. Tirole, *Game theory*, MIT Press, Cambridge, MA, 1991. MR 1124618 (92m:90001)
18. M. L. Gromov, *Isometric imbeddings and immersions*, Dokl. Akad. Nauk SSSR **192** (1970), 1206–1209. MR 0275456 (43 #1212)
19. ______, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 9, Springer-Verlag, Berlin, 1986. MR 864505 (90a:58201)
20. M. L. Gromov and V. A. Rohlin, *Imbeddings and immersions in Riemannian geometry*, Uspehi Mat. Nauk **25** (1970), no. 5 (155), 3–62. MR 0290390 (44 #7571)
21. M. Günther, *Zum Einbettungssatz von J. Nash*, Math. Nachr. **144** (1989), 165–187. MR 1037168 (91m:58014)
22. ______, *Isometric embeddings of Riemannian manifolds*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 1137–1143. MR 1159298 (93b:53049)
23. R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 1, 65–222. MR 656198 (83j:58014)
24. G. Herglotz, *Über die Starrheit der Eiflächen*, Abh. Math. Sem. Univ. Hamburg **15** (1943), no. 1, 127–129. MR 3069737
25. M. Janet, *Sur la possibilité de plonger un espace riemannien donné à n dimensions dans un espace euclidien à $\frac{n(+1)}{2}$ dimensions.*, C. R. Acad. Sci., Paris **183** (1926), 942–943 (French).
26. S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. **8** (1941), 457–459. MR 0004776 (3,60c)
27. V. Yu. Kaloshin, *Generic diffeomorphisms with superexponential growth of number of periodic orbits*, Comm. Math. Phys. **211** (2000), no. 1, 253–271. MR 1757015 (2001e:37035)
28. N. H. Kuiper, *On C^1-isometric imbeddings. I, II*, Nederl. Akad. Wetensch. Proc. Ser. A. **58** (Indag. Math.) **17** (1955), 545–556, 683–689. MR 0075640 (17,782c)
29. D. Levine and J. Zheng, *The relationship of economic theory to experiments*, The Methods of Modern Experimental Economics, ed. Guillame Frechette & Andrew Schotter, Oxford University Press, 2010.
30. W. F. Lucas, *A game with no solution*, Bull. Amer. Math. Soc. **74** (1968), 237–239. MR 0220522 (36 #3581)
31. J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591. MR 0159138 (28 #2356)
32. ______, *A rapidly convergent iteration method and non-linear differential equations. II*, Ann. Scuola Norm. Sup. Pisa (3) **20** (1966), 499–535. MR 0206461 (34 #6280)
33. ______, *A rapidly convergent iteration method and non-linear partial differential equations. I*, Ann. Scuola Norm. Sup. Pisa (3) **20** (1966), 265–315. MR 0199523 (33 #7667)
34. S. Müller and V. Šverák, *Convex integration for Lipschitz mappings and counterexamples to regularity*, Ann. of Math. (2) **157** (2003), no. 3, 715–742. MR 1983780 (2005i:35028)
35. R. Myerson, *Nash equilibrium and the history of economic theory*, Journal of Economic Literature **37** (1999), 1067–1082.
36. J. Nachbar, *Learning and evolution in games: Belief learning*, The New Palgrave Dictionary of Economics, ed. Steven N. Durlauf & Lawrence E. Blume. Second Edition, Palgrave Mcmillan, 2008.
37. J. F. Nash, Jr., *Equilibrium points in n-person games*, Proc. Nat. Acad. Sci. U. S. A. **36** (1950), 48–49. MR 0031701 (11,192c)
38. ______, *Non-cooperative games*, Ann. of Math. (2) **54** (1951), 286–295. MR 0043432 (13,261g)
39. ______, *Real algebraic manifolds*, Ann. of Math. (2) **56** (1952), 405–421. MR 0050928 (14,403b)
40. ______, *C^1 isometric imbeddings*, Ann. of Math. (2) **60** (1954), 383–396. MR 0065993 (16,515e)
41. ______, *The imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), 20–63. MR 0075639 (17,782b)
42. ______, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954. MR 0100158 (20 #6592)
43. ______, *Analyticity of the solutions of implicit function problems with analytic data*, Ann. of Math. (2) **84** (1966), 345–355. MR 0205266 (34 #5099)
44. ______, *The essential John Nash*, Princeton University Press, Princeton, NJ, 2002, Edited by Harold W. Kuhn and Sylvia Nasar. MR 1888522 (2002k:01044)
45. J. F. Nash, Jr., *Le problème de Cauchy pour les équations différentielles d'un fluide général*, Bull. Soc. Math. France **90** (1962), 487–497. MR 0149094 (26 #6590)
46. J. F. Nash, Jr., *Arc structure of singularities*, Duke Math. J. **81** (1995), no. 1, 31–38 (1996), A celebration of John F. Nash, Jr. MR 1381967 (98f:14011)
47. L. Onsager, *Statistical hydrodynamics*, Nuovo Cimento (9) **6** (1949), no. Supplemento, 2 (Convegno Internazionale di Meccanica Statistica), 279–287. MR 0036116 (12,60f)

48. M. J. Osborne and A. Rubinstein, *A course in game theory*, MIT Press, Cambridge, MA, 1994. MR 1301776 (95k:90002)
49. W. Sandholm, *Learning and evolution in games: An overview*, The New Palgrave Dictionary of Economics, ed. Steven N. Durlauf & Lawrence E. Blume. Second Edition., Palgrave Mcmillan, 2008.
50. V. Scheffer, *An inviscid flow with compact support in space-time*, J. Geom. Anal. **3** (1993), no. 4, 343–401. MR 1231007 (94h:35215)
51. J. Schwartz, *On Nash's implicit functional theorem*, Comm. Pure Appl. Math. **13** (1960), 509–530. MR 0114144 (22 #4971)
52. R. Serrano, *Nash program*, The New Palgrave Dictionary of Economics, ed. Steven N. Durlauf & Lawrence E. Blume. Second Edition, Palgrave Mcmillan, 2008.
53. A. Tognoli, *Su una congettura di Nash*, Ann. Scuola Norm. Sup. Pisa (3) **27** (1973), 167–185. MR 0396571 (53 #434)
54. ______, *Any compact differentiable submanifold of* $\mathbf{R}^n$ *has an algebraic approximation in* $\mathbf{R}^n$, Topology **27** (1988), no. 2, 205–210. MR 948183 (89i:14016)
55. J. v. Neumann, *Zur Theorie der Gesellschaftsspiele*, Math. Ann. **100** (1928), no. 1, 295–320. MR 1512486
56. J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, N. J., 1947, 2d ed. MR 0021298 (9,50f)
57. A. H. Wallace, *Algebraic approximation of manifolds*, Proc. London Math. Soc. (3) **7** (1957), 196–210. MR 0087205 (19,320a)

"Perhaps the best undergraduate course, the course in which I learned the most, was the junior full year course in real analysis. The teacher was John F. Nash Jr. He was brilliant, arrogant, and eccentric. At this time he was in the midst of his spectacular work on embedding theorems, nevertheless, his course was meticulously prepared and beautifully presented. The course started with an introduction to mathematical logic and set theory and covered, with great originality, the central topics of analysis culminating in the study of differential and integral equations."

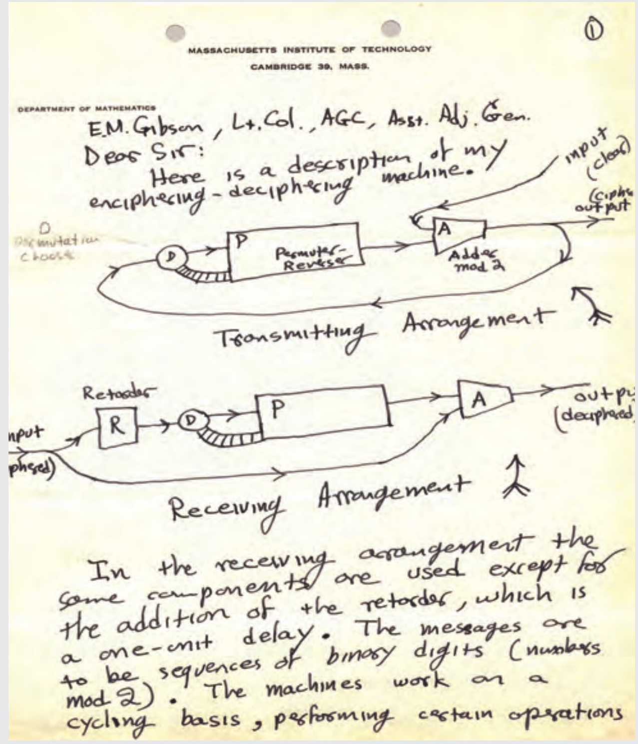
Joseph J. Kohn, "Mathematical Encounters", *All That Math*, Real Sociedad Matemática Española, 2011.

Nash and the NSA



Now my general conjecture is as follows: For almost all sufficiently complex types of enciphering, especially where the ~~information~~ instructions given by different portions of the key interact complexly with each other in the determination of their ultimate effects on the ~~enc~~ enciphering, the me key computation length increases exponentia with the length of the key, or in other words, with the information content of the key.

The significance of this general conjecture, assuming its truth, is easy to see. It means that it is quite feasible to design ciphers that are effectively unbreakable. As ciphers become more sophisticated the game of cipher breaking by skilled teams, etc., should become a thing of the past.



MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE 39, MASS.

DEPARTMENT OF MATHEMATICS

1

E.M. Gibson, Lt. Col., AGC, Asst. Adj. Gen.

Dear Sir:

Here is a description of my enciphering-deciphering machine.

In the receiving arrangement the same components are used except for the addition of the retarder, which is a one-unit delay. The messages are to be sequences of binary digits (numbers mod 2). The machines work on a cycling basis, performing certain operations

At left: An excerpt from a six-page letter Nash wrote to the NSA describing a conjecture that captures the transformation to modern cryptography, which occurred two decades after he wrote this letter.
At right: Diagrams Nash drew, as part of another multi-page letter to the NSA, describing an enciphering machine he invented.
Both formerly classified letters are now available in full at `https://www.nsa.gov/public_info/_files/nash_letters/nash_letters1.pdf`.

Above are excerpts from two Nash letters that the National Security Agency (NSA) declassified and made public in 2012. In these extraordinary letters sent to the agency in 1955, Nash anticipated ideas that now pervade modern cryptography and that led to the new field of complexity theory. (In the obituary for Nash that appears in this issue of the *Notices*, page 492, John Milnor devotes a paragraph to these letters.)

Nash proposed to the NSA the idea of using computational difficulty as a basis for cryptography. He conjectured that some encryption schemes are essentially unbreakable because breaking them would be computationally too difficult. He cannot prove this conjecture, he wrote, nor does he expect it to be proved, "[but] that does not destroy its significance." As Noam Nisan wrote in a February 2012 entry in the blog Turing's Invisible Hand (`https://agtb.wordpress.com`), "[T]his is exactly the transformation to modern cryptography made two decades later by the rest of the world (at least publicly…)."

Nash also discussed in the letters the distinction between polynomial time and exponential time computations, which is the basis for complexity theory. "It is hard not to compare this letter to Gödel's famous 1956 letter to von Neumann also anticipating complexity theory (but not cryptography)," Nisan writes. "That both Nash and Gödel passed through Princeton may imply that these ideas were somehow 'in the air' there."

The handwriting and the style of Nash's letters convey a forceful personality. One can imagine that the letters might not have been taken seriously at first by the NSA. "I hope my handwriting, etc. do not give the impression that I am just a crank or a circle-squarer," Nash wrote, noting that he was an assistant professor at the Massachusetts Institute of Technology.

After receiving a reply from the NSA, Nash sent another letter describing a specific "enciphering-deciphering machine" he had developed while at the RAND Corporation. At the Eurocrypt 2012 conference, Ron Rivest and Adi Shamir presented an analysis of the actual security level of Nash's proposed machine and found it was not as strong as Nash had thought (`www.iacr.org/conferences/eurocrypt2012/Rump/nash.pdf`). Their conclusion: "John Nash foresaw in 1955 many theoretical developments which would appear in complexity theory and cryptography decades later. However, he was a much better game theorist than a cryptographer…".

—Allyn Jackson

Open Problems in Mathematics

Just before he left to collect his Abel Prize in Oslo in May 2015, Nash was working with Princeton postdoc Michael Th. Rassias to finish up the preface to an extraordinary book they edited together called *Open Problems in Mathematics*. The book will be published later this year by Springer.

The book consists of seventeen expository articles, written by outstanding researchers, on some of the central open problems in the field of mathematics today. Each article is devoted to one problem or a "constellation of related problems," the preface says. Nash and Rassias do not claim the book represents all of the most important problems in mathematics; rather, it is "a collection of beautiful mathematical questions which were chosen for a variety of reasons. Some were chosen for their undoubtable importance and applicability, others because they constitute intriguing curiosities which remain unexplained mysteries on the basis of current knowledge and techniques, and some for more emotional reasons. Additionally, the attribute of a problem having a somewhat *vintage flavor* was also influential in our decision process."

Photo Princeton University, Office of Communications, Danielle Alio (2015).

Rassias talks to 2014 Abel Laureate Yakov Sinai as 2015 Abel Laureate Nash looks on.

Here is another taste of the book, this one from the introduction, titled "John Nash: Theorems and Ideas" and written by Mikhail Gromov: "Nash was solving classical mathematical problems, difficult problems, something that nobody else was able to do, not even to imagine how to do it… But what Nash discovered in the course of his constructions of isometric embeddings is far from 'classical'—it is something that brings about a dramatic alteration of our understanding of the basic logic of analysis and differential geometry. Judging from the classical perspective, what Nash has achieved in his papers is as impossible as the story of his life… [H]is work on isometric immersions…opened a new world of mathematics that stretches in front of our eyes in yet unknown directions and still waits to be explored."

Nash and Rassias first met in September 2014 in the common room of the Princeton mathematics building, Fine Hall. Nash was eighty-six years old and probably the most famous mathematician in the world, and Rassias a twenty-seven-year-old Princeton postdoc who hails from Greece and had just finished his PhD at the ETH in Zurich. A chemistry developed between the two mathematicians and precipitated their collaboration on *Open Problems in Mathematics*. A Princeton News article that appeared on the occasion of Nash receiving the 2015 Abel Prize discussed Rassias's interactions with Nash (`www.princeton.edu/main/news/archive/S42/72/29C63/index.xml?section=topstories`). Rassias is quoted as saying: "Working with him is an astonishing experience—he thinks differently than most other mathematicians I've ever met. He's extremely brilliant and has all this experience. If you were a musician and had an opportunity to work with Beethoven and compose music with him, it'd be astonishing. It's the same thing."

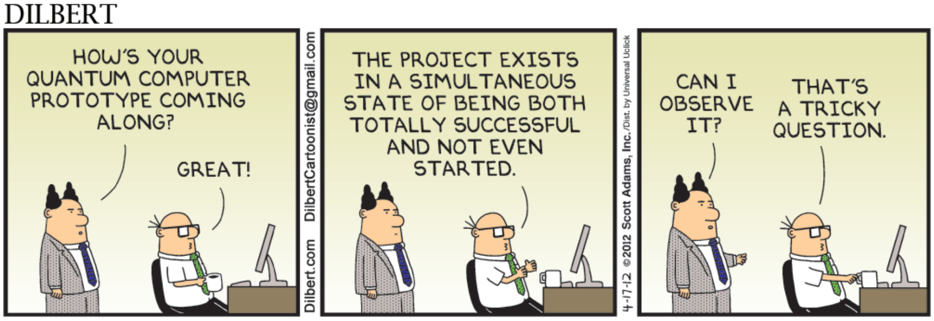
Table of Contents of *Open Problems in Mathematics* edited by John F. Nash Jr. and Michael Th. Rassias

The Quantum Computer Puzzle

Gil Kalai

Communicated by Joel Hass

DILBERT



Quantum computers are hypothetical devices, based on quantum physics, which would enable us to perform certain computations hundreds of orders of magnitude faster than digital computers. This feature is coined "quantum supremacy", and one aspect or another of such quantum computational supremacy might be seen by experiments in the near future: by implementing quantum error-correction or by systems of noninteracting bosons or by exotic new phases of matter called anyons or by quantum annealing, or in various other ways. We concentrate in this paper on the model of a universal quantum computer that allows the full computational potential for quantum systems, and on the restricted model, called "BosonSampling", based on noninteracting bosons.

A main reason for concern regarding the feasibility of quantum computers is that quantum systems are inherently noisy. We will describe an optimistic hypothesis regarding quantum noise that will allow quantum computing and a pessimistic hypothesis that won't. The *quantum computer puzzle* is to decide between these two hypotheses. We list some remarkable consequences of the optimistic hypothesis, giving strong reasons for the intensive efforts to build quantum computers, as well as good reasons for suspecting that this might not be possible. For systems of noninteracting bosons, we explain how quantum supremacy achieved without noise is replaced, in the presence of noise, by a very low yet fascinating computational power.[1] Finally, we describe eight predictions about quantum physics and computation from the pessimistic hypothesis.[2]

Are quantum computers feasible? Is quantum supremacy possible? My expectation is that the pessimistic hypothesis will prevail, leading to a negative answer. Rather than regarding this possibility as an unfortunate failure that impedes the progress of humanity, I believe that the failure of quantum supremacy itself leads to important consequences for quantum physics, the theory of computing, and mathematics. Some of these will be explored here.

A Brief Summary

Here is a brief summary of the author's pessimistic point of view as explained in the paper: understanding quantum computers in the presence of noise requires consideration of behavior at different scales. In the small scale, standard models of noise from the mid-90s are suitable, and quantum evolutions and states described by them manifest a very low-level computational power. This small-scale behavior has far-reaching consequences for the behavior of noisy quantum systems at larger scales. On the one hand, it does not allow reaching the starting points for quantum fault tolerance and quantum supremacy, making them both impossible at all scales. On the other hand, it leads to novel implicit ways for modeling noise at larger scales and to various predictions on the behavior of noisy quantum systems.

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[1] *Based on G. Kalai and G. Kindler, "Gaussian noise sensitivity and BosonSampling", arXiv:1409.3093.*

[2] *Based on G. Kalai, "How quantum computers fail: quantum codes, correlations in physical systems, and noise accumulation", arXiv:1106.0485, and a subsequent Internet debate with Aram Harrow and others.*

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The Vision of Quantum Computers and Quantum Supremacy

Circuits and Quantum Circuits

The basic memory component in classical computing is a bit, which can be in two states, “0” or “1”. A computer (or circuit) has n bits, and it can perform certain logical operations on them. The NOT gate, acting on a single bit, and the AND gate, acting on two bits, suffice for *universal* classical computing. This means that a computation based on another collection of logical gates, each acting on a bounded number of bits, can be replaced by a computation based only on NOT and AND. Classical circuits equipped with random bits lead to *randomized algorithms*, which are both practically useful and theoretically important.

Quantum computers (or circuits) allow the creation of probability distributions that are well beyond the reach of classical computers with access to random bits. A qubit is a piece of quantum memory. The state of a qubit can be described by a unit vector in a two-dimensional complex Hilbert space H. For example, a basis for H can correspond to two energy levels of the hydrogen atom or to horizontal and vertical polarizations of a photon. Quantum mechanics allows the qubit to be in a *superposition* of the basis vectors, described by an arbitrary unit vector in H. The memory of a quantum computer consists of n qubits. Let H_k be the two-dimensional Hilbert space associated with the kth qubit. The state of the entire memory of n qubits is described by a unit vector in the tensor product $H_1 \otimes H_2 \otimes \cdots \otimes H_n$. We can put one or two qubits through *gates* representing unitary transformations acting on the corresponding two- or four-dimensional Hilbert spaces, and as for classical computers, there is a small list of gates sufficient for universal quantum computing. Each step in the computation process consists of applying a unitary transformation on the large 2^n-dimensional Hilbert space, namely, applying a gate on one or two qubits, tensored with the identity transformation on all other qubits. At the end of the computation process, the state of the entire computer can be *measured*, giving a probability distribution on 0–1 vectors of length n.

A few words on the connection between the mathematical model of quantum circuits and quantum physics: In quantum physics, states and their evolutions (the way they change in time) are governed by the Schrödinger equation. A solution of the Schrödinger equation can be described as a unitary process on a Hilbert space, and quantum computing processes as we just described form a large class of such quantum evolutions.

A Very Brief Tour of Computational Complexity

Computational complexity is the theory of *efficient computations*, where “efficient” is an asymptotic notion referring to situations where the number of computation steps (“time”) is at most a polynomial in the number of input bits. The complexity class **P** is the class of algorithms that can be performed using a polynomial number of steps in the size of the input. The complexity class **NP** refers to nondeterministic polynomial time. Roughly speaking, it refers to questions where we can *provably* perform the task in a polynomial number of operations in the input size, provided we are given a certain polynomial-size “hint” of the solution. An algorithmic task A is **NP**-hard if a subroutine for solving A allows solving any problem in **NP** in a polynomial number of steps. An **NP**-complete problem is an **NP**-hard problem in **NP**. A useful analog is to think about the gap between **NP** and **P** as similar to the gap between finding a proof of a theorem and verifying that a given proof of the theorem is correct. **P** and **NP** are two of the lowest computational complexity classes in the *polynomial hierarchy* **PH**, which is a countable sequence of such classes, and there is a rich theory of complexity classes beyond **PH**.

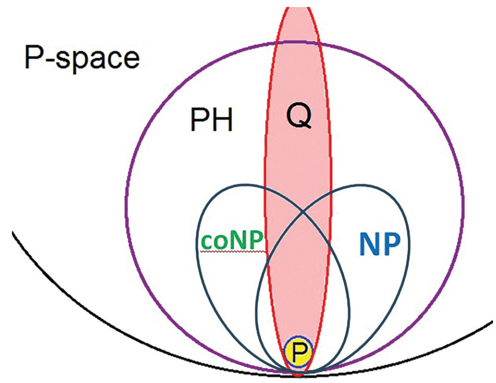

Figure 1. The (conjectured) view of some main computational complexity classes. The red ellipse represents efficient quantum algorithms.

There are intermediate problems between **P** and **NP**. Factoring an n-digit integer is not known to be in **P**, as the best algorithms are exponential in the cube root of the number of digits. Factoring is in **NP**, but it is unlikely that factoring is **NP**-complete. Shor’s famous algorithm shows that quantum computers can factor n-digit integers efficiently—in $\sim n^2$ steps! Quantum computers are not known to be able to solve efficiently **NP**-complete problems, and there are good reasons to think that they cannot. Yet, quantum computers can efficiently perform certain computational tasks beyond **NP**.

Two comments: First, our understanding of the computational complexity world depends on a whole array of conjectures: **NP** $\neq$ **P** is the most famous one, and a stronger conjecture asserts that **PH** does not collapse, namely, that there is a strict inclusion between the computational complexity classes defining the polynomial hierarchy. Second, computational complexity insights, while asymptotic, strongly apply to finite and small algorithmic tasks. Paul Erdős famously claimed that finding the value of the Ramsey function $R(n,n)$ for $n = 6$ is well beyond mankind’s ability. This statement is supported by computational complexity insights that consider the difficulty of computations as $n \to \infty$, while not directly implied by them.

Noise

Noise and Fault-Tolerant Computation

The main concern regarding the feasibility of quantum computers has always been that quantum systems are inherently noisy: we cannot accurately control them, and we cannot accurately describe them. To overcome this difficulty, a theory of quantum fault-tolerant computation based on quantum error-correction codes was developed.[3] Fault-tolerant computation refers to computation in the presence of errors. The basic idea is to represent (or "encode") a single piece of information (a bit in the classical case or a qubit in the quantum case) by a large number of physical components so as to ensure that the computation is robust even if some of these physical components are faulty.

The main concern regarding the feasibility of quantum computers has always been that quantum systems are inherently noisy.

What is noise? Solutions of the Schrödinger equation (quantum evolutions) can be regarded as unitary processes on Hilbert spaces. Mathematically speaking, the study of noisy quantum systems is the study of *pairs* of Hilbert spaces (H, H'), $H \subset H'$, and a unitary process on the larger Hilbert space H'. Noise refers to the general effect of neglecting degrees of freedom, namely, approximating the process on a large Hilbert space by a process on a small Hilbert space. For controlled quantum systems and, in particular, quantum computers, H represents the controlled part of the system, and the large unitary process on H' represents, in addition to an "intended" controlled evolution on H, also the uncontrolled effects of the *environment*. The study of noise is relevant not only to controlled quantum systems but also to many other aspects of quantum physics.

A second, mathematically equivalent way to view noisy states and noisy evolutions is to stay with the original Hilbert space H but to consider a mathematically larger class of states and operations. In this view, the state of a noisy qubit is described as a classical probability distribution on unit vectors of the associated Hilbert spaces. Such states are referred to as *mixed states*. It is convenient to think about the following form of noise, called *depolarizing noise*: in every computer cycle a qubit is not affected with probability $1 - p$, and, with probability p, it turns into the *maximal entropy mixed state*, i.e., the average of all unit vectors in the associated Hilbert space. In this example, p is the error rate, and, more generally, the error rate can be defined as the probability that a qubit is corrupted at a computation step conditioned on it surviving up to this step.

[3] *M. A. Nielsen and I. L. Chuang,* Quantum Computation and Quantum Information, *Cambridge University Press, Cambridge, 2000, Ch. 10.*

Two Alternatives for Noisy Quantum Systems

The quantum computer puzzle is, in a nutshell, deciding between two hypotheses regarding properties of noisy quantum circuits: the *optimistic hypothesis* and the *pessimistic hypothesis*.

Optimistic Hypothesis: It is possible to realize universal quantum circuits with a small bounded error level regardless of the number of qubits. The effort required to obtain a bounded error level for universal quantum circuits increases moderately with the number of qubits. Therefore, large-scale fault-tolerant quantum computers are possible.

Pessimistic Hypothesis: The error rate in every realization of a universal quantum circuit scales up (at least) linearly with the number of qubits. The effort required to obtain a bounded error level for any implementation of universal quantum circuits increases (at least) exponentially with the number of qubits. Thus, quantum computers are not possible.

Some explanations: For the optimistic hypothesis, we note that the main theorem of quantum fault tolerance asserts that (under some natural conditions on the noise) if we can realize universal quantum circuits with a sufficiently small error rate (where the threshold is roughly between 0.001 and 0.01), then quantum fault tolerance and hence universal quantum computing are possible. For the pessimistic hypothesis, when we say that the rate of noise per qubit scales up linearly with the number of qubits, we mean that when we double the number of qubits in the circuit, the probability for a single qubit to be corrupted in a small time interval doubles. The pessimistic hypothesis does not require new modeling for the noise for universal quantum circuits, and it is just based on a different assumption on the rate of noise. However, it leads to interesting predictions and modeling and may lead to useful computational tools, for more general noisy quantum systems. We emphasize that both hypotheses are assertions about physics (or physical reality), not about mathematics, and both of the hypotheses represent scenarios that are compatible with quantum mechanics.

The constants are important, and the pessimistic view regarding quantum supremacy holds that every realization of universal quantum circuits will fail for a handful of qubits long before any quantum supremacy effect is witnessed and long before quantum fault tolerance is possible. The failure to reach universal quantum circuits for a small number of qubits and to manifest quantum supremacy for small quantum systems is crucial for the pessimistic hypothesis, and Erdős's statement about $R(6,6)$ is a good analogy for this expected behavior.

Both on the technical and conceptual levels we see here what we call a "wide-gap dichotomy". On the technical level, we have a gap between small constant error rate per qubit for the optimistic view and linear increase of rate

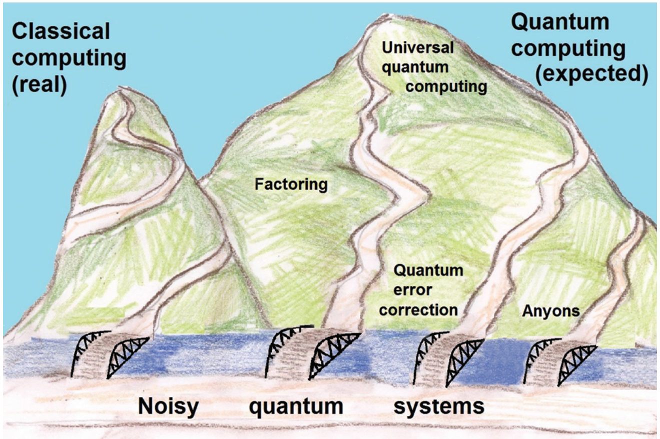

Figure 2. The optimistic hypothesis: Classical fault-tolerance mechanisms can be extended, via quantum error-correction, allowing robust quantum information and computationally superior quantum computation. Drawing by Neta Kalai.

per qubit (in terms of the number of qubits in the circuit) on the pessimistic side. We also have a gap between the ability to achieve large-scale quantum computers on the optimistic side and the failure of universal quantum circuits already for a handful of qubits on the pessimistic side. On the conceptual level, the optimistic hypothesis asserts that quantum mechanics allows superior computational powers, while the pessimistic hypothesis asserts that quantum systems without specific mechanisms for robust classical information that leads only to classical computing are actually computationally inferior. We will come back to both aspects of this wide-gap dichotomy.

Potential Experimental Support for Quantum Supremacy

A definite demonstration of quantum supremacy of controlled quantum systems—namely, building quantum systems that outperform, even for specific computational tasks, classical computers—or a definite demonstration of quantum error correction will falsify the pessimistic hypothesis and will give strong support for the optimistic hypothesis. (The optimistic hypothesis will be completely verified with full-fledged universal quantum computers.) There are several ways people plan, in the next few years, to demonstrate quantum supremacy or the feasibility of quantum fault tolerance.

(1) Attempts to create small universal quantum circuits with up to "a few tens of qubits."
(2) Attempts to create stable logical qubits based on surface codes.
(3) Attempts to have BosonSampling for 10–50 bosons.
(4) Attempts to create stable qubits based on anyonic states.
(5) Attempts to demonstrate quantum speed up based on quantum annealing.

Each of attempts (1)–(4) represents many different experimental directions carried out mainly in academic institutions, while (5) represents an attempt by a commercial company, D-wave.[4] There are many different avenues for realizing qubits, of which ion-trapped qubits and superconducting qubits are perhaps the leading ones. Quantum supremacy via nonabelian anyons stands out as a very different direction based on exotic new phases of matter and very deep mathematical and physical issues. BosonSampling (see the next section) stands out in the quest to demonstrate quantum supremacy for narrow physical systems without offering further practical fruits.

The pessimistic hypothesis predicts a decisive failure for *all* of these attempts to demonstrate quantum supremacy or very stable logical qubits and that this failure will be witnessed for small systems. A reader may ask how the optimistic hypothesis can be falsified beyond repeated failures to demonstrate universal quantum computers or partial steps toward them as those listed above. My view is that the optimistic hypothesis can be largely falsified if we can understand the absence of quantum supremacy and quantum error correction as a physical principle with predictive power that goes beyond these repeated failures, both in providing more detailed predictions about these failures themselves (such as scaling-up of errors, correlations between errors, etc.) and in providing predictions for other natural quantum systems. Mathematical modeling of noisy quantum systems based on the pessimistic hypothesis is valuable, not only if it represents a general physical principle, but also if it represents temporary technological difficulties or if it applies to limited classes of quantum systems.

BosonSampling

Quantum computers allow the creation of probability distributions that are beyond the reach of classical computers with access to random bits. This is manifested by BosonSampling, a class of probability distributions representing a collection of noninteracting bosons that quantum computers can efficiently create. It is a restricted subset of distributions compared to the class of distributions that a universal quantum computer can produce, and it is not known if BosonSampling distributions can be used for efficient integer factoring or for other "useful" algorithms. BosonSampling was introduced by Troyansky and Tishby in 1996 and was intensively studied by Aaronson and Arkhipov,[5] who offered it as a quick path for experimentally showing that quantum supremacy is a real phenomenon.

Given an n by n matrix A, let $det(A)$ denote the determinant of A, and let $per(A)$ denote the permanent of A. Thus $det(A) = \sum_{\pi \in S_n} sgn(\pi) \prod_{i=1}^{n} a_{i\pi(i)}$, and $per(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i\pi(i)}$. Let M be a complex $n \times m$

[4] *D-wave is attempting to demonstrate quantum speedup for* **NP***-hard optimization problems and even to compute Ramsey numbers.*

[5] *S. Aaronson and A. Arkhipov, "The computational complexity of linear optics", Theory of Computing* **4** *(2013), 143–252; arXiv:1011.3245.*

matrix, $m \geq n$. Consider all $\binom{m}{n}$ subsets S of n columns, and for every subset consider the corresponding $n \times n$ submatrix A. The algorithmic task of sampling subsets S of columns according to $|det(M')|^2$ is called FermionSampling. Next consider all $\binom{m+n-1}{n}$ submultisets S of n columns (namely, allow columns to repeat), and for every submultiset S consider the corresponding $n \times n$ submatrix A (with column i repeating r_i times). BosonSampling is the algorithmic task of sampling those multisets S according to $|per(A)|^2/(r_1!\, r_2! \cdots r_n!\,)$. Note that the algorithmic task for BosonSampling and FermionSampling is to sample according to a specified probability distribution. They are not decision problems, where the algorithmic task is to provide a yes/no answer.

Let us demonstrate these notions by an example for $n = 2$ and $m = 3$. The input is a 2×3 matrix:

$$\begin{pmatrix} 1/\sqrt{3} & i/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}.$$

The output for FermionSampling is a probability distribution on subsets of two columns, with probabilities given according to absolute values of the square of determinants. Here we have probability $1/6$ for columns $\{1, 2\}$, probability $1/6$ for columns $\{1, 3\}$, and probability $4/6$ for columns $\{2, 3\}$. The output for BosonSampling is a probability distribution according to absolute values of the square of permanents of submultisets of two columns. Here, the probabilities are: $\{1, 1\} \to 0$, $\{1, 2\} \to 1/6$, $\{1, 3\} \to 1/6$, $\{2, 2\} \to 2/6$, $\{2, 3\} \to 0$, $\{3, 3\} \to 2/6$.

FermionSampling describes the state of n noninteracting fermions, where each individual fermion is described as a superposition of m "modes". BosonSampling describes the state of n noninteracting fermions, where each individual fermion is described by m modes. A few words about the physics: Fermions and bosons are the main building blocks of nature. Fermions, such as electrons, quarks, protons, and neutrons, are particles characterized by Fermi–Dirac statistics. Bosons, such as photons, gluons, and the Higgs boson, are particles characterized by Bose–Einstein statistics.

Moving to computational complexity, we note that Gaussian elimination gives an efficient algorithm for computing determinants, but computing permanents is very hard: it represents a computational complexity class called #**P** (in words, "number **P**" or "sharp **P**") that extends beyond the entire polynomial hierarchy. It is commonly believed that even quantum computers cannot efficiently compute permanents. However, a quantum computer can efficiently create a bosonic (and a fermionic) state based on a matrix M and therefore perform efficiently both BosonSampling and FermionSampling. A classical computer with access to random bits can sample FermionSampling efficiently, but, as proved by Aaronson and Arkhipov, a classical computer with access to random bits cannot perform BosonSampling unless the polynomial hierarchy collapses!

Predictions from the Optimistic Hypothesis

Barriers Crossed. Quantum computers would dramatically change our reality.

(1) A universal machine for creating quantum states and evolutions will be built.
(2) Complicated evolutions and states with global interactions, markedly different from anything witnessed so far, will be created.
(3) It will be possible to experimentally time-reverse every quantum evolution.
(4) The noise will not respect symmetries of the state.
(5) There will be fantastic computational complexity consequences.
(6) Quantum computers will efficiently break most current public-key cryptosystems.

Items (1)–(4) represent a vastly different experimental reality than that of today, and items (5) and (6) represent a vastly different computational reality.

Magnitude of Improvements. It is often claimed that quantum computers can perform certain computations that even a classical computer of the size of the entire universe cannot perform! Indeed it is useful to examine not only things that were previously impossible and that are now made possible by a new technology but also the improvement in terms of orders of magnitude for tasks that could have been achieved by the old technology. Quantum computers represent enormous, unprecedented order-of-magnitude improvement of controlled physical phenomena as well as of algorithms. Nuclear weapons represent an improvement of 6–7 orders of magnitude over conventional ordnance: the first atomic bomb was a million times stronger than the most powerful (single) conventional bomb at the time. The telegraph could deliver a transatlantic message in a few seconds compared to the previous three-month period. This represents an (immense) improvement of 4–5 orders of magnitude. Memory and speed of computers were improved by 10–12 orders of magnitude over several decades. Breakthrough algorithms at the time of their discovery also represented practical improvements of no more than a few orders

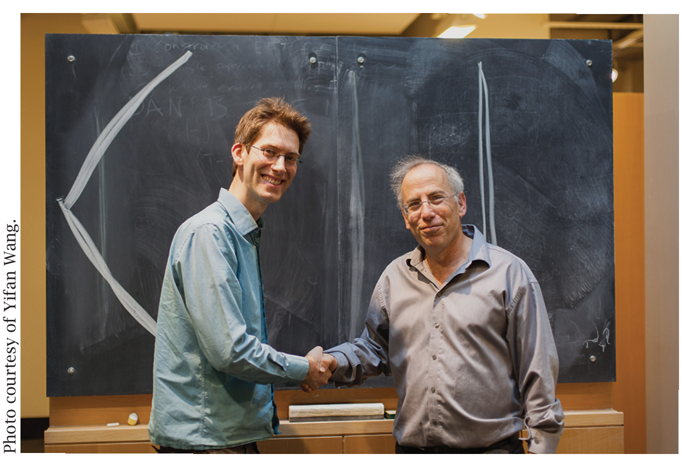

Photo courtesy of Yifan Wang.

Aram Harrow and Gil Kalai shake hands at MIT after their internet debate.

of magnitude. Yet implementing BosonSampling with a hundred bosons represents more than a hundred orders of magnitude improvement compared to digital computers, and a similar story can be told about a large-scale quantum computer applying Shor's algorithm.

Computations in Quantum Field Theory. Quantum electrodynamics (QED) computations allow one to describe various physical quantities in terms of a power series

$$\sum c_k \alpha^k,$$

where c_k is the contribution of Feynman's diagrams with k loops and α is the fine structure constant (around 1/137). Quantum computers will (likely)[6] allow one to compute these terms and sums for large values of k with hundreds of digits of accuracy, similar to computations of the digits of e and π on today's computers, even in regimes where they have no physical meaning!

My Interpretation. I regard the incredible consequences from the optimistic hypothesis as solid indications that quantum supremacy is "too good to be true" and that the pessimistic hypothesis will prevail. Quantum computers would change reality in unprecedented ways, both qualitatively and quantitatively, and it is easier to believe that we will witness substantial theoretical changes in modeling quantum noise than that we will witness such dramatic changes in reality itself.

BosonSampling Meets Reality

How Does Noisy BosonSampling Behave?

BosonSampling and Noisy BosonSampling (i.e., BosonSampling in the presence of noise) exhibit radically different behavior. BosonSampling is based on n noninteracting, indistinguishable bosons with m modes. For noisy Boson Samplers these bosons will not be perfectly noninteracting (accounting for one form of noise) and will not be perfectly indistinguishable (accounting for another form of noise). The same is true if we replace bosons by fermions everywhere. The state of n bosons with m modes is represented by an algebraic variety of decomposable symmetric tensors of real dimension $2mn$ in a huge relevant Hilbert space of dimension $2m^n$. For the fermion case this manifold is simply the Grassmannian.

We have already discussed the rich theory of computational complexity classes beyond **P**, and there is also a rich theory below **P**. One very low-level complexity class consists of computational tasks that can be carried out by bounded-depth polynomial-size circuits. In this model the number of gates is, as before, at most polynomial in the input side, but an additional severe restriction is that the entire computation is carried out in a bounded number of rounds. Bounded-depth polynomial-size circuits cannot even compute or approximate the parity of n bits, but they can approximate real functions described by bounded-degree polynomials and can sample approximately according to probability distributions described by real polynomials of bounded degree.

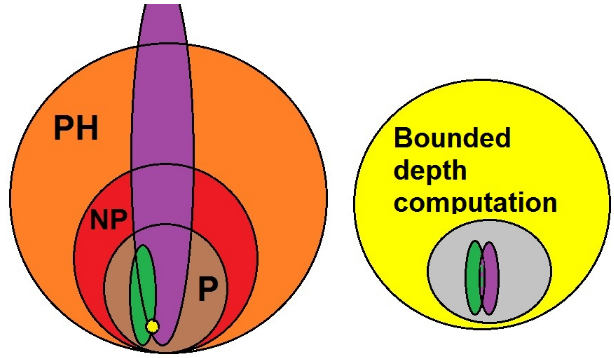

Figure 3. The huge computational gap (left) between BosonSampling (purple) and FermionSampling (green) vanishes in the noisy versions (right).

Theorem 1 (Kalai and Kindler)**.** *When the noise level is constant, BosonSampling distributions are well approximated by their low-degree Fourier–Hermite expansion. Consequently, noisy BosonSampling can be approximated by bounded-depth polynomial-size circuits.*

It is reasonable to assume that for all proposed implementations of BosonSampling, the noise level is at least a constant, and therefore an experimental realization of BosonSampling represents, asymptotically, bounded-depth computation. The next theorem shows that implementation of BosonSampling will actually require pushing down the noise level below $1/n$.

Theorem 2 (Kalai and Kindler)**.** *When the noise level is $\omega(1/n)$ and $m \gg n^2$, BosonSampling is very sensitive to noise, with a vanishing correlation between the noisy distribution and the ideal distribution.*[7]

Theorems 1 and 2 give evidence against expectations of demonstrating "quantum supremacy" via BosonSampling: experimental BosonSampling represents an extremely low-level computation, and there is no precedence for a "bounded-depth machine" or a "bounded-depth algorithm" that gives a practical advantage, even for small input size, over the full power of classical computers, not to mention some superior powers.

Bounded-Degree Polynomials

The class of probability distributions that can be approximated by low-degree polynomials represents a severe restriction below bounded-depth computation. The description of noisy BosonSampling with low bounded-degree polynomials is likely to extend to small noisy quantum circuits and other similar quantum systems, and this would support the pessimistic hypothesis. This

[6] *This plausible conjecture, which motivated quantum computers to start with, is supported by the recent work of Jordan, Lee, and Preskill and is often taken for granted. A mathematical proof is still beyond reach.*

[7] *The condition $m \gg n^2$ can probably be removed by a more detailed analysis.*

description is relevant to important general computational aspects of quantum systems in nature, as we now discuss.

Why Is Robust Classical Information Possible? The ability to approximate low-degree polynomials still supports robust *classical* information. The ("Majority") Boolean function[8] $f(x_1, x_2, \dots, x_n) = sgn(x_1+x_2+\cdots+x_n)$ allows for very robust bits based on a large number of noisy bits and admits excellent low-degree approximations. Quantum error correction is also based on encoding a single qubit as a function $f(q_1, q_2 \dots, q_n)$ of many qubits, and also for quantum codes the quality of the encoded qubit grows with the number of qubits used for the encoding. But for quantum error-correction codes, implementation with bounded-degree polynomial approximations is not available, and I conjecture that no such implementation exists. This would support the insight that quantum mechanics is limiting the information one can extract from a physical system in the absence of mechanisms leading to robust classical information.

Why Can We Learn the Laws of Physics from Experiments? Learning the parameters of a process from examples can be computationally intractable, even if the process belongs to a low-level computational task. (Learning even a function described by a depth-two Boolean circuit of polynomial size does not admit an efficient algorithm.) However, the approximate value of a low-degree polynomial can efficiently be learned from examples. This offers an explanation for our ability to understand natural processes and the parameters defining them.

Predictions from the Pessimistic Hypothesis

Under the pessimistic hypothesis, universal quantum devices are unavailable, and we need to devise a specific device in order to implement a specific quantum evolution. A sufficiently detailed modeling of the device will lead to a familiar detailed Hamiltonian modeling of the quantum process that also takes into account the environment and various forms of noise. Our goal is different: we want to draw from the pessimistic hypothesis predictions on noisy quantum circuits (and, at a later stage, on more general noisy quantum processes) that are common to *all* devices implementing the circuit (process).

The basic premises for studying noisy quantum evolutions when the specific quantum devices are not specified are as follows: First, modeling is implicit; namely, it is given in terms of conditions that the noisy process must satisfy. Second, there are systematic relations between the noise and the entire quantum evolution and also between the target state and the noise.

In this section we assume the pessimistic hypothesis, but we note that the previous section proposes the following picture in support of the pessimistic hypothesis: evolutions and states of quantum devices in the small scale are described by low-degree polynomials. This allows, for a larger scale, the creation of robust classical information and computation but does not provide the necessary starting point for quantum fault tolerance or for any manifestation of quantum supremacy.

[8] *A Boolean function is a function from* $\{-1, 1\}^n$ *to* $\{-1, 1\}$.

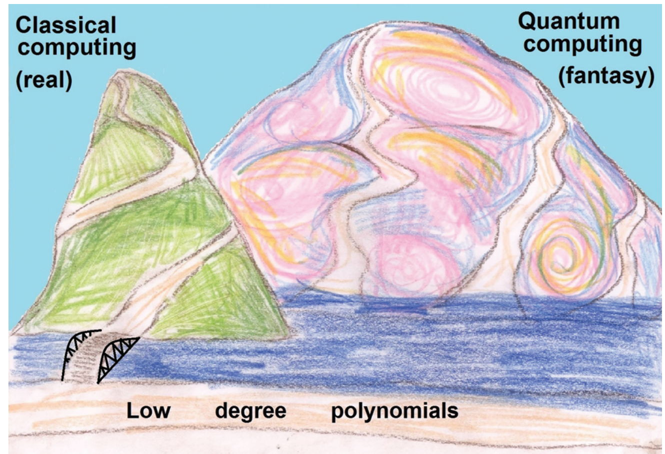

Figure 4. The pessimistic hypothesis: Noisy quantum evolutions, described by low-degree polynomials, allow via the mechanisms of averaging/repetition robust *classical* information and computation but do not allow reaching the starting points for quantum supremacy and quantum fault tolerance. Drawing by Neta Kalai.

No Quantum Fault Tolerance: Its Simplest Manifestation

Entanglement and Cat States. Entanglement is a name for quantum correlation, and it is an important feature of quantum physics and a crucial ingredient of quantum computation. A *cat state* of the form $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ represents the simplest form of entanglement between two qubits. Let me elaborate: the Hilbert space H representing the states of a single qubit is two-dimensional. We denote by $|0\rangle$ and $|1\rangle$ the two vectors of a basis for H. A pure state of a qubit is a *superposition* of basis vectors of the form $a|0\rangle + b|1\rangle$, where a, b are complex and $|a|^2 + |b|^2 = 1$. Two qubits are represented by a tensor product $H \otimes H$, and we denote it by $|00\rangle = |0\rangle \otimes |0\rangle$. Now, a superposition of two vectors can be thought of as a quantum analog of a coin toss in classical probability—a superposition of $|00\rangle$ and $|11\rangle$ is a quantum analog of correlated coin tosses: two heads with probability 1/2, and two tails with probability 1/2. The name "cat state" refers, of course, to Schrödinger's cat.

Noisy Cats. The following prediction regarding noisy entangled pairs of qubits (or "noisy cats") is perhaps the simplest prediction on noisy quantum circuits under the pessimistic hypothesis.

Prediction 1: Two-qubits behavior. Any implementation of quantum circuits is subject to noise, for which errors for a pair of entangled qubits will have substantial positive correlation.

Prediction 1, which we will refer to as the "noisy cat prediction", gives a very basic difference between the optimistic and pessimistic hypotheses. Under the optimistic hypothesis gated qubits will manifest correlated noise, but when quantum fault tolerance is in place, such correlations will be diminished for most pairs of qubits. Under the pessimistic hypothesis quantum fault-tolerance is not possible, and without it there is no mechanism to remove correlated noise for entangled qubits. Note that the condition on noise for a pair of entangled qubits is implicit, as it depends on the unknown process and unknown device leading to the entanglement.

Further Simple Manifestations of the Failure of Quantum Fault Tolerance.

Prediction 2: Error synchronization. For complicated (very entangled) target states, highly synchronized errors will occur.

Error synchronization refers to a substantial probability that a large number of qubits, much beyond the average rate of noise, are corrupted. Under the optimistic hypothesis error synchronization is an extremely rare event.

Prediction 3: Error rate. For complicated evolutions, and for evolutions approximating complicated states, the error rate, in terms of qubit-errors, scales up linearly with the number of qubits.

The three predictions 1–3 are related. Under natural assumptions, the noisy cat prediction implies error synchronization for quantum states of the kind involved in quantum error correction and quantum algorithms. Roughly speaking, the noisy cat prediction implies positive correlation between errors for every pair of qubits, and this implies a substantial probability for the event that a large fraction of qubits (well above the average rate of errors) will be corrupted at the same computer cycle. Error synchronization also implies, again under some natural assumptions, that error rate in terms of qubit errors is at least linear in the number of qubits. Thus, the pessimistic hypothesis itself can be justified from the noisy cat prediction, together with natural assumptions on the rate of noise. Moreover, this also explains the wide-gap dichotomy in terms of qubit errors.

The optimistic hypothesis allows creating via quantum error correction very stable "logical" qubits based on stable raw physical qubits.

Prediction 4: No logical qubits. Logical qubits cannot be substantially more stable than the raw qubits used to construct them.

No Quantum Fault-Tolerance: Its Most General Manifestation[9]

We can go to the other extreme and try to examine consequences of the pessimistic hypothesis for the most general quantum evolutions. We start with a prediction related to the discussion in the section "BosonSampling Meets Reality".

Prediction 5: Bounded-depth and bounded-degree approximations. Quantum states achievable by any implementation of quantum circuits are limited by bounded-depth polynomial-size quantum computation. Even stronger: low-entropy quantum states in nature admit approximations by bounded-degree polynomials.

The next items go beyond the quantum circuit model and do not assume that the Hilbert space for our quantum evolution has a tensor product structure.

Prediction 6: Time smoothing. Quantum evolutions are subject to noise, with a substantial correlation with time-smoothed evolutions.

Time-smoothed evolutions form an interesting restricted class of noisy quantum evolutions aimed to model evolutions under the pessimistic hypothesis when fault tolerance is unavailable to suppress noise propagation. The basic example for time-smoothing is the following: Start with an ideal quantum evolution given by a sequence of T unitary operators, where U_t denotes the unitary operator for the tth step, $t = 1, 2, \dots, T$. For $s < t$ we denote $U_{s,t} = \prod_{i=s}^{t-1} U_i$ and let $U_{s,s} = I$ and $U_{t,s} = U_{s,t}^{-1}$. The next step is to add noise in a completely standard way: consider a noise operation E_t for the tth step. We can think about the case where the unitary evolution is a quantum computing process and E_t represents a depolarizing noise with a fixed rate acting independently on the qubits. And finally, replace E_t with a new noise operation E'_t defined as the average

$$E'_t = \frac{1}{T} \cdot \sum_{s=1}^{T} U_{s,t} E_s U_{s,t}^{-1}. \tag{1}$$

Prediction 7: Rate. For a noisy quantum system a lower bound for the rate of noise in a time interval is a measure of noncommutativity for the projections in the algebra of unitary operators in that interval.

Predictions 6 and 7 are implicit and describe systematic relations between the noise and the evolution. We expect that time-smoothing will suppress high terms for some Fourier-like expansion, thus relating Predictions 5 and 6. We also note that Prediction 7 resembles the picture about the "unsharpness principle" from symplectic geometry and quantization.[10]

Locality, Space and Time

The decision between the optimistic and pessimistic hypotheses is, to a large extent, a question about modeling locality in quantum physics. Modeling natural quantum evolutions by quantum computers represents the important physical principle of "locality": quantum interactions are limited to a few particles. The quantum circuit model enforces local rules on quantum evolutions and still allows the creation of very nonlocal quantum states. This remains true for noisy quantum circuits under the optimistic hypothesis. The pessimistic hypothesis suggests that quantum supremacy is an artifact of incorrect modeling of locality. We expect modeling based on the pessimistic hypothesis, which relates the laws of the "noise" to the laws of the "signal", to imply a strong form of locality for both.

We can even propose that spacetime itself emerges from the absence of quantum fault tolerance. It is a

[9] *This section is more technical and assumes more background on quantum information.*

[10] *L. Polterovich, "Symplectic geometry of quantum noise", Comm. Math. Phys* ***327*** *(2014), 481–519; arXiv:1206.3707.*

familiar idea that since (noiseless) quantum systems are time reversible, time emerges from quantum noise (decoherence). However, also in the presence of noise, with quantum fault tolerance, every quantum evolution that can experimentally be created can be time-reversed, and, in fact, we can time-permute the sequence of unitary operators describing the evolution in an arbitrary way. It is therefore both quantum noise and the absence of quantum fault tolerance that enable an arrow of time.

Next, we note that with quantum computers one can emulate a quantum evolution on an arbitrary geometry. For example, a complicated quantum evolution representing the dynamics of a four-dimensional lattice model could be emulated on a one-dimensional chain of qubits. This would be vastly different from today's experimental quantum physics, and it is also in tension with insights from physics, where witnessing different geometries supporting the same physics is rare and important. Since a universal quantum computer allows the breaking of the connection between physics and geometry, it is noise and the absence of quantum fault tolerance that distinguish physical processes based on different geometries and enable geometry to emerge from the physics.

Classical Simulations of Quantum Systems

Prediction 8: Classical simulations of quantum processes. Computations in quantum physics can, in principle, be simulated efficiently on a digital computer.

This bold prediction from the pessimistic hypothesis could lead to specific models and computational tools. There are some caveats: heavy computations may be required for quantum processes that are not realistic to start with, for a model in quantum physics representing a physical process that depends on many more parameters than those represented by the input size, for simulating processes that require knowing internal parameters of the process that are not available to us (but are available to nature), and when we simply do not know the correct model or relevant computational tool.

Photo courtesy of Matas Šileikis.

Gil Kalai lecturing at Adam Mickiewicz University in Poland.

Quid est Noster Computationis Mundus?[11]

Deciding between the optimistic and pessimistic hypotheses reflects a far-reaching difference in the view of our computational world. Is the wealth of computations we witness in reality only the tip of the iceberg of a supreme computational power used by nature and available to us, or is it the case that the wealth of classical computations we witness represents the full computational power that can be extracted from natural quantum physics processes?

I expect that the pessimistic hypothesis will prevail, yielding important outcomes for physics, the theory of computing, and mathematics. Our journey through probability distributions described by low-degree polynomials, implicit modeling for noise, and error synchronization may provide some of the ingredients needed for solving the quantum computer puzzle.

[11] *What is our computational world?*

COMMUNICATION

Communicating Mathematics to Children

Rich Schwartz

Communicated by Joel Hass

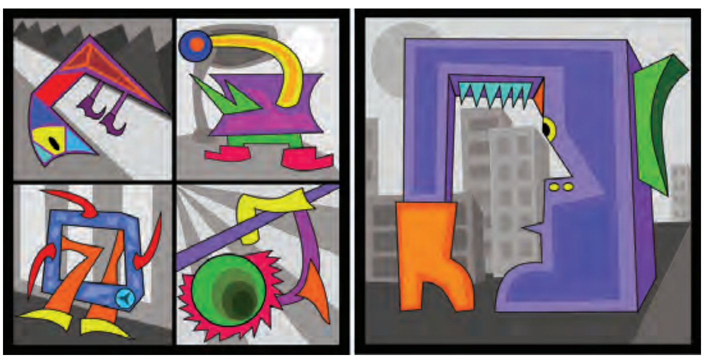

Figure 1. A few inkscape-drawn aliens from my unpublished comic book, *Guardian of the Blue Metropolis*.

Introduction

I think of myself mostly as a research mathematician, but I also like to draw goofy pictures and explain things. Eventually these interests led me to write and illustrate some children's math books. In this article I will describe some aspects of my experience communicating mathematics to children through these books. I think that there is a great need for children's math books written by creative mathematicians, books which can show the vibrant, exciting, awe-inspiring nature of the subject. In case you are interested in doing this, I hope that my account will help you along.

In the first part of this article, I will describe how I got started in this business, as well as some obstacles I faced along the way. In the second part I will discuss my vision for what a children's book written by a creative mathematician might look like and why it doesn't fit into the conventional mold for a children's educational book.

How I Got Started

I have always enjoyed drawing comic books, and sometimes I take some time off from my research to do it. The books are usually about strange topics, like the collective intelligence of ants, or the Star Trek transporter problem, or alien life, or a monster made out of chicken lo mein. I sometimes draw the pictures by hand, but usually I use the computer programs **xfig** and **inkscape**. These are free drawing programs which I, like many mathematicians, use to illustrate my math papers. Figure 1 shows a few of the aliens I drew using inkscape.

I got interested in writing and illustrating comic books for children, naturally enough, when I had children of my own. In 2002, when my older daughter, Lucina, turned five, I drew a short booklet for her designed to teach her about prime numbers. Figure 2 shows roughly what the first version looked like.

The idea was for her to figure out the pattern from the pictures. Why are some of the numbers smiling?

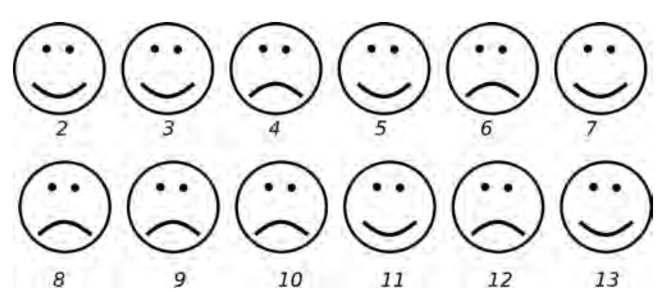

Figure 2. A drawing I made for my 5-year-old daughter to teach her about primes and composites.

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Figure 3. Some of my xfig pictures of the numbers.

Figure 4. A group photo involving the 2-monster, the 3-monster, and the 5-monster illustrates the number $30 = 2 \times 3 \times 5$**.**

Eventually I decided to jazz up the pictures a bit. Figure 3 shows the new pictures I drew using xfig.

I soon grew tired of drawing monsters, and it was then that I hit on the idea for my first children's math book, *You Can Count on Monsters*. Instead of having to create a new monster for every number, I decided to show a composite picture for composite numbers. The idea is to factor composite numbers into primes and then to arrange the corresponding prime monsters into a group photo. Figure 4 shows one example from the book.

It took me about five years to finish *You Can Count on Monsters* because I worked on it very sporadically. I don't mind saying that I did some of it while occupying Marcel Berger's office at IHES in the summer of 2002.

The Recursive Problem of Publishing

I tried occasionally to publish my comic books with big conventional publishers or smaller comic book publishers, but I never had a glimmer of success. The experience was always one of sending a manuscript off into a black hole. I had heard that it was a good idea to get an agent to help pitch your books to publishers, but I could never figure out how to get an agent. It seemed to me that you needed to know someone to get an agent. In short, you needed an agent to get an agent.

I went through similar struggles with the question of how to illustrate my books. Not having any formal training as an artist, I was very self-conscious about illustrating them myself. I had the persistent notion that my own illustrations would just be preliminary ones and that some great illustrator would go back through the book and re-do the pictures. So, sometimes I held off trying to publish something because I wanted to find an illustrator first. I could never find any illustrators, and so I started thinking about finding people who could introduce me to them....

Eventually I decided to do the whole thing myself. With encouragement from my artist friends, my wife amongst them, I started thinking of my own illustrations as the final product. Also, I started using self-publishing companies like CreateSpace. With CreateSpace, you send in a PDF file containing your manuscript and then a few weeks later they send you a printed proof. You then tweak the manuscript and send it back. And so on. When you are happy with the manuscript, CreateSpace will sell it for you on amazon.com, print it on demand, and give you a small royalty. You may not get rich from this, but you can see a high-quality version of your book in print and for sale. A few of my comics ended up on amazon.com this way.

Write about what interests you, but think about the mind of the child.

Shortly after making the original book for my daughter, I reimagined *You Can Count on Monsters* as a poster, where one could see all the pictures at once. Five years later, my wife convinced me to go back to the book format. I did this, and I was about to send it to CreateSpace, but on a whim I decided to ask Alice and Klaus Peters about it. At the time, Alice and Klaus owned A K Peters, a math publishing house which has published some wonderful and offbeat math books. They both liked the book and finally it was published.

Sadly, A K Peters no longer exists, but I think that there are still opportunities for publishing offbeat children's math books. I published my second book, *Really Big Numbers*, with the American Mathematical Society (AMS), and they plan to publish two more I wrote, *Gallery of the Infinite* and *Life on the Infinite Farm*. The AMS seems

Figure 5. Gracie, a cow with infinitely many feet, loves shoes.

interested in this sort of thing. I'd also like to mention that MSRI has a new program, called Mathical Books, which highlights and promotes mathematics in children's literature.

The Target Audience

Often when I tell someone that I have written a math book for children, they ask, *What is the age range for your book?* This is a very sensible question. When you start out on a commercial venture, you ought to know your target audience. Publishers, booksellers, and teachers will all want to know how to categorize, pitch, and place your book.

I hate this question. In my experience it has been the most discouraging thing anyone has asked me. Probably if I paid attention to the question I would never have written anything at all. Let me explain why I think that this question is not so appropriate for children's math books of the kind I imagine and how I dodged the question in one of my books.

I think that the kind of book a creative research mathematician might write for children would not appeal to the bulk of the children of any age. Rather, it will appeal to a few of the students at the top end of whatever grade they happen to be in, students who like the subject already but who are perhaps dissatisfied with what they are learning in school. I like the idea of writing books that have layers of meaning and detail, books whose depth unfolds before the reader as he or she spends more time with it.

Let me give an example from my book *Life on the Infinite Farm.* The book features a number of animals who are infinitely extended in one way or another. Figure 5 shows one of the characters, Gracie, an infinite cow who loves shoes. She has a shoe on every foot. Her dilemma is that she wants to wear the new shoes she gets as gifts from the other animals, but she doesn't want to take off her other shoes.

Anyone familiar with the famous Hilbert hotel will see the solution to her dilemma. She just places the shoes in front of her and steps out of each of the old shoes and into the new ones. All the shoes have moved back and she has the new shoes on her front feet. (The book has illustrations of this. If you are curious, you can find a link on my website.) It seems that a construction like this, which involves the nature of infinity, does not really have a target age; it ought to appeal to some people of all ages.

As a deeper example, I talk about Delores and Beena, two squids who have infinite branching trees of tentacles. The left half of Figure 6 shows Delores and the right half shows Delores and Beena together. Beena is asking Delores whether she can borrow some of Delores's jewelry.

Delores loves jewelry and wants to keep herself more or less completely covered in it. At the same time, Beena wants to be outfitted in a similar fashion. How does Delores keep herself covered in jewelry and outfit Beena as well? To solve this problem, Delores transfers some of her jewelry to Beena using local moves enabled by little fish. The fish move every piece of jewelry one unit towards Delores's head and thereby "double" the amount of jewelry she has. Then the fish transfer half the jewelry to Beena. Figure 7 shows two snapshots of the local move.

Some readers might recognize this as the heart of the proof that the free group on two generators is not an amenable group. I first saw this argument in a talk Shmuel Weinberger gave in Berkeley back in 1992, and it stayed with me all these years.

After presenting these kinds of problems and solutions, I raise the open-ended question as to how the animals

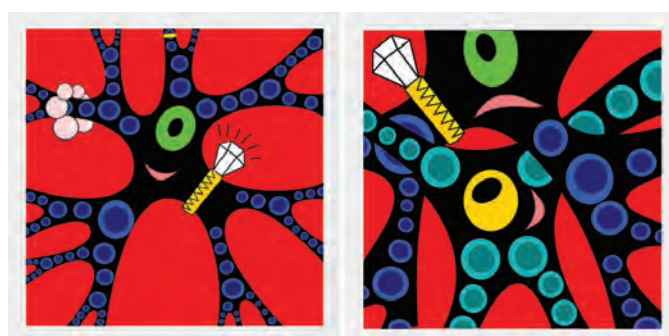

Figure 6. Delores and Beena are squids with infinite branching trees of tentacles. Beena asks Delores if she can borrow some of her jewelry.

Figure 7. Delores "doubles" her jewelry using local moves that transport all her jewelry closer to her head.

Figure 8. My book is like a bucking bronco, which you are supposed to ride until you fall off.

manage to move around on the farm, given their infinite sizes. How do they avoid crashing into each other? I suggest that the animals exist at many different scales, that they are quite acrobatic, and that some have features like trap doors and detachable parts. I also had in my mind that the infinite farm is a negatively curved space, so that the animals would not be stymied by the parallel postulate.

I imagine the proof of the Cantor-Bernstein Theorem as an analysis of the way cats and dogs are chasing each other.

On the one hand, I can imagine that some very young children would be intrigued by the infinite and might like the pictures of the animals—I hope so, anyway. On the other hand, I can imagine older kids discussing how space might be designed to make something like this possible, or biologists wondering about an infinite periodic digestive tract, or physicists complaining about the nonrelativistic nature of the farm. It is hard to figure out an age range for *Life on the Infinite Farm*.

My book *Really Big Numbers* also presents material at many different levels of difficulty. It runs all the way from counting dots to recursive definitions akin to the Ackerman function. Figure 8 shows the method I used to dodge the problem of needing to write for a specific age or maturity level. At the beginning of the book, I explain that the book is a lot like the game of bucking bronco I used to play with my own children: The ride starts out slow and gradually gets faster until they fall off. The goal of the game is to stay on as long as possible, but it is no big deal if you fall off.

Once people get over the idea that they have to read everything in the book, the possibilities for exposition open up quite a bit.

Math in Slow Motion

In principle, all of mathematics reduces to a series of logical conclusions drawn from simple axioms. In practice, mathematicians take big strides through the system in an effort to reach deep and surprising results. One of the main issues involved in learning mathematics is the speed with which it piles up.

It might appear to take just a few pages to define, say, an integrable system on a smooth manifold, but if you wanted to explain it to a college student, you would have to slow down and explain what a manifold is and how calculus on manifolds works. If you wanted to reach a high school student, you would probably want to explain about the real numbers, continuity, linear algebra, and so on. Eventually the sheer length of time it would take you to explain the whole thing would make it impossible.

Given that children have very little math background, a children's math book which aims to impart real understanding has to be about a very small segment of mathematics. Fortunately, even very small pieces of mathematics—the Pythagorean Theorem, Heron's Formula, Pascal's triangle, the platonic solids, the infinitude of primes, the definition of Langton's ant, the hypercube,

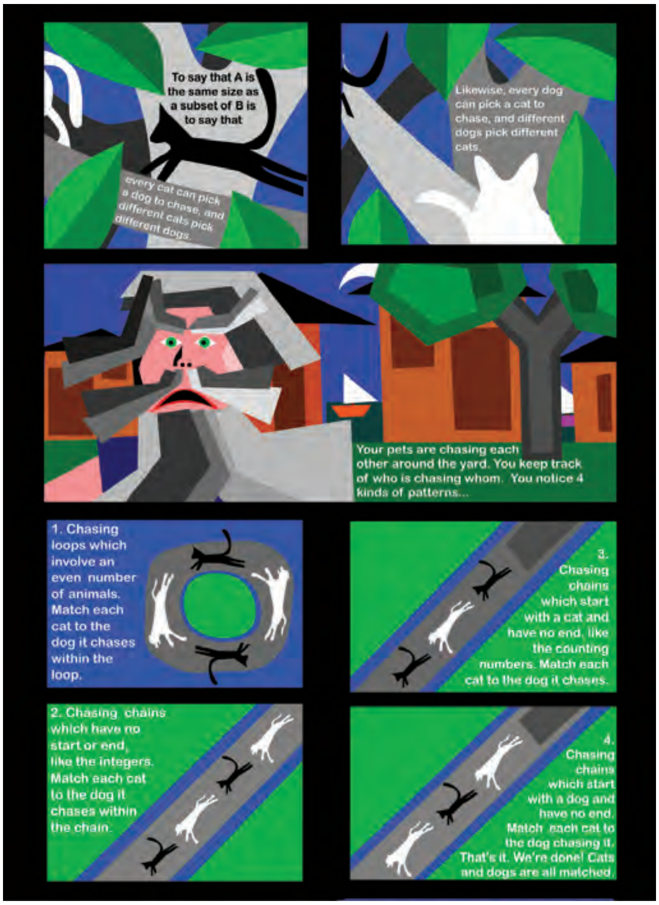

Figure 9. I imagine the proof of the Cantor-Bernstein Theorem as an analysis of the way cats and dogs are chasing each other.

the definition of continued fractions, scissors congruence of polygons, complex numbers, the Cantor diagonal argument, etc., etc., etc.—are beautiful and interesting. I think that a successful children's book about math should take a topic like this and present it vividly and in slow motion, so that a child could see every step.

In my book *Gallery of the Infinite* (which isn't quite for children) I tried to do this for some classic theorems in set theory, especially the famous Cantor diagonal argument. Figure 9 shows some of my exposition of the Cantor-Bernstein Theorem, which says that two sets A and B are bijective if there is an injection from A into B and an injection from B into A. I don't think that what I did is quite suitable for children, but I tried pretty hard to prove the result in a playful and engaging way.

Once people get over the idea that they have to read everything in the book, the possibilities for exposition open up quite a bit.

It is hard to say exactly how to keep things slow and simple yet still present interesting mathematics, but it is easy to say what not to do. Like most mathematicians currently walking the planet, I have had the unpleasant experience (many times) of listening to a lecture in which the speaker assumes that the audience knows as much about the topic as the speaker does. Usually I am too embarrassed to stop the speaker, and I end up wasting an hour and coming out of the room with a headache. I can understand this happening when the speaker is addressing a large audience—perhaps the speaker has other listeners in mind—but sometimes this happens when it is just the two of us! Don't blow past your audience; write with empathy.

One positive suggestion I have is that you should illustrate your book lavishly. Children love catchy pictures. Also, the discipline of having a picture for every key concept in the book keeps the exposition going at a slow, measured pace. When I think about math, I often think in little cartoons, which later, in my papers, I have to transcribe into the written word. The reader then has to absorb all the words and formulas and (I hope) reassemble the cartoon pictures. It would be nice to be able to communicate the pictures directly. Maybe for simple topics this can be done.

The main idea is to understand the math all the way to the bottom, think about exactly how you understand it, and then put it all down on the page in a friendly and engaging way. Write about what interests you, but think about the mind of the child.

Courtesy of Brienne Brown.

Rich Schwartz and his daughters.

Melanie Wood Interview

Photo by Joe Rabinoff.

Melanie Wood *is assistant professor at the University of Wisconsin-Madison and an American Institute of Mathematics Five-Year Fellow.*

Diaz-Lopez: *When did you know you wanted to be a mathematician?*

Wood: My mathematics research experiences as an undergraduate at the REU [Research Experiences for Undergraduates] at the University of Minnesota-Duluth and through the PRUV [Program for Research for Undergraduates] program at Duke University, where I was an undergraduate, were really the tipping point for me in deciding I wanted to be a mathematician. I had always liked math, but until these experiences I didn't really have any sense of what math as a career would be like. I had so much fun working on my own research problems that I knew it was something I would want to do as a job.

Diaz-Lopez: *Who encouraged or inspired you?*

Wood: I have been extraordinarily lucky to have so many wonderful teachers and mentors that have encouraged me and inspired me along my path to becoming a mathematician. Some who particularly stand out from my youth: Bob Fischer, who was the Indiana MATHCOUNTS® coach when I was in 7th and 8th grade and the first person I can remember giving me math problems I didn't know how to solve; Joanne Black, a teacher at my high school who was incredibly supportive of my mathematical development; Zvezdelina Stankova, who particularly inspired me as a teacher at the Math Olympiad Summer Program. Then throughout college and graduate school the mathematicians who have encouraged and inspired me are too numerous to mention all of them. I had a lot of wonderful math professors as an undergraduate at Duke. My PhD advisor, Manjul Bhargava, was incredibly supportive, not to mention inspiring, through the somewhat rocky path of graduate school. When I was a postdoc, Ravi Vakil was an important mentor and inspiration.

Diaz-Lopez: *How would you describe your research to a graduate student?*

Wood: I work on a lot of different questions, mostly focused in number theory but also in algebraic geometry, algebraic topology, and probability. I am interested in the most basic objects in number theory, number fields, which are finite extensions of the rational numbers. I want to understand how many number fields there are and how often they have various properties. As basic as it is, this sort of question can be incredibly difficult and require ideas from a broad spectrum of mathematics. I also am interested in questions about the number of solutions of polynomial equations, both solutions that are rational numbers and solutions that lie in a finite field. These questions are deeply connected to the geometry of the space of solutions of the equations.

Diaz-Lopez: *What theorem are you most proud of, and what was the most important idea that led to this breakthrough?*

Wood: I am most proud of my results proving the distribution of sandpile groups of random graphs or, relatedly, cokernels of symmetric random matrices. First, let's talk about the cokernel of a matrix. Take a free abelian group on n generators, and then pick n relations (sums of those n generators). If

Practice talking about your work to a wide range of mathematicians.

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you put the n^2 coefficients of the relations in a matrix, the abelian group you get by taking generators mod relations is the cokernel of the matrix. It is just a natural way to build a random abelian group. Many different properties of random matrices have been well studied, and in general they are much easier to understand when the entries of the matrices are all independent. My work had to tackle two kinds of dependency in the matrices and still understand their random behavior. I got interested in thinking about random abelian groups this way because class groups of number fields, which measure the failure of unique factorization in those number fields, occur naturally this way via matrices whose entries are dependent in deep and mysterious ways. I wouldn't say that there was a single most important idea that led to the breakthrough. The work took a long time, with many different breakthroughs (and some antibreakthroughs!) along the way, and required developing several kinds of new methods.

Diaz-Lopez: *What advice do you have for graduate students?*

Wood: Figure out what you want to get out of graduate school and what it takes to get that. Tell your advisor what you want and ask his or her advice for what it will take to get it, and ask other faculty for advice as well. The figuring-out part might be a significant project for some people, and you should undertake it as real work. Talk to finishing students about what kinds of jobs they got and what it took to get them—what kinds of skills they had to develop, what kind of experience they needed, what kinds of theorems they proved, what kinds of teaching evaluations or feedback. Talk to mathematicians in a range of different jobs about what their jobs are like and what it takes to get them. Go to conferences and talk to a wide range of people there. These are some of the questions you can ask them.

Practice talking about your work to a wide range of mathematicians, and figure out what it takes to communicate what you do and why it is interesting to them. Ask your advisor for advice about this, and pay attention in seminar talks to how people motivate their work. Learn how to put your work in context at many different levels. What are the overarching goals of your field and how does your work fit into that? You can ask a similar question about your subfield or more specialized area. Remember that what you say about your work should depend on your audience. Graduate students generally tend to assume other people know way more about the topic they are studying for their thesis than anyone actually does. Talk to other graduate students and tell them to stop you if you say something they don't know.

Diaz-Lopez: *All mathematicians feel discouraged occasionally. How do you deal with discouragement?*

Wood: I keep a list of positive experiences—when I proved a result, had a great mathematical conversation, got inspired by a talk—and go to it when I feel discouraged. I also have mathematical colleagues who are good friends whose advice and support help keep me afloat.

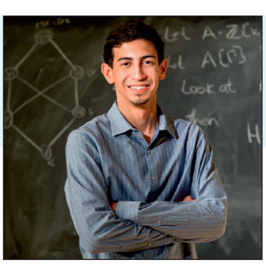
Diaz-Lopez: *You have won several honors and awards. Which one has been the most meaningful and why?*

Wood: The American Institute of Mathematics (AIM) Five-Year Fellowship has meant the most to me. As I was finishing graduate school, I wasn't even sure if becoming a professional mathematician was the right career for me. I found graduate school lonely and discouraging in parts. The AIM Five-Year Fellowship was greatly needed positive feedback about my work and allowed me to start a postdoc in ideal conditions. My experience as a postdoc was exciting and encouraging, as I started new collaborations and took my research in new directions.

I keep a list of positive experiences and go to it when I feel discouraged.

Diaz-Lopez: *If you were not a mathematician, what would you be?*

Wood: It is hard to say because I have always had a lot of interests. I started college thinking I would go into cognitive science. Lately, I've gotten interested in supreme court jurisprudence, and so I could imagine really enjoying going to law school.



Alexander Diaz-Lopez is a PhD student at the University of Notre Dame. Diaz-Lopez is the first graduate student member of the *Notices* Editorial Board.

WHAT IS...

Nash Equilibrium?

Rajiv Sethi and Jörgen Weibull

Communicated by Cesar E. Silva

In game theory, a *Nash equilibrium* is an array of strategies, one for each player, such that no player can obtain a higher payoff by switching to a different strategy while the strategies of all other players are held fixed. The concept is named after John Forbes Nash Jr.

For example, if Chrysler, Ford, and GM choose production levels for pickup trucks, a commodity whose market price depends on aggregate production, an equilibrium is an array of production levels, one for each firm, such that none can raise its profits by making a different choice.

Formally, an n-player game consists of a set $I = \{1, \dots, n\}$ of players, a set S_i of strategies for each player $i \in I$, and a set of goal functions $g_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ that represent the preferences of each player i over the n-tuples, or profiles, of strategies chosen by all players. A strategy profile has a higher goal-function value, or *payoff*, than another if and only if the player prefers it to the other. Let $S = S_1 \times \cdots \times S_n$ denote the set of all strategy profiles, with generic element s, and let (t_i, s_{-i}) denote the strategy profile $(s_1, \dots, s_{i-1}, t_i, s_{i+1}, \dots, s_n)$ obtained from s by switching player i's strategy to $t_i \in S_i$ while leaving all other strategies unchanged. An equilibrium point of such a game is a strategy profile $s^* \in S$ with the property that, for each player i and each strategy $t_i \in S_i$,

$$g_i(s^*) \geq g_i(t_i, s^*_{-i}).$$

That is, a strategy profile is an equilibrium point if no player can gain from a unilateral deviation to a different strategy.

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The invention and succinct formulation of this concept, along with the establishment of its existence under very general conditions, reshaped the landscape of research in economics and other social and behavioral sciences.

Nash's existence theorem pertains to games in which the strategies S_i available to each player are probability distributions over a finite set of alternatives. Typically, each alternative specifies what action to take under each and every circumstance that the player may encounter during the play of the game. The alternatives are referred to as *pure strategies* and the probability distributions over these as *mixed strategies*. Players' randomizations, according to their chosen probability distributions over their own set of alternatives, are assumed to be statistically independent. Any n-tuple of mixed strategies then induces a probability distribution or lottery over n-tuples of pure strategies. Provided that a player's preferences over such lotteries satisfy certain completeness and consistency conditions—previously identified by John von Neumann and Oskar Morgenstern—there exists a real-valued function with the n-tuples of *pure* strategies as its domain such that the expected value of this function represents the player's preferences over n-tuples of *mixed* strategies. Given only this restriction on preferences, Nash was able to show that every game has at least one equilibrium point in mixed strategies.

> *Nash equilibrium reshaped the landscape of research in economics.*

Emile Borel had a precursory idea, concerning symmetric pure conflicts of interest between two parties with very few alternatives at hand. In 1921 he defined the notion of a finite and symmetric zero-sum two-player game. In such a game each player has the same number of pure strategies, the gain for one player equals the loss to the other, and they both have the same probability of winning whenever they use the same pure strategy. Borel also formalized the concept of a mixed strategy, and for games in which each player has three pure strategies, proved the existence of what would later come to be called a *maxmin pair* of mixed strategies. This is a pair of strategies such that one player's strategy maximizes his own gain while his opponent simultaneously minimizes this gain. He subsequently extended this result to the case of five strategies per player, but seems to have doubted that general existence results could be achieved.

A few years later, and apparently unaware of Borel's partial results, von Neumann formalized the notion of finite zero-sum games with an arbitrary (finite) number of players, where each player has an arbitrary (finite) number of pure strategies. For all such games involving two players he proved the existence of a maxmin strategy pair, presented the result in Göttingen in 1927, and published it in 1928.

In comes Nash, a young doctoral student in mathematics at Princeton University. Nash defined a much more general class of games and a more general equilibrium concept. He allowed for any (finite) number of players, each having an arbitrary (finite) number of pure strategies at his or her disposal and equipped with any goal function. In particular, players may be selfish, altruistic, spiteful, moralistic, fair-minded, or have any goal function whatsoever. His definitions and his existence result contain those of Borel and von Neumann as special cases. Previously restricted to pure conflicts of interest, game theory could now be addressed to any (finite) number of parties with arbitrary goal functions in virtually any kind of strategic interaction. Nash published this in a one-page article in the *Proceedings of the National Academy of Sciences* in 1950.

His existence proof—merely sketched in this short paper—is based upon Kakutani's fixed-point theorem (established some years earlier). Kakutani's theorem states that if a subset X of $\mathbb{R}^m$ is nonempty, compact and convex, and a (set-valued) correspondence $\Gamma : X \rightrightarrows X$ is nonempty-valued, convex-valued and has a closed graph, then there exists $x \in X$ such that $x \in \Gamma(x)$. That is, there exists a fixed point of the correspondence. Nash's existence proof relies on the construction of what today is called the *best-reply correspondence*, which can then be shown to satisfy the conditions of Kakutani's theorem.

Given any n-tuple of mixed strategies, Nash defined a *countering* n-tuple as a mixed-strategy profile that obtains for each player the highest payoff given the strategies chosen by other players in the original, *countered* n-tuple. By associating with each n-tuple of mixed strategies the set of all countering n-tuples, one obtains a self-correspondence on the set of all mixed-strategy profiles. Since any n-tuple of mixed strategies is a point in the product space S obtained by taking the Cartesian product of the individual strategy spaces S_i, the domain of this correspondence is a nonempty, compact and convex subset of $\mathbb{R}^m$ for some m. In fact, it is a polyhedron, the Cartesian product of finitely many unit simplexes. Furthermore, the correspondence thus constructed is convex-valued, since a convex combination of countering n-tuples must itself be a countering n-tuple. And since the payoff functions are all continuous (in fact, polynomial) functions with closed domain, the correspondence has a closed graph. The existence of a fixed point follows from Kakutani's theorem, and any such fixed point is a *self-countering* n-tuple, or an equilibrium point of the game.

A year later Nash published an alternative existence proof in the *Annals of Mathematics* that instead is based on Brouwer's fixed-point theorem. Since Kakutani's theorem is derived from Brouwer's, Nash was more satisfied with the latter. This second proof has a touch of genius. It is simple and intuitive in retrospect but completely unexpected beforehand.

In order to use Brouwer's theorem, Nash needed to construct a self-map on the space of mixed-strategy profiles with the property that a strategy profile is an equilibrium point if and only if it is a fixed point of this map. But the best-reply correspondence could not be used for this purpose, since it need not be single-valued and does not permit a continuous selection in general.

This is how he did it. Consider any n-tuple of mixed strategies s, and recall that the payoff to a player i at this strategy profile is $g_i(s)$. Let $g_{ih}(s)$ denote the payoff that player i would receive if he were to switch to the *pure* strategy h while all other players continued to use the strategies specified in s. Define the continuous function

$$\phi_{ih}(s) = \max\{0, g_{ih}(s) - g_i(s)\}.$$

Each function value $\phi_{ih}(s)$ represents the "excess payoff" obtained by pure strategy $h \in S_i$, as compared with the payoff obtained under strategy profile s. Letting s_{ih} denote the probability with which pure strategy h is played under s, the function ϕ may be used to obtain a new n-tuple of mixed strategies, s', from s by setting

$$s'_{ih} = T_{ih}(s) = \frac{s_{ih} + \phi_{ih}(s)}{1 + \sum_h \phi_{ih}(s)}.$$

This defines a self-map T on the space of mixed strategy profiles. As long as there exists a pure strategy with positive excess payoff, T lowers the probabilities with which pure strategies having zero excess payoff are played. It is clear that if s is an equilibrium point, it must be a fixed point of T, since no pure strategy h can yield player i a higher payoff, forcing $\phi_{ih}(s) = 0$ for all i and h. It is easily verified that the converse is also true: if s is a fixed point of T, so that $\phi_{ih}(s) = 0$ for all i and h, then s must be an equilibrium point of the game.

To complete the proof, one need only use the fact that T is a continuous self-map on the compact and convex set of mixed-strategy n-tuples. This is sufficient, from Brouwer's theorem, for the existence of a fixed point.

The second proof has a touch of genius.

Nash's equilibrium concept lies at the heart of contemporary theoretical research on strategic interactions in economics and other fields. One especially fruitful area of application has been to auction theory, as the following example illustrates. Many strategic interactions—including lobbying, arms races, contests, and wars of attrition—can be modeled as *all-pay auctions* in which the highest bidder obtains an object of value but all players must pay their bids. (If there are multiple highest bidders they each get the object with the same probability.) Consider an object with value $v > 0$ and $n \geq 2$ bidders, each of whom is constrained to bid from the nonnegative integers. Players submit their bids simultaneously, without knowledge of any opponent's bid. This is a symmetric n-player game with countably infinite pure-strategy sets. However, Nash's existence result still applies, since no bid above v is ever a best reply to the bids of others, and hence the game has the same set of Nash equilibria as the finite game in which bids are bounded from above by v.

Nash's result tells us that there must be an equilibrium in pure or mixed strategies in this game. For instance, if $n = 2$ and $v = 5/2$, then it can be shown that no pure strategy equilibrium exists, but if each player chooses the distribution $(1/5, 3/5, 1/5)$ over the bids $\{0, 1, 2\}$, then neither can obtain a higher payoff by deviating unilaterally to any other strategy. Furthermore, each player's expected payoff in equilibrium is $1/4$, which is lower than the $5/4$ that each could secure if they colluded to bid zero. This example illustrates that equilibrium behavior, while individually optimal, can cause players to impose costs on each other that are wasteful in the aggregate.

The 1994 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel was awarded to Nash, along with Reinhard Selten and John C. Harsanyi, for their "pioneering analysis of equilibria in the theory of noncooperative games."

About the Authors

Photo courtesy of Barnard College.

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Rajiv Sethi's research interests include evolutionary game theory and applications, financial economics, and the economics of inequality.

Photo courtesy of the Stockholm School of Economics.

Jörgen Weibull

Jörgen Weibull's main field of research is noncooperative and evolutionary game theory, with applications to economics, political science, and evolutionary biology. He is a member of the Royal Swedish Academy of Sciences and Fellow of the Econometric Society.

MY PROFESSOR

Artwork by Sam White.

AMS Blogs
AMERICAN MATHEMATICAL SOCIETY

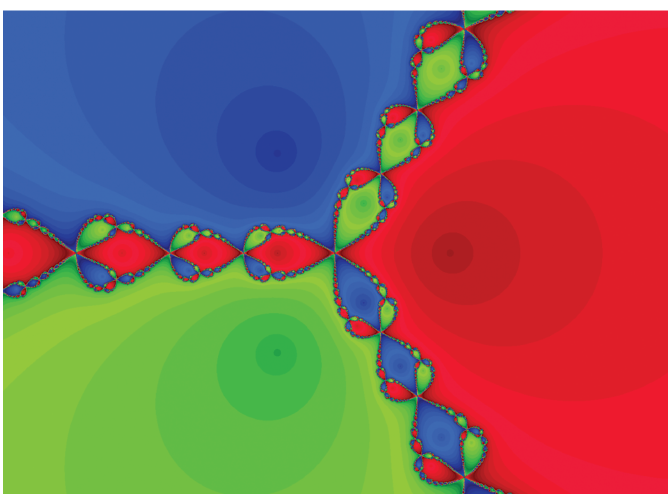
Graduate Student Blog
by and for math graduate students

The AMS Graduate Student Blog, by and for math graduate students, includes puzzles and a variety of interesting columns. *January 2016 posts include "The Academic Job Search IV: Negotiating an Offer", an interview with number theorist Leo Goldmakher, and those sampled below.*
`blogs.ams.org/mathgradblog`.

Visualizing Newton's Method

by djbruce, University of Wisconsin

...One of the topics commonly covered in a first or second semester of calculus is the use of Newton's method to approximate roots of functions.... When I've taught Newton's method, I tried to stress that this method is not guaranteed to always work and can be fairly sensitive to the initial condition.... One way to visualize some of these complexities is via a cool program called FractalStream. FractalStream takes each point in the complex plane and iterates it under the given map until the sequence seems to stop. It then colors that initial point depending on where the sequence of iterates ended.



Notice that while there is a large area round each root in which Newton's method converges quickly to that root the areas sort of between each root show more complex behavior. In particular, in this region we see just how sensitive to the initial condition Newton's method becomes....

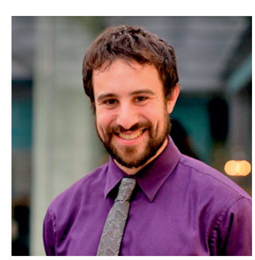
djbruce is a second-year graduate student at the University of Wisconsin.

Matrices and MLK Day

by Matthew Simonson

In February 2013, the *Wall Street Journal* reported, "Prison sentences of black men were nearly 20% longer than those of white men for similar crimes in recent years...". Is this evidence of racism, intentional or subconscious, on the part of judges? ...that is what we will try to suss out here using matrix multiplication.... I'm not claiming that you can fully explain racial sentencing disparities in one lesson, [but] there is plenty of room in the curricula of introductory college math courses to tackle race, class, and social justice. And indeed, there is no excuse to stand on the sidelines in an age of such inequality and injustice. Many math, science, and engineering students badly need to be exposed to the reality of these inequities, and what better a place than in a course that they value, in a context they find engaging? Students from other disciplines merely trying to fulfill their quantitative requirement might suddenly find that math is important to the world they live in and the values they hold. Moreover, both types of students will learn how math can serve as a valuable tool for fighting injustice....

Matthew Simonson is a first-year Network Science doctoral student at Northeastern University in Boston and Editor-in-Chief of the Graduate Student Blog.



CALL FOR NOMINATIONS

The selection committees for these prizes request nominations for consideration for the 2017 awards, which will be presented at the Joint Mathematics Meetings in Atlanta, GA in January 2017. Information about past recipients of these prizes may be found in the April 2014 and 2015 issues of the *Notices*, pp. 398–404 and 427–429, respectively, and at www.ams.org/profession/prizes-awards/prizes.

BÔCHER MEMORIAL PRIZE

The Bôcher Prize, awarded for a notable paper in analysis published during the preceding six years, is awarded every three years. To be eligible, papers must be either authored by an AMS member or published in a recognized North American journal.

FRANK NELSON COLE PRIZE IN NUMBER THEORY

The Frank Nelson Prizes are now presented at three-year intervals for outstanding contributions in algebra and number theory published in the preceding six years. The award in January 2017 will be the Frank Nelson Cole Prize in Number Theory.

LEVI L. CONANT PRIZE

The Levi L. Conant Prize, first awarded in January 2001, is presented annually for an outstanding expository paper published in either the *Notices* or the *Bulletin of the American Mathematical Society* during the preceding five years.

EACH OF THE PRIZES BELOW IS AWARDED EVERY TWO OR THREE YEARS.

JOSEPH L. DOOB PRIZE

The Doob Prize recognizes a single, relatively recent, outstanding research book that makes a seminal contribution to the research literature, reflects the highest standards of research exposition, and promises to have a deep and long-term impact in its area. The prize is awarded every three years and the book must have been published within the six calendar years preceding the year in which it is nominated. Books may be nominated by AMS members, members of the selection committee, members of AMS editorial committees, or by publishers.

LEONARD EISENBUD PRIZE FOR MATHEMATICS & PHYSICS

The Leonard Eisenbud Prize for Mathematics and Physics honors a work or group of works that brings mathematics and physics closer together. Thus, for example, the prize might be given for a contribution to mathematics inspired by modern developments in physics or for the development of a physical theory exploiting modern mathematics in a novel way. The prize is awarded every three years for a work published in the preceding six years.

RUTH LYTTLE SATTER PRIZE IN MATHEMATICS

The Ruth Lyttle Satter Prize is presented every two years in recognition of an outstanding contribution to mathematics research by a woman in the previous six years.

Further information about AMS prizes can be found at the Prizes and Awards website:
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Further information and instructions for submitting a nomination can be found at the prize nominations website: **www.ams.org/profession/prizes-awards/nominations**

For questions contact the AMS Secretary at secretary@ams.org

The nomination period ends June 30, 2016.

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Generosity was also expressed through several thoughtful tribute gifts, as well as estate gifts from Richard M. Cohn, Isidore Fleischer, Trevor James McMinn, and Franklin P. Peterson. Their dedication to mathematics will benefit the mathematics community now and for years to come.

The future of the AMS and how it serves mathematics is bright. Thank you for your charitable giving that helps it to be so.

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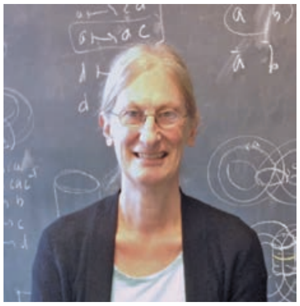

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MATHEMATICAL IMAGERY

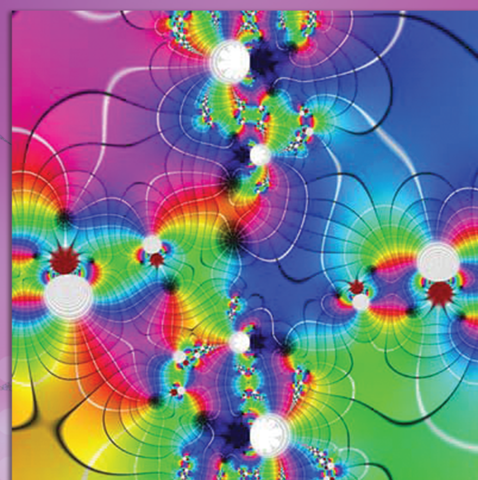
View and share hundreds of images!

The connection between mathematics and art goes back thousands of years. Mathematics has been used in the design of Gothic cathedrals, Rose windows, oriental rugs, mosaics, and tilings. Geometric forms were fundamental to the cubists and many abstract expressionists, and award-winning sculptors have used topology as the basis for their pieces. Dutch artist M.C. Escher represented infinity, Möbius bands, tessellations, deformations, reflections, Platonic solids, spirals, symmetry, and the hyperbolic plane in his works.

Mathematicians and artists continue to create stunning works in all media and to explore the visualization of mathematics--origami, computer-generated landscapes, tessellations, fractals, anamorphic art, and more.

A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

—G. H. Hardy, A Mathematician's Apology



"Exploring Complex Domain Functions Using Domain Coloring," by Konstantin Poelke and Konrad Polthier, Free University of Berlin

"Praying Mantis, opus 416," by Robert J. Lang.
One uncut square of paper, 4", composed and folded 2002.
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"Trifurcation," by Robert Fathauer
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www.ams.org/mathimagery

2016 AMS Award for Impact on the Teaching and Learning of Mathematics

MICHAEL GAGE and ARNOLD PIZER have received the 2016 AMS Award for Impact on the Teaching and Learning of Mathematics.

Photo courtesy of Sandra Cherin.

Michael Gage

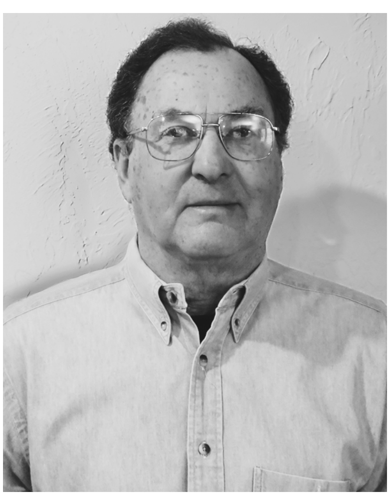

Photo courtesy of Carol Pizer.

Arnold Pizer

Citation

Michael Gage and Arnold Pizer at the University of Rochester are the mathematicians who created and developed WeBWorK, one of the first web-based systems that assign and grade homework problems in mathematics and science courses and the most successful that is nonprofit, free, open source, and textbook/publisher independent. There are now almost 1,000 institutions (high schools, colleges, large research universities) using WeBWorK, and its Open Problem Library contains more than 30,000 problems: college algebra through linear algebra, complex analysis, probability, and statistics.

Gage and Pizer began working on WeBWorK in the mid-1990s and launched it with the "Calculus with Foundations" class of twenty-nine students in fall, 1996. They received the first NSF [National Science Foundation] grant for the support of WeBWorK in 1999, the same year that WeBWorK received the International Conference on Technology in Collegiate Mathematics Award for Excellence and Innovation with the Use of Technology in Collegiate Mathematics. Since then, WeBWorK has received three additional NSF grants and is currently supported by the Mathematical Association of America. James Glimm, former president of the AMS, has written about the improvement to student learning that can come from the use of WeBWorK: "The key mechanism for this improvement seems to be that the students find their homework to be far more rewarding and do more of it, and, not surprisingly, do learn more." Instructors praise its flexibility in terms of the types of questions that can be posed and the benefits of its open source software that make it possible for individuals to add onto its capabilities.

Biographical Sketches

Michael Gage received his bachelor's degree from Antioch College (1971) and his PhD from Stanford University (1978), where his advisor was Robert Osserman. After five years in postdoctoral and visiting positions, he joined the faculty at the University of Rochester and assumed his present position as professor of mathematics in 1993. In 2014 Gage was a plenary speaker at the conference "WeBWorK and Math Support Center Workshop", held at the Hong Kong University of Science & Technology. He has served on the AMS Committee on Education (2008–2011).

Arnold Pizer received his bachelor's degree from Yale University (1967) and his PhD from Yale University (1971), where his advisor was T. Tamagawa. After assistant professor positions at the University of California, Los Angeles, and at Brandeis University, he joined the faculty of the University of Rochester in 1976 and became a full professor in 1989. In 2007 he became a professor emeritus at Rochester.

In 1999 Gage and Pizer received the International Congress on Technology in Collegiate Mathematics (ICTCM) award for creating WeBWorK. This award recognizes an individual or group for excellence and innovation in using technology to enhance the teaching and learning of mathematics.

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Information about hosting WeBWorK is available at `webwork.maa.org` and about the WeBWork community at `webwork.maa.org/wiki`.

Response

We are extremely honored to accept the AMS Award for Impact on the Teaching and Learning of Mathematics for the development of the WeBWorK homework system. We have been gratified by the positive benefit that WeBWorK has had on student homework performance. We, along with our co-principal investigator Dean Vicki Roth, director of the Center for Excellence in Teaching and Learning at the University of Rochester, wish to acknowledge the support of the National Science Foundation and the Mathematical Association of America in helping to create an active open source academic support community around the WeBWorK software. The mathematicians in this community who are augmenting and upgrading the open source software and contributing to and curating the collection of Creative Commons licensed questions in the Open Problem Library continue to improve the resources made freely available for the teaching of mathematics. We are grateful for the recognition that the AMS has given to us for initiating the WeBWorK project and fostering the growth of this WeBWorK community.

About the Award

The Award for Impact on the Teaching and Learning of Mathematics was established by the AMS Committee on Education (COE) in 2013. The Award is given annually to a mathematician (or group of mathematicians) who has made significant contributions of lasting value to mathematics education. Priorities of the award include recognition of (a) accomplished mathematicians who have worked directly with pre-college teachers to enhance teachers' impact on mathematics achievement for all students or (b) sustainable and replicable contributions by mathematicians to improving the mathematics education of students in the first two years of college. The US$1,000 award is given annually. The endowment fund that supports the award was established in 2012 by a contribution from Kenneth I. and Mary Lou Gross in honor of their daughters, Laura and Karen. The award is presented by the COE acting on the recommendation of a selection subcommittee. The members of the subcommittee were Matt Baker, David Bressoud, Jennifer Taback (Chair), and Karen Vogtmann.

Previous recipients of the Impact Award were Paul J. Sally Jr. (2014) and W. James Lewis (2015).

—AMS Committee on Education

AMS EXEMPLARY PROGRAM AWARD

CALL FOR NOMINATIONS

The AMS Award for Exemplary Program or Achievement in a Mathematics Department is presented annually to a department that has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of the society. Examples might include a department that runs a notable minority outreach program, a department that has instituted an unusually effective industrial mathematics internship program, a department that has promoted mathematics so successfully that a large fraction of its university's undergraduate population majors in mathematics, or a department that has made some form of innovation in its research support to faculty and/or graduate students, or which has created a special and innovative environment for some aspect of mathematics research.

The award amount is $5,000. All departments in North America that offer at least a bachelor's degree in the mathematical sciences are eligible.

The Award Selection Committee requests nominations for this award, which will be announced in Spring 2017. Letters of nomination may be submitted by one or more individuals. Nomination of the writer's own institution is permitted. The letter should describe the specific program(s) for which the department is being nominated as well as the achievements that make the program(s) an outstanding success, and may include any ancillary documents which support the success of the program(s). The letter should not exceed two pages, with supporting documentation not to exceed an additional three pages.

Nominations with supporting information should be submitted to www.ams.org/profession/prizes-awards/nominations. Those who prefer to submit by regular mail may send nominations to the AMS Secretary, Professor Carla D. Savage, North Carolina State University, Department of Computer Science, Campus Box 8206, Raleigh, NC 27695-8206. The nominations will be forwarded by the Secretary to the Prize Selection Committee.

Deadline for nominations is September 15, 2016.

2016 Award for an Exemplary Program or Achievement in a Mathematics Department

The Department of Mathematics at California State University at Northridge is the recipient of the 2016 Award for an Exemplary Program or Achievement in a Mathematics Department.

Citation

The American Mathematical Society is pleased to recognize the Department of Mathematics at California State University at Northridge (CSUN) with the 2016 Award for an Exemplary Program or Achievement in a Mathematics Department. CSUN is being recognized for its program "Preparing Undergraduates through Mentoring towards PhDs" (PUMP). The diversity efforts at all levels of the PUMP program have been truly exemplary.

The PUMP program was created in 2005 by a group of faculty members at CSUN with the aim of increasing access to PhD programs in the Mathematical Sciences for underrepresented minority students. CSUN is a large Hispanic-serving institution in an ethnically and economically diverse region. Before PUMP, the number of mathematics majors at CSUN was tiny and essentially none continued to PhD programs. While PUMP began as a program at CSUN, it was expanded in 2013 to include 10 Cal State campuses. The program has two main features: a centralized residential summer boot camp and a research experience during the academic year at the students' home institutions (across the 10 participating Cal State campuses). Throughout the program, students are closely mentored. They participate in regional and national conferences, and gain a strong sense of community.

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Students tend to enter the program thinking that the only career for math majors is high school teaching. Many have not declared math majors before the program. After the program, most become math majors; they are enthusiastic about mathematics research. As an example, more than 80 percent of PUMP participants from 2013 (who graduated college in 2015) have now started graduate programs in the mathematical sciences. The 2013 cohort included 52 percent women, 54 percent Hispanic, 10 percent African American, and 2 percent Native American students.

This program is a true gem. It offers a model that can be adopted nationwide. The program creates a mathematics research culture that is inviting and inclusive for undergraduates from underrepresented groups. Its impact is enormous. The California State Universities involved in this program should be proud of this achievement and their administrations should capitalize on this success by institutionalizing the program with local support.

For the many ways in which the PUMP program at California State University Northridge has had a large impact on underrepresented groups in the mathematical sciences, we are happy to present the AMS Award for Exemplary Program or Achievement to the Department of Mathematics at California State University at Northridge.

About the Award

The Award for an Exemplary Program or Achievement in a Mathematics Department was established by the AMS Council in 2004 and was given for the first time in 2006. The purpose is to recognize a department that has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of society. Departments of mathematical sciences in North

America that offer at least a bachelor's degree in mathematical sciences are eligible. Through the generous support of an anonymous donor, the award carries a cash prize of US$5,000.

The award is presented by the AMS Council acting on the recommendation of a selection committee. For the 2016 award, the members of the selection committee were: Michael Dorff, Eric Grinberg, Aloysius Helminck (Chair), Monica Jackson, and Cesar Silva.

The previous recipients of the award are:

- Harvey Mudd College (2006)
- The University of California, Los Angeles (2007)
- The University of Iowa (2008)
- The University of Nebraska, Lincoln (2009)
- North Carolina State University (2010)
- The Math Center at the University of Arizona (2011)
- Bryn Mawr College (2012)
- The University of Texas at Arlington (2013)
- Williams College (2014)
- Iowa State University (2015).

—Exemplary Program Award Selection Committee

COMMUNICATION

PUMPed about Math: CSU Northridge Wins Exemplary Program Award

Allyn Jackson

For many kids growing up in the San Fernando Valley of Los Angeles County, the default university is California State University Northridge (CSUN). It's not a "destination campus"; it's the campus down the street. CSUN (pronounced "see-sun") has more than 40,000 students, around half of them low-income and around 40 percent Hispanic. Most commute to campus, and many work part-time or even full-time and juggle complex family lives. Many are the first in their families to attend college, so their aspirations don't stretch beyond a bachelor's degree. Those who choose mathematics as a major typically see exactly one career option: teaching high school math.

So how did it come about that, in the past decade, more than fifty CSUN math majors enrolled in PhD programs? And how is it that many of them landed in top-quality mathematics programs like Georgia, Utah, UCLA, Johns Hopkins, University of Illinois Urbana-Champaign, and University of Texas Austin? And how is it that some of them took postdocs at the likes of Dartmouth, Rice, and the Institut des Hautes Études Scientifiques in Paris? Perhaps the CSUN mathematics department is becoming a "destination department".

These surprising and inspiring developments stem from PUMP (Preparing Undergraduates through Mentoring towards PhDs), a program that originated in the CSUN mathematics department. Now encompassing an alliance of ten Cal State institutions, PUMP is making a significant contribution to increasing the number of mathematicians who are members of groups traditionally underrepresented in the field. It is also profoundly changing the lives of the students it touches. For initiating PUMP and overseeing the program after its expansion to other campuses, the CSUN mathematics department has received the AMS Award for an Exemplary Program or Achievement in a Mathematics Department.

Allyn Jackson is senior writer and deputy editor of Notices. *Her email address is* axj@ams.org.

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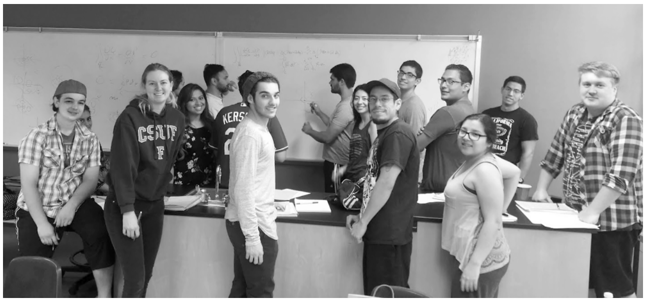
Upping Commitment to Underrepresented Students

The CSUN mathematics department, which grants bachelor's and master's degrees, has always had a commitment to encouraging students underrepresented in mathematics. After all, these are the students the department sees every day. With PUMP, this commitment has reached an entirely new level over the past decade.

PUMP started on the initiative of CSUN mathematics professor Helena Noronha, a native of Brazil who earned her PhD at the Universidade Estadual de Campinas in 1983 and joined the CSUN faculty in 1990. Having served as a program officer in the Division of Mathematical Sciences of the National Science Foundation (NSF) from 2000 until 2002 (and later from 2009 to 2011), Noronha saw the potential for her department to have a national impact. She and three CSUN colleagues, Alberto Candel, Rabia Djellouli, and Werner Horn, developed the idea of PUMP and collaborated on a successful proposal to the NSF. Noronha was the principal investigator and the other three were co-principal investigators. With Noronha as director, PUMP started in 2005. Its main goal: prepare underrepresented minority students from the San Fernando Valley for success in PhD programs in the mathematical sciences.

"In the beginning we made probably every mistake we could have made, like any program startup," said Horn, who served as director of PUMP from 2009 to 2012.

Photo courtesy of Helena Noronha.



PUMP students discuss mathematics together.

Undiscouraged, the PUMP personnel used the knowledge gained to improve and fine-tune the program. For example, at the beginning PUMP emphasized to students the offer of stipends to help them prepare for graduate work in mathematics. The stipends were certainly attractive, as many CSUN students work and receive financial aid, but they didn't always understand the career benefits of a PhD, and many struggled with the more immediate goal of graduating from CSUN.

So PUMP changed its strategy to emphasize how it would help students do better in their classes and reach graduation. As a side benefit, the students would get information about PhD programs and the career options such a degree opens. "After that they could decide whether graduate school was for them," Noronha said. This change in emphasis changed students' perception of PUMP. "It went from the students thinking that maybe the program was not for them, it was a difficult program, to 'It's cool to be in PUMP.'"

> *"It's cool to be in PUMP."*

A Winning Program Structure

By 2007 the PUMP personnel had come to understand well the needs of the students and hit upon a winning structure for the program. This structure serves as a model for the multicampus version, called the CSU Alliance for PUMP, which began in 2013 with a new five-year NSF grant (see sidebar for a list of the Alliance institutions). A centerpiece of PUMP in both incarnations is the PUMP Summer Institute. Originally the institute took place on the CSUN campus, and now it rotates among the various campuses in the CSU Alliance. Students receive stipends to attend the four-week institute, in which they take intensive, rigorous courses in linear algebra and analysis. Noronha said that those two subjects were chosen because they are fundamental to any area of specialization the students might choose. During the institute, she said, "They live in the dorms, they work together on mathematics night and day, Saturday and Sunday." The shared experience forges long-lasting bonds among the students and greatly increases their motivation and confidence.

At the summer institute, participants also learn about doctoral programs in mathematics through presentations by faculty from nearby universities, who explain what they look for in graduate school applicants. In addition, PUMP alumni who are succeeding in graduate school give talks, and what they have to say really hits home. Horn said that many of the more recent PUMP students have told them that this part of the program changed their lives. "The current students now see someone who has a demographic background similar to their own and is doing something they never dreamed of," Horn said. "That gives them the idea, 'I can do that as well.'"

> *"It is unusual that an undergraduate student in the department does not do any type of research."*

Another component of the PUMP structure is its undergraduate research projects. These take place during the academic year, with PUMP providing a bit of financial support to the faculty and students involved. Djellouli, a co-PI on the original PUMP grant and now chair of the CSUN mathematics department, has worked extensively with students on research projects. He said the projects have become an integral part of the department. "We are now in a situation where it is unusual that an undergraduate student in the department does *not* do any type of research activity," he said. "Faculty have seen that it works, it helps students, it motivates them and lifts their aspirations." The enthusiasm is now spreading in the CSU Alliance departments, where faculty can apply to PUMP for small grants to support Undergraduate Research Groups (URGs) consisting of a faculty member and at least two students.

The PUMP research projects have a ripple effect in that even after support for the projects ends, the faculty and students often continue to work together. The activity fosters closer engagement between faculty and students, thereby leading naturally to better mentoring. In addition, students gain experience in presenting their work in the PUMP Symposium, which brings together all the PUMP URG students across the CSU Alliance campuses.

Another component of PUMP is its systematic efforts to get students to apply for off-campus opportunities, such as the many Research Experiences for Undergraduates programs that take place in various locations across the United States. PUMP provides support for students to attend regional and national conferences, such as meetings of chapters of the Mathematical Association of America and the Joint Mathematics Meetings. Many PUMP students attend the Field of Dreams Conference, sponsored by the National Alliance for Doctoral Studies in the Mathematical Sciences. Held annually, Field of Dreams has become one of the major annual events where faculty come together with students from groups underrepresented in the mathematical sciences.

Photo courtesy of Helena Noronha.

PUMP students attend the 2014 Field of Dreams conference.

Positive Effects on the Department

The CSUN mathematics department soon saw the positive effects of PUMP. Faculty reported that they could teach at a deeper level than before. More undergraduate students were turning up in the department's master's-level courses. The department was able to augment its degree offerings to include a bachelor of science option, which specifically prepares majors for graduate school. Prior to PUMP, most math majors took the department's secondary teaching option for their degrees. Today there are over 110 majors in the BS/BA option versus around fifty in the secondary teaching option; around 35 students are in the applied mathematics/statistics option.

Students are also finding the department a more welcoming place. CSUN is largely a commuter campus, and prior to PUMP many mathematics students would come for two hours to attend a class and then leave to work jobs or take care of their families. "They did not stick around on campus," Horn remarked. But with PUMP, more students are staying on campus for a full day. The department has helped in various ways, such as providing tutoring jobs for advanced undergraduates and designating a room where students can hang out and work together. Now, said Horn, "There is an undergraduate and beginning graduate mathematical community" within the department. "And it all started with the original PUMP grant from 2005."

An increasing number of CSUN math majors are entering and progressing in doctoral programs. Between 2005 and 2015, over 50 CSUN math majors entered PhD programs. They are well equipped for the rigors of graduate school; very few have dropped out. Moreover, Horn said that in the few cases where students have dropped out, the reasons were always personal. "I know of no case where it was for academic reasons," he noted. Prior to PUMP, the number of CSUN math majors going to graduate school was far smaller. For example, NSF statistics show that between 1996 and 2000, only one CSUN math major completed a PhD. Each year between 2007 and 2012, an average of five CSUN math majors completed a PhD.

PUMP students have been accepted into some of the top graduate programs in mathematics in the nation and are continuing on to good positions as postdoctoral or junior faculty or in industry. Three PUMP students received NSF Graduate Fellowships. One is CSUN student Evan Randles, who went to Cornell. The other two are from Cal Poly Pomona and participated in the CSU-Alliance for PUMP: Kristin Dettmers, in applied mathematics at MIT, and Natalie Gasca, in statistics at the University of Washington. Another PUMP student, William Yessen, received an NSF Postdoctoral Fellowship and is at Rice University. Such distinctions for Cal State students were extremely rare prior to PUMP.

Photo courtesy of CSU Channel Islands.

Cynthia Flores lectures to her students at Cal State University Channel Islands.

Student Successes

There are many inspiring stories among the PUMP students, and one is that of Cynthia Flores. To say that she was the first in her family to attend college seems inadequate to describe the enormous leap she has made. Her parents fled El Salvador as civil war raged, got amnesty in the United States, and settled in Los Angeles. Although she grew up a few blocks from the University of Southern California, she had no idea what it was, and no one around her did either. She suspected it was a church, "since everyone who went there dressed nice," she said. When she was identified as gifted at the age of six, she and her parents were not sure what the term meant, other than that she was given extra assignments. She loved math and excelled in the subject, but by high school she found her math classes dull. On the advice of a guidance counselor, she enrolled in college algebra at a nearby community college and enjoyed it greatly. This was her first inkling of what college would be like.

Intending to become a high school math teacher, Flores enrolled in CSUN. She did well academically but held many misconceptions about career paths. She thought that college professors were people who had been outstanding high school teachers and had been promoted. It was not until Flores enrolled in PUMP that Noronha explained the actual career path of a mathematics professor. Given Flores's excellent grades, Noronha recommended she stay on for a master's degree at CSUN and then apply to graduate school. PUMP "changed my life," Flores said. A major factor was the dedication of the PUMP faculty. "They strike a balance between mentoring, advising, and rigorous teaching" that provided a strong foundation for graduate school, she said.

Today, as assistant professor at California State University Channel Islands, Flores is once again involved with PUMP, through the CSU Alliance. Like the other universities in the alliance, Channel Islands is a Hispanic-serving institution. PUMP allows the mathematics faculty there to capitalize on their experience mentoring students from underrepresented groups. Said Flores, "I am so excited that PUMP is making a difference in the lives of many underserved undergraduate students."

Another PUMP student, Sam Fleischer, has a very different story. Out of high school he was accepted into a top-level art conservatory in New York. He studied theater there for a year and a half before deciding that wasn't what he wanted to do. He ended up at CSUN mainly because his mother works there and he got a tuition reduction. After entering PUMP, he attended a Field of Dreams conference and began exploring predator-prey models under the guidance of CSUN math professor Jing Li. "Clearly, PUMP has had a direct, positive effect on my career as a mathematician," he said.

Photo courtesy of Sam Fleischer.

Sam Fleischer leafs through a newspaper in a coffee house in San Francisco.

Fleischer is now a doctoral student and teaching assistant at UC Davis. He noted that if his original goal had been an academic career, he might have tried for one of the many fancier institutions in the Los Angeles area. But after taking part in PUMP at CSUN, he has no regrets. "I ended up having a great experience, since there was less competition to do research with professors and I received lots of one-on-one attention when I sought it out in office hours," he said. His experience at UC Davis made him appreciate CSUN and PUMP all the more. For undergraduates at big research-oriented universities, he observed, "there is much greater potential to get lost and not receive adequate attention from professors."

Keys to Success

One reason PUMP is so successful in getting underrepresented minority students into mathematics is that PUMP lives where these students live. While other programs make heroic efforts to recruit qualified students from underrepresented groups, CSUN does essentially no recruiting for PUMP. The same is true for the institutions in the CSU Alliance for PUMP, where the percentage of Hispanic students ranges between 25 and 50 percent.

Programs that recruit underrepresented students often must put in a lot of effort to strengthen the students' backgrounds to prepare them for graduate school. PUMP works differently in that its summer program reaches students much earlier, when they are sophomores or juniors. As Djellouli put it, PUMP operates at a "crossroads" point for the students. By the time PUMP students are seniors, "they know so many things about mathematics, about the beauty of math, the variety of the math, and also about possible jobs, whether in academia or industry," he said. "Also, they know themselves.... Those who decide to go into PhD programs really know that this is what they want. These are not students who are going to start and drop."

The success of PUMP has put CSUN on the map in various ways. Students at community colleges in California who are interested in mathematics often want to transfer to CSUN for bachelor's degrees, to join PUMP. The department is able to attract very good job candidates who fit its elevated profile as a center for undergraduate research. For the first time, the department is getting job applications from overseas candidates. Also for the first time, CSUN's very own alumni are applying for jobs in the department.

The effects of PUMP have also raised the profile of the mathematics department on the CSUN campus. As the largest department in terms of enrollments, mathematics used to be viewed solely as a service teaching department. That view has shifted, as the department has become more research-oriented, partly because of PUMP and partly because of new hires. Also, the number of students the mathematics department sends to graduate programs is now on a par with, for example, the biology department.

"We are a little more respected now as a valid academic department," said Horn.

What PUMP has done at CSUN and is now spreading to the other campuses in the CSU Alliance is to foster a "doctoral culture" in an undergraduate institution. PUMP students thrive in this culture. And in the end, the students are the real reason for the success of PUMP. PUMP provides a structure and setting where the students' knowledge and aspirations can blossom. Said Djellouli, "The students when we talk to them, they are so happy, so thankful that they had this opportunity."

CSU Alliance for PUMP

The CSU Alliance for PUMP comprises the following campuses in the California State University system:

Channel Islands	Los Angeles
Dominguez Hills	Northridge
Fresno	Pomona
Fullerton	San Marcos
Long Beach	San Bernardino

In Her Own Words: A PUMP Success Story

PUMP had a very strong impact on my academic goals and achievements, so I am very passionate about this program.

I was accepted into the PUMP Summer Program in 2011, during my sophomore year. Until then I was set on becoming a math teacher, mainly because teaching was familiar, since my mom is a teacher. But I wanted to learn more about graduate school because academics is what I really enjoyed and excelled in.

A week before PUMP started, I found out I was expecting my first child—I was only nineteen years old. I had always hoped to become a mother, but at such a young age and in an unplanned situation, I felt as if I had failed. I felt I was no longer a good role model for my family. I am from the Northridge area and am the oldest of my cousins and siblings. We grew up in a low-income area with not so many great influences, so I had a lot of pressure to be "perfect". How could I consider graduate school when I was unsure I would even be able to finish my undergraduate degree?

Photo by Jesus Reyes.

Brianna Amador teaches her son some mathematics by counting fruit snacks.

I decided to attend PUMP anyway, and I am glad I did. Many PUMP guest speakers told of their experiences. One of them was earning her PhD and had her first child at age eighteen. Meeting her gave me hope. After the program I really considered making a PhD my goal and sought more opportunities that could help me get there. However, while trying to talk about my goals with professors, some of them put me down. One reacted with rolling eyes when I said I'd be taking a semester off because I was pregnant. Another lectured me about how long it took some women to earn their PhDs. Another just sighed. I was filled with guilt, anger, sadness. But I wanted to prove them wrong. I had to keep reminding myself of the women I had met in PUMP.

I had my son in January 2013. Although I took some time off classes, I found another opportunity to go after: The PUMP Undergraduate Research Group (URG). That was a perfect fit for me because it allowed me to gain research experience and did not require me to move out of state.

The URG helped me realize that I really enjoy computational biology, analyzing data, and building mathematical models. This experience led me to consider graduate programs in computational biology and biostatistics. My URG research advisor Bruce Shapiro was very supportive of my career path and my new status as a mother. He offered me an opportunity to continue the research at Caltech, where I collaborated with biologists, presented my work at a large conference, and learned new programming skills.

After that, I was accepted to do the PUMP Summer Program and PUMP URG a second time. I was mentored by Ramin Vakilian in multiple research projects and attended more conferences to present our work. Both he and PUMP founder Helena Noronha supported me and wanted to see me succeed.

During my last year, I applied to graduate programs, mostly teaching credential and master of education programs. I wanted to retreat to something that was more familiar and that I could do closer to home. Also, the negative responses I'd had from some of the professors during my pregnancy made me wary. However, Helena really pushed me to apply to master's programs in biostatistics, including the one at Duke University.

All but one of the programs I applied to accepted me with scholarships. My mind was set on UCLA, because it was nearest to home and it was a teaching program. But I also went to visit Duke. I did not feel there that negativity I had encountered before---people actually wanted to know more about my son and my experience. I kept reminding myself of what I had learned at PUMP and how many of the people I met there had struggles but overcame them and were succeeding. During the last week before decision deadlines, my scholarship from Duke was increased.

I am now in the master of biostatistics program at Duke University. I owe so much of where I'm at today to PUMP. It paved the way for many opportunities and gave me confidence to keep pushing through. It has definitely been challenging, but I'm making it just fine. As for my son, he is now three and is loving the new sights in North Carolina.

—Brianna Amador

A View from a CSU Alliance Member

I first heard about PUMP when I met Helena Noronha at a conference at Harvey Mudd College in 2013. I could hardly believe that such a program exists: stipends for Cal State faculty and students to learn mathematics and conduct research together. I got involved in the CSU Alliance for PUMP and ran the 2014 PUMP Summer Program at my university, Cal Poly Pomona. I also applied for and received support to conduct research with a pair of undergrads (one of whom, Kristin Dettmers, is now at MIT pursuing her PhD in applied math).

What makes PUMP so successful is the confidence it ignites in its students. PUMP applicants are typically underrepresented students who may not have ever thought about pursuing a graduate degree in math. The PUMP Summer Program provides a unique opportunity at a critical point in their development to learn about proof-based mathematics beyond the computations of calculus courses in a challenging and supportive environment. The stipend and lodging support the Summer Program provides are vital: without this monetary support, many applicants might, for example, be under pressure to make money by taking a part-time job to support themselves or their families.

Photo courtesy of Helena Noronha.

Students at the PUMP Summer Program in July 2015. In the front row are PUMP director Helena Noronha on the far left and, on the far right, two Cal Poly Pomona faculty members who are co-directors of the CSU Alliance for PUMP, John Rock and Arlo Caine.

CSU Alliance member departments see improvement in the performance of students who have taken part in the PUMP Summer Program and an increase in the number of math faculty who conduct research with undergraduates. Cal State faculty members are always encouraged to conduct research with students, but this is frequently not feasible due to our significant teaching and service responsibilities. The stipends faculty receive through the PUMP Undergraduate Research Groups (URG) are crucial.

Similarly, the stipends provided to the students allow them to forgo part-time work and focus on their mathematical development. Plus, PUMP supports the URG-supported faculty and students as they travel to conferences to give talks or present posters at mathematics conferences.

The future of PUMP is very bright. Word has begun to spread through the Cal State System that PUMP is a program that makes a difference. The recognition of PUMP provided by Phil Kutzko and the Math Alliance has sparked interest in PhD-granting institutions and other four-year and master's-granting institutions. There is significant potential for the PUMP program to grow beyond the Cal State System and become a force for strengthening mathematics programs across the country.

—John Rock, Cal Poly Pomona, co-director,
CSU Alliance for PUMP

American Mathematical Society

CALL FOR NOMINATIONS

AMS Award for *Mathematics Programs that Make a Difference*





Deadline: September 15, 2016

This award was established in 2005 in response to a recommendation from the AMS's Committee on the Profession that the AMS compile and publish a series of profiles of programs that:

1. aim to bring more persons from underrepresented backgrounds into some portion of the pipeline beginning at the undergraduate level and leading to advanced degrees in mathematics and professional success, or retain them once in the pipeline;
2. have achieved documentable success in doing so; and
3. are replicable models.

Preference will be given to programs with significant participation by underrepresented minorities.

One or two programs are highlighted annually.

Nomination process: Letters of nomination may be submitted by one or more individuals.

Nomination of the writer's own program or department is permitted. The nomination should describe the specific program which is being nominated as well as the achievements that make the program an outstanding success. The letter of nomination should not exceed two pages, with supporting documentation not to exceed three more pages. Up to three supporting letters may be included in addition to these five pages.

Send nominations to:
Programs that Make a Difference
c/o Associate Executive Director, Meetings and Professional Services
American Mathematical Society
201 Charles Street
Providence, RI 02904
or via email to AED-MPS@ams.org

Recent Winners:

2016: Department of Mathematics, Morehouse College

2015: Pacific Coast Undergraduate Math Conference (PCUMC);
Center for Undergraduate Research in Mathematics (CURM)

2014: Carleton College Summer Math Program;
Rice University Summer Institute of Statistics

2013: Nebraska Conference for Undergraduate Women in Mathematics (NCUWM)

2012: Mathematical Sciences Research Institute

AMS
American Mathematical Society
www.ams.org

2016 Mathematics Programs That Make a Difference

Photo courtesy of the Morehouse College Department of Mathematics.

The Harriett J. Walton Symposium on Undergraduate Mathematics Research, held annually at Morehouse College, brings mathematics students from Georgia and beyond to share oral presentations about their work.

Each year, the AMS Committee on the Profession (CoProf) chooses outstanding programs to be designated as Mathematics Programs That Make a Difference. For 2016 CoProf has selected the Department of Mathematics at Morehouse College.

Citation

Be it resolved that the American Mathematical Society and its Committee on the Profession recognize the Department of Mathematics at Morehouse College for its significant efforts to encourage students from underrepresented groups to continue in the study of mathematics.

Morehouse College is a private, all-male, historically black college in Atlanta, Georgia, with enrollment of approximately 2,200 students. In recent years its Department of Mathematics has graduated an average of fourteen mathematics majors per year. This places Morehouse as the nation's top producer of black male mathematics degree recipients (and one of the top producers of all black mathematics graduates). Roughly half of recent mathematics majors have gone on to graduate programs in STEM disciplines, a majority of those in the mathematical sciences. Notably, three alumni earned mathematics PhDs in 2015 (and a total of six in the past seven years); for comparison, a total of fifteen black male US citizens earned a PhD in mathematics nationwide in 2013–14.

Alumni laud Morehouse faculty for fostering a welcoming and caring environment

According to one of the letters in support of its nomination, "The Morehouse program emphasizes a culture of mentoring and strong personal interactions between faculty and students.… When I talk to Morehouse students, the common theme is not *what* led to their success but *who* did so." Alumni of the program laud the Morehouse faculty for fostering a welcoming and caring environment while at the same time establishing and maintaining high expectations for the majors.

The AMS commends the members of the Department of Mathematics at Morehouse College for their high level of

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DOI: http://dx.doi.org/10.1090/noti1378

commitment and successful efforts to improve diversity in the profession of mathematics in the United States.

Program Description

Each year, the Department of Mathematics at Morehouse College hosts the Harriett J. Walton Symposium on Undergraduate Mathematics Research. Now in its fourteenth year, the conference gives undergraduate math majors in Georgia and neighboring states a venue in which to present their own research and to make contact with others sharing their interests. The conference was named to honor Harriett J. Walton, a remarkable black woman who joined the Morehouse faculty in 1958. By that time she held master's degrees in mathematics from both Howard and Syracuse Universities. While teaching full-time at Morehouse and raising four children, she earned her PhD in mathematics education from Georgia State University in 1974. A teacher and mentor of exceptional dedication and a beloved member of the department, she retired from Morehouse in 2000, after forty-two years of service.

That the conference was named after this inspirational figure gives a hint of the soul of the Morehouse department. This is a place that values and draws on its heritage and traditions while striving to improve itself today and plant seeds for the future.

Founded in 1867, Morehouse College is the nation's only institution of higher education dedicated to black men. Its approximately 2,200 students are all male, and nearly all of them are black. With an emphasis on top-quality academics, the college aims to produce highly educated and morally conscious graduates who are, as the Morehouse website puts it, "the heart, soul and hope of the community."

The Department of Mathematics at Morehouse is oriented to this ideal. Its faculty strive to provide a challenging yet nurturing environment for all of its students. Much of the department's energy is focused on excelling in the ordinary business of all mathematics departments: delivering high-quality instruction in rigorous courses. Faculty provide plenty of office hours for one-on-one help, students can join study sessions to work together, and the Mathematics Lab offers a sociable setting where students provide or receive tutoring. Encouragement and support are given both to students who are struggling and to students who are excelling.

A mathematical bond that creates a brotherly sense of community

Through interactions with faculty and colloquium speakers, Morehouse mathematics students gain new perspectives on the opportunities a degree in mathematics opens up. They also start to see beyond their coursework and to get a taste of what research in mathematics is like. Mathematics majors are encouraged to participate in Research Experiences for Undergraduates programs, and some also do research projects with Morehouse faculty. Students make presentations in departmental poster sessions and at local and national mathematics conferences, including the Harriett J. Walton Symposium.

Photo courtesy of the Morehouse College Department of Mathematics.

Morehouse mathematics professor Ulrica Wilson (second from left) with students Aquia Richburg, Terrell Glenn, and Caleb Bugg on the Morehouse Mathematics Awards Day in April 2015.

The department hosts various social events that build community among the students and foster a sense of camaraderie centered on shared interest in mathematics. When a mathematics education researcher, Christopher Jett, was invited by the department to interview its majors in 2014, he found that "[the students'] mathematical bond created a brotherly sense of community among them" (quotation from a summary report submitted to the Morehouse department). He described the atmosphere in the Morehouse department as a "Mathematical Brotherhood."

Today the department has about sixty mathematics majors, and an average of fourteen of them graduate each year, making Morehouse the nation's top producer of black male mathematics bachelor's degree recipients. Increasingly, Morehouse math majors are continuing on to graduate school. Of the twenty-nine majors who finished in 2014 and 2015, half are now in graduate programs, most of them in the mathematical sciences and nearly all in science, engineering, or technology disciplines.

Photo courtesy of the Morehouse College Department of Mathematics.

Morehouse students do mathematics at an "analysis party".

The year 2015 was a banner year for the department, as three of its alumni received PhDs in mathematics: Kevin Buckles (PhD, Tufts University), now at Henry Ford College; Bobby Wilson (PhD, University of Chicago), now a Moore

Instructor at the Massachusetts Institute of Technology; and Samuel J. Ivy (PhD, North Carolina State University), now an assistant professor at the United States Military Academy in West Point. They are among a total of six Morehouse alumni who received mathematics doctorates over the past seven years. To put those numbers in context, consider that, between 2007 and 2014, an average of 13 mathematics PhDs went to black men each year, out of an average yearly total of 1,700 mathematics PhDs.

Photo courtesy of the Morehouse College Department of Mathematics.

Morehouse mathematics students at the Joint Mathematics Meetings: Tre Willis, Zerotti Woods, Aquia Richburg, Jordan Clark, Jerrell Mure, Octavious Talbot, and Malachi Morgan.

In a letter supporting the nomination of Morehouse for the Programs That Make a Difference award, Ivy wrote that the Morehouse Department of Mathematics has great professors who "perpetuate the ideals of Morehouse in producing great leaders and mathematical scholars….I only wish to continue the efforts of this program and its faculty within my career." In this way, the seeds the Morehouse department is planting are bearing fruit beyond the borders of its own institution.

About the Award

CoProf created the Mathematics Programs That Make a Difference designation in 2005 as a way to bring recognition to outstanding programs that successfully address the issue of underrepresented groups in mathematics. Each year CoProf identifies one or two exemplary programs that:

1. aim to bring more individuals from underrepresented minority backgrounds into some portion of the pipeline beginning at the undergraduate level and leading to an advanced degree and professional success in mathematics or retain them in the pipeline,
2. have achieved documentable success in doing so, and
3. are replicable models.

The CoProf subcommittee making the selection for this year's awards consisted of Michael Dorff, Pamela Gorkin, Kendra Killpatrick, William McCallum, and David Savitt (Chair).

Previously designated Mathematics Programs That Make a Difference are: Graduate Program at the University of Iowa, and Summer Institute in Mathematics for Undergraduates at Universidad de Puerto Rico, Humacao (2006); Enhancing Diversity in Graduate Education (EDGE) at Bryn Mawr College and Spelman College, and Mathematical Theoretical Biology Institute (MTBI) at Arizona State University (2007); Mathematics Summer Program in Research and Learning (Math SPIRAL) at University of Maryland, and Summer Undergraduate Mathematical Science Research Institute (SUMSRI) at Miami University (Ohio) (2008); Department of Mathematics at University of Mississippi, and Department of Statistics at North Carolina State University (2009); Department of Computational and Applied Mathematics at Rice University, and Summer Program in Quantitative Sciences at Harvard School of Public Health (2010); Center for Women in Mathematics and the Center's Post-Baccalaureate Program at Smith College, and Department of Mathematics at North Carolina State University (2011); Mathematical Sciences Research Institute in Berkeley (2012); Nebraska Conference for Undergraduate Women in Mathematics (2013); Carleton College Summer Mathematics Program, and Rice University Summer Institute of Statistics (2014); and Center for Undergraduate Research in Mathematics at Brigham Young University, and Pacific Coast Undergraduate Mathematics Conference (2015).

—Allyn Jackson

Photo courtesy of the Morehouse College Department of Mathematics.

Morehouse mathematics professors Curtis Clark (far left) and Duane Cooper (far right) flank students Arman Green, Aquia Richburg, and Johnny Gillings Jr. during commencement in May 2015.

2016 Joint Policy Board of Mathematics Communications Awards

Two Communications Awards of the Joint Policy Board for Mathematics (JPBM) were presented at the Joint Mathematics Meetings in Seattle, Washington, in January 2016. The Museum of Mathematics received the 2016 Communications Award for Public Outreach, and Simon Singh was presented the 2016 JPBM Communications Award for Expository and Popular Books. The JPBM Communications Award is presented annually to reward and encourage journalists and other communicators who, on a sustained basis, bring mathematical ideas and information to nonmathematical audiences. JPBM represents the American Mathematical Society, the American Statistical Association, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics. Each award carries a cash prize of US$1,000.

Previous recipients of the JPBM Communications Award are:

- James Gleick (1988)
- Hugh Whitemore (1990)
- Ivars Peterson (1991)
- Joel Schneider (1993)
- Martin Gardner (1994)
- Gina Kolata (1996)
- Philip J. Davis (1997)
- Constance Reid (1998)
- Ian Stewart (1999)
- John Lynch and Simon Singh (special award, 1999)
- Sylvia Nasar (2000)
- Keith J. Devlin (2001)
- Claire and Helaman Ferguson (2002)
- Robert Osserman (2003)
- Barry Cipra (2005)
- Roger Penrose (2006)
- Steven H. Strogatz (2007)
- Carl Bialik (2008)
- George Csicsery (2009)
- Marcus du Sautoy (2010)
- Nicolas Falacci and Cheryl Heuton (2011)
- Dana Mackenzie (2012)
- John Allen Paulos (2013)
- Danica McKellar (2014)
- Nate Silver (2015)

Citation: MoMath

The 2016 JPBM Communications Award for Public Outreach is presented to the Museum of Mathematics, "MoMath", for its innovative approach to presenting fundamental mathematical ideas to the public in a variety of creative, informative, and entertaining exhibits and events that engage audiences with the beauty and utility of mathematics in daily life.

Biographical Sketch of MoMath

To a mathematician, math is a world of discovery and exploration. It's a place where one can wonder "what if" and then seek the answers. It's a world of color, imagination, and beauty, a place where one can be creative and discover a host of unexpected connections to the world around us. But this place, this idea of math as an unbounded realm yet to be fully explored, is foreign to many people. Mathematics is often portrayed as a tool; a series of steps one performs to solve a particular problem. And those problems may have nothing to do with the world around us, or with the human experience.

The National Museum of Mathematics was founded to share the real world of mathematics with the public and to allow everyone to experience the sense of the wonder and beauty that can be found within this world. For more than seven years, the Museum has been working to create a sense of community, to bring together seasoned professionals with bright young students, academic mathematicians with their counterparts in education, senior citizens with wide-eyed toddlers, and people from all walks of life and all

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Photo by Buck Ennis.

Glen Whitney, MoMath Treasurer, and Cindy Lawrence, MoMath Executive Director and CEO.

backgrounds, sharing together a unique experience as they view the world around them through a new, mathematical lens.

Response from the Museum of Mathematics

It is with great honor and appreciation that the National Museum of Mathematics accepts the 2016 Communications Award for Public Outreach of the Joint Policy Board for Mathematics. Since its inception, MoMath has been warmly welcomed by the mathematical societies and indeed by the entire mathematical community. Your members have provided support, encouragement, ideas, and feedback, and the Museum simply would not exist today and would not continue to flourish without the continued involvement and collaboration of each of your organizations and its members. It is our sincere hope that together we will continue to foster a greater understanding of mathematics and to provide a place that encourages one to step into this exciting world.

Citation: Singh

The 2016 JPBM Communications Award for Expository and Popular Books is presented to Simon Singh for his fascinating books on mathematical topics, including *Fermat's Enigma, The Code Book*, and *The Simpsons and Their Mathematical Secrets*, which have opened up the beauty of mathematics and mathematical thinking to broad audiences with clear and charming prose.

Biographical Sketch of Singh

Simon Singh is a writer and broadcaster who lives in London. Having completed his PhD in particle physics at the University of Cambridge and CERN, Singh joined the BBC's Science Department in 1990. He was a producer and director on programs such as *Tomorrow's World, Horizon*, and *Earth Story*. His documentary about Fermat's Last Theorem was titled "The Proof" in North America and broadcast as part of the *Nova* series on PBS. The film was nominated for an Emmy and won a British Academy of Film and Television Arts (BAFTA) award.

In 1997 he wrote a book on the same subject, titled *Fermat's Last Theorem* in the United Kingdom and *Fermat's Enigma* in North America, which was the first mathematics book to become a number one bestseller in Britain. It has been translated into over twenty-five languages.

Singh published *The Code Book*, a history of codes and codebreaking, in 1999. His most recent book is *The Simpsons and Their Mathematical Secrets*, which explores the numerous references to mathematics hidden in the world's most successful TV show. The references are the result of a writing team that contains several people with strong mathematical backgrounds. His other books are *Big Bang*, a history of cosmology, and *Trick or Treatment? Alternative Medicine on Trial*, coauthored with Edzard Ernst.

He has presented several radio and TV shows in the United Kingdom, most notably *The Science of Secrecy* (a five-part history of cryptography), *Mind Games* (a puzzle series), and *Five Numbers, Another Five Numbers*, and *A Further Five Numbers*.

His mathematical activities on stage have included *Theatre of Science* (which he performed with Richard Wiseman in London, Edinburgh, Dublin, and New York) and *The Uncaged Monkeys* (a show involving comedians and nerds, which played to forty thousand people across twenty-three shows in the United Kingdom). Online, he is a contributor to Brady Haran's very successful YouTube channel "Numberphile".

He has spoken to approximately five hundred school groups over the last twenty-five years, and his school-based projects include the Undergraduate Ambassadors Scheme, which currently runs in over one hundred STEM departments in the United Kingdom, sending one thousand undergraduates into schools each year in order to support pupils.

Response from Simon Singh

Photo by Nick Smith.

Simon Singh

I am delighted to receive this award, particularly as my background is in physics rather than mathematics.

Although I am very proud of my books and my other work, I am sometimes concerned that we place too much emphasis on popularizers such as myself while paying insufficient attention to what happens in high schools. And when we do look at the achievements of schools, my experience is that we tend to focus on supporting and encouraging the weak or average students, while perhaps ignoring the strong students.

While many keen, strong young mathematicians will read popular books on mathematics, they are not a replacement for a rich and challenging curriculum, presented day after day, year and year, something that will provide a springboard for the mathematicians (and scientists and engineers) of tomorrow.

2016 Breakthrough Prize and New Horizons in Mathematics Prizes Awarded

Photo courtesy of Ian Agol.

Ian Agol

Breakthrough Prize: Ian Agol

Ian Agol of the University of California Berkeley and the Institute for Advanced Study has been selected as the recipient of the 2016 Breakthrough Prize in Mathematics by the Breakthrough Prize Foundation. Agol was honored "for spectacular contributions to low dimensional topology and geometric group theory, including work on the solutions of the tameness, virtual Haken and virtual fibering conjectures."

The *Notices* asked David Gabai of Princeton University to comment on the work of Agol. Gabai responded: "Ian Agol is a brilliant mathematician who has made many important and fundamental contributions to such areas as 3-dimensional topology, geometric group theory, hyperbolic geometry, foliation theory, and knot theory. His work utilizes an uncommonly wide range of techniques and methods. Using hyperbolic geometry and 3-manifold topology he proved the Marden Tameness conjecture (independently proved by Danny Calegari and Gabai). That result is crucial to many other results in hyperbolic geometry, such as the ending lamination theorem of Brock-Canary-Minsky. It also proved the Ahlfors measure conjecture, which had been reduced to Marden's conjecture by Thurston and Canary.

"Agol, with Storm and Thurston, was the first to apply Perelman's work on Ricci flow to a problem outside of geometrization to address many important questions about volumes of hyperbolic 3-manifolds. In particular, their paper gave a new proof of a result of Agol-Dunfield that was used to find the minimal volume closed orientable hyperbolic 3-manifold (Gabai-Meyerhoff-Milley). Agol, with Codá Marques and Neves, used the min-max theory and the Willmore conjecture (Codá Marques-Neves) to solve a long-standing conjecture of Freedman-He-Wang on Möbius energy of links. Using ideas from sutured manifold theory, Agol found an elegant criterion for a 3-manifold to have a finite sheeted cover that fibers over the circle. This, together with the deep work of Dani Wise on quasi-convex virtual hierarchy groups and the foundational work of Haglund-Wise on special cube complexes, proved Thurston's virtual fibering conjecture for Haken hyperbolic 3-manifolds. Agol used geometric group theory (in part with Groves and Manning) and his 3-manifold intuition to solve a conjecture of Wise that implied that all closed hyperbolic 3-manifold groups are quasi-convex virtual hierarchy groups. (This used the Kahn-Markovic solution to the surface subgroup problem and the seminal work of Sageev, as shown by Bergeron and Wise, who proved that fundamental groups of closed hyperbolic 3-manifolds satisfy the hypothesis of Wise's conjecture.) In combination with Agol's virtual fibering criterion, this proved the full Thurston virtual fibering conjecture and hence Waldhausen's virtual Haken conjecture. (That Waldhausen's conjecture reduces to one on hyperbolic 3-manfolds relies on Perelman's geometrization theorem.)

"Beyond solving famous long-standing conjectures, this monumental work proves that the group

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theory of hyperbolic 3-manifolds has tremendous structure not incomparable with that of two-dimensional manifolds. Surfaces have great structure in large part because (except for the sphere) they can be reduced to the disc by sequentially cutting along essential curves. This enables one to prove theorems via induction arguments (often very sophisticated). A consequence of this work is that given any closed hyperbolic 3-manifold, one can first pass to a finite sheeted cover and then cut by a single surface and be left with the product of a surface with the interval. That in turn is the starting point for many deep results in 3-manifold theory."

Biographical Sketch

Ian Agol was born in 1970 in Hollywood, California. He obtained his PhD in 1998 from the University of California San Diego under the direction of Michael Freedman. He taught at the University of Illinois at Chicago from 2001 to 2007 before joining the faculty at Berkeley. He was awarded the 2009 Clay Research Award (with Danny Calegari and David Gabai). He received a Guggenheim Fellowship in 2005. In 2013 he was awarded, with Daniel Wise, the AMS Veblen Prize in Geometry. He was elected a Fellow of the AMS in 2012.

Response from Ian Agol

I would like to acknowledge my teachers who encouraged my interest in mathematics. This includes my doctoral advisor, Mike Freedman, whose example encouraged me to be fearless—to work on hard problems and to not be afraid to admit what I do not know. I would also like to acknowledge the many mathematicians whose work mine builds on and to which I've added only a small amount. Their vision encouraged me to go places mathematically that I would not have considered otherwise. I will only single out Daniel Groves and Jason Manning, with whom it has been a pleasure to collaborate. Finally, I'd like to thank my family, especially my wife, Michelle, for her support, and my mother, Maureen, for all the sacrifices she made to get the best education for my brother and me.

About the Prize

The Breakthrough Prize in Mathematics was created by Mark Zuckerberg and Yuri Milner in 2013. It aims to recognize major advances in the field, to honor the world's best mathematicians, to support their future endeavors, and to communicate the excitement of mathematics to the general public. The prize is accompanied by a cash award of US$3 million. As have all five past math laureates, Agol plans to give US$100,000 of his prize winnings to support graduate students from developing countries through the Breakout Graduate Fellowships administered by the International Mathematical Union. Previous winners of the Breakthrough Prize are Simon Donaldson, Maxim Kontsevich, Jacob Lurie, Terence Tao, and Richard Taylor (2015).

New Horizons in Mathematics Prize: Larry Guth

Photo courtesy of MIT.

Larry Guth

LARRY GUTH of the Massachusetts Institute of Technology has been awarded a New Horizons in Mathematics Prize for his "ingenious and surprising solutions to long-standing open problems in symplectic geometry, Riemannian geometry, harmonic analysis, and combinatorial geometry."

The *Notices* asked Terence Tao of the University of California Los Angeles to comment on the work of Guth. Tao responded: "By introducing powerful techniques from geometry and topology, Guth has solved or made key breakthroughs on many fundamental and difficult problems in harmonic analysis, combinatorics, and (most recently) analytic number theory. For instance, he ingeniously used algebraic topology tools relating to the Ham Sandwich Theorem to prove an important inequality now known as the endpoint multilinear Kakeya inequality, which, roughly speaking, describes the extent to which 'transverse' families of tubes (or tubelike objects) can intersect each other. He has since used this inequality (and variants thereof) for many further applications, most recently in his solution (with Bourgain and Demeter) of the Vinogradov main conjecture on exponential sums, which is connected to such fundamental problems in analytic number theory as Waring's problem (on representing large natural numbers as sums of k^{th} powers) and bounds on the Riemann zeta function. With Nets Katz, he unexpectedly deployed tools from algebraic geometry (such as the classification of ruled surfaces) to almost fully resolve a famous problem of Erdős on the minimum number of distinct distances in the plane generated by n points in the plane. Guth's work has given us several new toolboxes of techniques that we will be using to attack further problems in analysis and combinatorics for many years to come." (Note: Tao was among those who nominated Guth for this prize.)

Biographical Sketch

Larry Guth received his PhD from the Massachusetts Institute of Technology in 2005 under the direction of Tomasz Mrowka. He held a professorship at the Courant Institute before joining the MIT faculty. He received a Sloan Fellowship in 2010. He was an Invited Speaker at the International

Congress of Mathematicians in 2010 and a Simons Investigator in 2014. He received the Salem Prize in 2013 and the Clay Research Award (with Nets Katz) in 2015.

Response from Larry Guth

I want to take this moment to thank my teachers, my collaborators, and my family. I feel very fortunate to have spent time with so many special teachers. I think of them often, especially now that I also teach and I get to try to pass on to my students some of the things that I learned. I've also been very fortunate in my collaborations, and I think that my best work has been collaborative. It has opened up new doors and changed the direction of my career. Finally, I want to thank my family for all their love and support.

New Horizons in Mathematics Prize: André Arroja Neves

André Arroja Neves

André Arroja Neves of Imperial College London has been awarded a New Horizons in Mathematics Prize for his "outstanding contributions to several areas of differential geometry, including work on scalar curvature, geometric flows, and his solution with Codá Marques of the 50-year-old Willmore Conjecture."

The *Notices* asked Simon Donaldson of the Simons Center for Geometry and Physics at Stony Brook University to comment on the work of Neves. Donaldson responded: "In the ten years from his PhD, André Neves has produced a series of deep results, cutting a wide swath through differential geometry. In one direction, he and Fernando Codá Marques have made spectacular use of variational methods, bringing new ideas and techniques into this venerable field. Fifty years ago, Willmore made a conjecture whose statement can be understood in a first course in surface geometry: the L^2 norm of the mean curvature of an immersed surface of positive genus in 3-space is at least $\sqrt{2}\pi$. This was a major problem in the subject, attacked by many leading experts and finally solved by Codá Marques and Neves. Their proof is a technical masterpiece of great subtlety. They also used variational methods to establish the existence of infinitely many minimal hypersurfaces in certain Riemannian manifolds, a result to set alongside the renowned classical literature on closed geodesics but which opens up completely new territory. In another direction, Neves is one of the leading experts on Lagrangian mean curvature flow, which is important in connection with Calabi-Yau geometry and mirror symmetry. Among other results, he showed that, in a certain sense, singularities in this flow are unavoidable.

"These are just a few highlights of Neves's work—there are many other important contributions, for example, to the Yamabe functional and to the asymptotic geometry of Riemannian manifolds—and we can be sure that there will be many more to come. His work is characterized by exceptional technical power, combining ingenious geometrical constructions with deep results from analysis and PDE."

Biographical Sketch

André Neves was born in 1975 in Lisbon, Portugal, and received his PhD from Stanford University in 2005 under Richard Schoen. He is a recipient of the Leverhulme Prize (2012), the Whitehead Prize of the London Mathematical Society (2013), and the AMS Veblen Prize (with Fernando Codá Marques) in 2016. He was an Invited Speaker at the International Congress of Mathematicians in 2014.

Response from André Arroja Neves

Twenty years ago I was very fortunate to find mathematics and fall in love with it. It has been a great journey so far, with its adequate share of surprises and disappointments, new paths and dead ends. I haven't traveled it alone, and two mathematicians have had a great influence on my career: my advisor, Rick Schoen, a constant source of inspiration, and my collaborator, Fernando Marques, with whom discovering mathematics together has been a tremendous pleasure. I must also thank Filipa, as none of this would have any meaning without her tremendous generosity and unyielding support, and our two little children, Eva and Tomás, who effortlessly fill my days with constant joy.

About the Prize

The New Horizons in Mathematics Prizes were created in 2015 to recognize promising junior researchers who have produced important work. Up to three of these prizes will be given each year. These prizes carry a cash award of US$100,000, each.

—Elaine Kehoe

Eliashberg Awarded 2016 Crafoord Prize in Mathematics

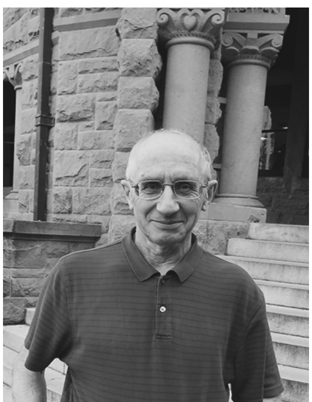

Photo by Rose Stauder. Courtesy of Y. Eliashberg.

Yakov Eliashberg

The 2016 Crafoord Prize in Mathematics has been awarded to YAKOV ELIASHBERG of Stanford University "for the development of contact and symplectic topology and groundbreaking discoveries of rigidity and flexibility phenomena".

Citation

Yakov Eliashberg has solved many of the most important problems in the field and found new and surprising results. He has further developed the techniques he used in contact geometry, a twin theory to symplectic geometry. While symplectic geometry deals with spaces with two, four, or other even dimensions, contact theory describes spaces with odd dimensions. Both theories are closely related to current developments in modern physics, such as string theory and quantum field theory.

Symplectic geometry's link to physics has old roots. For example, it describes the geometry of a space in a mechanical system, the space phase. For a moving object, its trajectory is determined each moment by its position and velocity. Together, they determine a surface element that is the basic structure of symplectic geometry. The geometry describes the directions in which the system can develop; it describes movement. Physics becomes geometry. One of Yakov Eliashberg's first and perhaps most surprising results was the discovery that there are regions where symplectic geometry is rigid and other regions where it is completely flexible. But where the boundary is between the flexible and the rigid regions, and how it can be described mathematically, is still a question that is awaiting an answer.

Biographical Sketch

Yakov Eliashberg is the Herald L. and Caroline L. Ritch Professor of Mathematics at Stanford University. Born in Russia in 1946, he received his PhD from Leningrad University in 1972 under the direction of Vladimir Rokhlin. He moved to the United States in 1988 and has been at Stanford since 1989. He received the Leningrad Mathematical Prize in 1972 and a Guggenheim Fellowship in 1995. He has been an invited speaker at the International Congress of Mathematicians in 1986 and 1998 and has been the recipient of a number of lectureships. He was awarded the AMS Veblen Prize in Geometry in 2001 and the Heinz Hopf Prize in 2013. Eliashberg was elected to the US National Academy of Sciences in 2003 and became an AMS Fellow in 2012.

About the Prize

The Crafoord Prize in Mathematics is awarded by the Royal Swedish Academy of Sciences approximately every three years. It is intended to promote international basic research in several disciplines of science, including astronomy, geosciences, biosciences (particulary ecology), and polyarthritis, as well as mathematics. These disciplines were chosen to complement those for which Nobel Prizes are awarded. The prize carries a cash award of 6 million Swedish krona (approximately US$700,000). The prize is awarded at a ceremony in Stockholm.

—*Elaine Kehoe*

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COMMUNICATION

Negotiating for Release Time and Leave

Maura Mast and Nathan Tintle

Communicated by Christina Sormani

Editor's Note: *We encourage readers to post their comments and suggestions at the* Notices *website,* `www.ams.org/notices`.

Many large-scale research projects require a course release or a complete reduction in teaching responsibilities for a semester, a year, or more. This article suggests some ways to get them.

Finding Funding

There are three main sources of funding for course releases and leaves for research-active faculty: institutional or internal funding, external public funding (typically in the form of federal support from agencies such as the National Science Foundation, the National Security Agency, or the National Institutes of Health), and external private funding (typically from foundations or industry).

Making the Case

Many institutions view their internal research funding as "venture capital" or seed money; often this funding takes the form of a course release or summer salary. Early in the life of a new research project, securing institutional funding may be the best way to find a small amount of time and resources to get preliminary work done to position yourself for external funding. Making this case explicit to your internal funding officer is a good idea.

Regarding internally funded course releases, the best opportunity to negotiate for a course release with your institution is when you are hired. It's much easier for your chair to agree to a reduced teaching load for several semesters than to agree to a higher salary or credit toward tenure. Definitely ask for what you need regarding salary and credit toward promotion, but know that if the answer is no to those requests, the chair may be willing to give you a reduced load as a compromise. You can help with this request by making the case that your research likely has relatively low start-up costs compared to lab scientists. You should also be prepared to outline a specific project that you can reasonably accomplish given the course releases.

Both new and continuing faculty who ask for course reductions should be prepared to make an argument that it is worthwhile to get this exception. Don't assume that the institution will understand how important this time is for you. Instead, be clear about what you would achieve if you had the course release and be ready to update the chair during the semester. Be specific, concrete, and realistic. It's better to say, "I will use this time to build on work I've done on affine root systems to prepare a paper for submission to the *Journal of Pure and Applied Algebra* by the end of the semester" than "I need time to get started on some work I've been thinking about for a while." If you have achievements that resulted from these requests in the past, remind the chair of this ("Last fall I used my reduced teaching load to organize a grant-funded international conference. I've been invited to edit a volume based on talks at that meeting, and a course release this fall would allow me to focus on that responsibility.") You know your work best and you can help the chair make the case to the dean by giving the chair some talking points to support your request. The administration will look for measurable outcomes—grants applied for, grants received, total funding amounts, papers submitted, talks given, conferences organized, etc.—and you should couch your request with that in mind. When your paper does get accepted or when

Be prepared to tell a good story about your work and why it's important.

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you do get the grant, notify the chair and the dean and thank them for their support.

Different offices in your institution may provide support in the form of a course release or summer salary for preparing grant or foundation applications. If you are part of a larger system, such as a state university system, you may be able to apply for a system-wide grant to support the development of a significant grant application or research project. Your sponsored-research office will likely view this type of program as a small expenditure with a large potential payback. Your international programs office may also provide small grants for international travel, either to a conference or for research work abroad. Even if these offices don't offer such support, talk to them about getting access to grant opportunities (such as a foundation database), help with grant writing, and offer ideas for making connections with other faculty with whom you could partner on proposals.

Like the sponsored-research office, the senior administrators at your institution may have seed money in the form of a course release or other activity to help faculty prepare a major grant proposal or start significant research work. Even if no such formal program exists, it's worth asking if the dean has discretionary funding to support your request. Be prepared to point to examples of how this has been successful, perhaps at your school's peer or aspirant peer institutions. And be prepared to tell a good story about your work and why it's important. The dean may be able to connect you to an external donor, such as an alum of your institution who shares your passion for this research and is willing to buy equipment or provide other support for it.

To be successful at obtaining external funding, you will need to do your research (find opportunities), learn how the system works (talk to grants officers, look at examples of successful proposals), and invest time in writing (and rewriting) a solid proposal that fits with the funding guidelines. While it may be initially more difficult to identify opportunities for funding from foundations or industry partners, it is worthwhile to look. Work with your institution's advancement/development office and your sponsored-research office early in this process.

Other Keys to Success

In this final section, we offer a few ideas for success in finding funding for your work.

Shotgunning

The funding landscape is much too fickle to let your entire research career and agenda ride on a single proposal. You must explore multiple funding options for each project. This likely will involve multiple different funding agencies and different spins on your core ideas. Just be careful that your proposal legitimately fits the request for proposals. Take the time to pitch your proposal carefully.

Developing a Research Portfolio

Just like any good investment portfolio, your research portfolio of projects should be well diversified. This includes projects that are mature and in a place where they are generating many important results, as well as new projects that may, some day, turn into a mature project. You also need to be willing to eliminate the dogs from your portfolio—projects that, despite initial promise, are no longer worthy of your time.

Momentum and Persistence

Once you receive your first funding for a release or leave, you should certainly enjoy the moment. However, you must now doubly commit to completing the projects you proposed and writing additional proposals to build momentum in your research. This commitment also needs to be able to persist in the face of unfunded proposals, rejected papers, and the general roller-coaster ride of research.

Learn to Be a Salesperson

Obtaining support for your research requires learning about sales. The product is you and your research, and you are trying to get the funding agency to buy your product. Some people are better at sales than others. Don't be discouraged if you're uncomfortable with this. It may be helpful to go to a course or do some online reading about grant writing in order to learn about the art of sales from the perspective of grant writing. Agencies such as the National Science Foundation regularly give grant-writing presentations at the national meetings. If you can't attend one of these, contact the presenter to ask for a copy of the presentation. It's always a good idea to have someone who has been funded before read your proposal and give their sales critique before sending it off to the funding agency. Another key to being in sales is networking. This includes talking with your program officer as well as the other major players in your field of research. Identify strategic partners who are already successful in your research area and begin to cultivate collaborative relationships with them.

Plan Ahead

Too many researchers are not thinking more than a couple of steps ahead in their work. But in a highly competitive funding environment you need to be thinking three to five years ahead at a minimum. Federal funding agencies have turnaround times of at least six months, often with only one call for proposals for a specific program each year. Thus, being rejected twice before being successful could mean three years before your proposal is funded. While turnaround times at the institutional level tend to be shorter, it will still take time to get the relevant parties onboard and convinced. Take the time to plan ahead.

Avoiding the Two Big Mistakes

In our experience, there are two big mistakes that we've seen made by junior research faculty who are trying to get support to start their research. The first is that they feel entitled, which leads to an unwillingness to compromise or a feeling of frustration at being rejected for funding. It's important to learn to listen carefully to peer reviewers and funding agencies, learn from them, and make sure you keep your options for funding open. The second mistake is not getting started. It's true that writing a proposal is a lot of work. It's true that it's not fun to get turned down. It's also true that if you don't start, you'll never get funded and you risk losing the excitement, momentum, and passion that you currently have for your research.

Small Funding Opportunities Shouldn't Be Overlooked

Look to different organizations for small funding. The Association for Women in Mathematics, for example, has regular competitions for travel grants and for mentoring grants. Applying for these grants will give you good practice in writing larger grant applications; success in obtaining these grants will help funding agencies and foundations see that you have a track record. Similarly, get on email distribution lists for grant opportunities, either from databases that your school subscribes to or from federal grant agencies. Don't forget private foundations and industry-sponsored opportunities!

Accountability and Moral Support

Consider forming a grant-writing group with other faculty, especially faculty from other disciplines. These networks can provide moral support and be a good resource for trying out ideas and proofreading proposals. These conversations may also lead to interdisciplinary projects.

Get Started!

Now that you've made it to the end of this article, it's time to get started. Create an action plan based on what you just read, including at least one specific to-do. Now go do it and enjoy the ride!

About the Authors

Photo courtesy of Jack Reynolds.

In the summer of 2015, Maura Mast moved with her family from Boston to the Bronx to take the position of Dean of Fordham College at Rose Hill, part of Fordham University. She is the first woman—and the first mathematician—to serve as dean of the college.

Photo courtesy of Lisa Tintle.

Nathan Tintle hiking in the Palisades of South Dakota with his two-year-old son, Mason.

Twenty Years Ago in the Notices

May 1996: *A Personal Perspective on Mathematics Research in Industry*, by Robert Calderbank.

Industrial labs where mathematics research is done have largely disappeared but still existed in 1996. His article gives insight into the rationale for having such labs.

`www.ams.org/notices/199605/comm-calderbank.pdf`

Liberal Arts Colleges: An Overlooked Opportunity?

David Damiano, Stephan Ramon Garcia, and Michele Intermont

Communicated by Steven J. Miller

Introduction

Liberal arts colleges have been part of the American academic landscape almost since the beginning of higher education in the United States; the oldest were founded in the late 1700s. They are largely an American invention, although there are fledgling liberal arts colleges being established around the world as we write. *US News & World Report* lists more than 175 institutions in its ranking of National Liberal Arts Colleges, and there are national organizations for both private (Consortium of Liberal Arts Colleges) and public liberal arts colleges (Council of Public Liberal Arts Colleges).

Liberal arts colleges, which typically enroll at most a few thousand students, are known for their attention to teaching and mentoring undergraduate students. These colleges are frequently residential, with a majority of students living on campus. They usually do not have graduate or professional programs, although some of them do. Liberal arts colleges aim to produce well-rounded graduates, armed with general intellectual skills rather than specific vocational skills.

The smaller, more intimate environment of liberal arts colleges shapes the experiences of the faculty as well as the students. As faculty members at three substantially different liberal arts colleges in different regions of the country, we would like to share our experiences with the broader mathematical community. This article is a brief overview of the opportunities, rewards, and challenges of working at a liberal arts college.

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Teaching at a Liberal Arts College

At a liberal arts college, teaching is as important as, if not more important than, research. This is reflected in the daily experience, which revolves around interactions with undergraduates. The time spent in the classroom is interspersed with office hours (Figure 1), although students are likely to drop by any time the office door

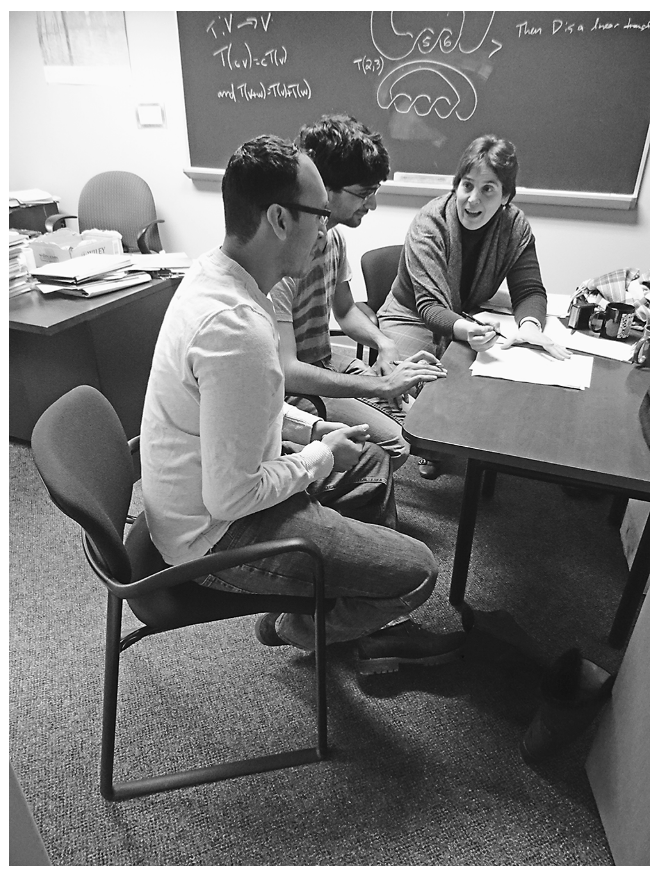

Photo courtesy of Andrew Niles.

Figure 1. The third author with students during office hours.

Photo by Alexa Miller. Photography courtesy of Pomona College.

Figure 2. Pomona College students working on an in-class assignment.

Photo by Dan Vaillancourt, College of the Holy Cross.

Figure 3. The first author with a research student.

is open. Faculty-student interaction outside the confines of the classroom or office is also common. There may be departmental lunches, a Putnam seminar, or an undergraduate colloquium, although there are many other possibilities. For instance, each semester the Pomona College mathematics department holds an overnight retreat at the college's cabin in the nearby San Jacinto mountains. At Holy Cross, the Math and Computer Science Club holds the weekly Tea and Games get-together and sponsors the annual Pi Mu Epsilon banquet, both of which are for students and faculty. Kalamazoo math students and faculty take occasional hikes along a trail and walk to the neighboring university for math events.

Faculty members teach a mix of courses each year. This likely includes lower-level courses and one or two intermediate (e.g., mulitvariable calculus and linear algebra) or upper-level courses (e.g., abstract algebra and real analysis). One can even teach an upper-level topics course as staffing allows. Class sizes are often capped around thirty, with more advanced courses sometimes significantly smaller. Faculty members grade exams and, in many instances, the homework in upper-level courses. However, it is not uncommon to have math majors grade homework for lower-level courses.

Although some introductory courses might have department-mandated syllabi, there is generally a great deal of freedom in designing syllabi and choosing pedagogical approaches. Small class sizes make it possible to employ various active-learning strategies, including group assignments (Figure 2), projects, and oral presentations. Since there are no graduate teaching assistants, students are inclined to attend office hours, either individually or in groups.

The terms "lower-level course" and "nonmajors course" at a liberal arts college do not refer to business calculus, college algebra, or precalculus. Those courses are probably not even in the course catalog at some liberal arts colleges. Instead, the introductory calculus sequence serves students majoring in all disciplines. Institutions may also offer a noncalculus course which meets a quantitative literacy requirement. The parameters of such courses often allow enough degrees of freedom for faculty members to tailor them to their interests.

There is a great deal of faculty ownership of the curriculum. Faculty members are concerned about how courses and departmental activities fit together as a whole and convey the dynamic nature of mathematics. Discussions often cross departmental boundaries in pursuit of educational goals. The experience that nonmajors have in the department is also of concern; there is an effort to ensure that students who will take only one math course have a positive and enlightening experience.

… there is generally a great deal of freedom in designing syllabi and choosing pedagogical approaches.

Working with individuals or small groups during the academic year is common. This can be in the context of a focused research project, senior thesis, or independent study course (Figure 3). Such small-scale interactions have been some of the most rewarding experiences for the third author, who has held some small, informal seminars over the years; exploring some of the key ideas of mathematics with a small group of motivated and interested students proved to be a great source of energy.

First-Year Seminars

Many liberal arts colleges, including Holy Cross, Kalamazoo, and Pomona, feature opportunities to teach outside the standard mathematics curriculum. One can develop courses for nonmajors on topics where quantitative literacy and the interests of the instructor overlap, such as sports and statistics, and it is sometimes possible to team-teach courses with colleagues from other disciplines. The most common opportunity, however, is to teach a first-year seminar.

First-year seminar programs have a variety of incarnations, but most feature small, discussion-based courses that reflect interests or topics not in the yearly mathematics course offering. Can Zombies Do Math?, Rap Music, Privacy in the Digital Age, and Ciphers and Heroes are some of the titles of seminars taught by mathematicians at our institutions recently.

Most first-year seminars have goals beyond content, such as developing writing and presentation skills and acclimating students to collegiate expectations. There is often institutional support for instructors who are new to this venture, such as guidelines to assess writing, librarians who teach students how to perform academic research in the library, and informal meetings with other seminar faculty to discuss their experiences. The third author has taught in this program several times. The experience included a mentoring aspect not present in other courses which produced a rewarding esprit de corps. She also enjoyed the opportunity to develop a seminar course that indulged her interest in writing and her enthusiasm for topology.

Research is an essential part of professional development at liberal arts colleges.

Research at Liberal Arts Colleges

Research is an essential part of professional development at liberal arts colleges, and all faculty members are expected to engage in research; the exact requirements vary between institutions. Maintaining the balance between research and teaching is a concern; the challenges of finding time to keep up with research must be addressed by each individual.

Some liberal arts colleges have, in the last decade or two, reduced their teaching loads in order to encourage their faculty to be more productive. Research expectations for new faculty members, who may have lower teaching loads than their predecessors did, are often higher than they were several decades ago. Most liberal arts colleges greatly prize those who are able to regularly incorporate undergraduates into their research.

Research expectations are often higher at liberal arts colleges with lower teaching loads than at those with higher loads, although there can be significant variations between otherwise comparable institutions. Output that might be sufficient for tenure at one institution might be insufficient to be hired at another. Each liberal arts college is different, and it is difficult to make any blanket statements regarding research expectations, except perhaps that research expectations at liberal arts colleges generally do not rise to the level of those encountered at R1 institutions.

Most liberal arts colleges greatly prize those who are able to regularly incorporate undergraduates into their research.

Although external research grants are not expected for tenure or promotion, there are National Science Foundation (NSF) programs that support research at primarily undergraduate institutions. These include the Research in Undergraduate Institutions (RUI) and Research Opportunity Awards (ROA) programs. For instance, several faculty members at Pomona and Holy Cross are supported by RUI grants.

Faculty members at liberal arts colleges are active in the AMS, MAA, and other professional organizations. They serve as editors of journals and on elected and appointed committees. For instance, the president-elect of the Association for Women in Mathematics (AWM) is from Pomona College, the editor-in-chief of the *Notices of the AMS* is from Williams College, and the president of the MAA is from Harvey Mudd College. In fact, the second author is on the editorial board of the *Proceedings of the AMS*. In the last ten years, Holy Cross (2011), Macalester College (2010), and Claremont McKenna College (2008) each hosted AMS Sectional Meetings.

One key difference between life at a liberal arts college and at a large, research-oriented institution is that faculty members must actively seek out research opportunities. Visiting scholars do not appear in the department as regularly as they do at research-intensive institutions. Liberal arts colleges typically do not boast specialized research seminars in which like-minded researchers keep up with the latest developments in the field, and there are no graduate seminars. Many liberal arts colleges, however, do have a colloquium series. Some have a colloquium requirement for their students and hence a habit of regular speakers; others have a more spontaneous approach. While the talks are geared toward undergraduate mathematics majors, for the faculty they provide accessible talks on a variety of topics and, more importantly, opportunities to network within the larger mathematical world. For many liberal arts colleges, neighboring research institutions may also be a source of inspiration; in such a setting, faculty members can attend seminars and collaborate with colleagues in research-oriented departments.

Undergraduate Research

Undergraduate research is now widely embraced by liberal arts colleges. While some colleges, such as Wabash College, Hope College, and Lafayette College, have their own NSF-funded Research Experience for Undergraduate (REU) programs, many now have college-wide undergraduate research initiatives that include summer research components. These are typically funded by established institutional budget lines and institutional grants, supplemented occasionally by individual grants. The summer programs provide experiences similar to federally funded REUs: students work with a faculty mentor, either individually or in small groups (Figure 4).

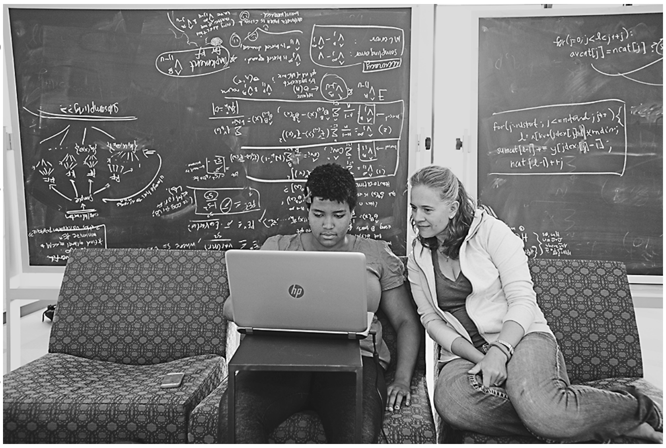

Photo by Carrie Rosema, courtesy of Pomona College.

Figure 4. Pomona College professor Johanna Hardin working with a research student.

Photo courtesy of Andrew Niles.

Figure 5. Holy Cross student presenters at the 2015 Joint Mathematics Meetings.

In-house summer opportunities can continue during the academic year in the form of independent study projects, capstone projects, or senior theses, or a research project might begin during the academic year as a senior thesis project or an independent study course. There are many opportunities for students to present posters or give talks on their work at regional conferences and national meetings, including the Joint Mathematics Meetings (Figure 5). In addition, schools that support internal research programs often have in-house poster sessions or conferences that bring together all the undergraduate research at the institution. Depending on the quality of the work, the students and mentor may write up the results for publication.

The second author has written dozens of articles with student coauthors; the vast majority of these articles stem from senior theses or academic-year research projects. Although many of these students went on to top graduate programs and several have earned NSF Graduate Research Fellowships, most of them did not start college intending to major in mathematics. One of the most rewarding aspects of working at a liberal arts college, in the opinion of the second author, is being able to reach out to students who, without individual guidance, might not envision themselves doing research or becoming mathematicians.

One of the most rewarding aspects … is being able to reach out to students who … might not envision themselves doing research or becoming mathematicians.

At some schools, such as Pomona and Kalamazoo, every mathematics major must write a senior thesis. Because the mathematics major is Pomona's most popular major, each faculty member supervises three to five senior theses per year. In contrast, at Holy Cross no more than a handful of students write a thesis in a given year. While these theses may be expository, many advisors pose projects involving original research, some of which will lead to publications. Although a thesis is not required at Holy Cross, all students are required to complete a project-based course in which students work in small groups on a project related to the topic of the course. Many involve original problem solving related to new examples or applications of mathematics. There are, of course, a wide variety of programs leading to student projects. For example, Carleton College's Comps program for mathematics and mathematics/statistics majors provides the flexibility of small group or individual work leading to a public presentation and a possible paper.

The time that faculty members spend on undergraduate projects during the academic year varies. Some meet with students once a week as a graduate advisor might, others meet more regularly as one does in an undergraduate course. While the students are talented and desire to be independent, they are not advanced graduate students. Supervisors should be ready to spend more time with their students as needed. Project topics can vary; they might be outgrowths of course-related projects that are essentially for the benefit of the student or they might be in areas that are central to the supervisor's own research. While theses and projects are time consuming for faculty advisors, they can also lead to stimulating interactions with students and can lead to new research directions.

Diversity

One of the challenges for many liberal arts colleges is diversifying the ethnic and socio-economic profile of their campuses. Among other issues, these institutions are simply not well known among many immigrant communities


Photo by Michele Intermont.

Figure 6. Some of the students in Kalamazoo's NSF STEM scholar program.

and underrepresented groups. For example, the second author is the son of immigrant parents and grew up in California but had never heard of the Claremont Colleges until he was on the job market after his postdoc. Over the last decade many liberal arts colleges have made strides in the direction of diversification, and they continue to work toward that goal.

The Posse Foundation is an important national player in promoting underrepresented students in higher education; roughly half of the fifty-three Posse partner institutions are liberal arts colleges. Posse partners with schools to identify and bring to campus students who might otherwise be overlooked in the admissions process. A cohort of talented students from public schools in a large urban area (e.g., Chicago, Los Angeles, or Miami) arrive on a campus together and navigate their four years together under the close mentorship of a faculty member. The Posse Scholars graduate from college at a rate of about 90 percent nationally.

Depending on geography and regional demographics, some liberal arts colleges have devised their own outreach programs for bringing students from underrepresented groups to campus. These may start with college preparatory or educational events, with the hope of eventually enrolling some of these students. They likely include programs to provide academic and social support on campus for their students from underrepresented groups.

Pomona College recently introduced the Pomona Scholars of Mathematics (PSM) program, a cohort model of holistic advising for students from traditionally underrepresented groups in the mathematical sciences. The PSM students enter the college with a strong interest in mathematics-based disciplines (e.g., mathematics, statistics, physics, economics, and computer science). Faculty members work closely with about sixteen students, meeting weekly with each individually. The entire PSM cohort meets as a group over lunch every week to discuss topics of interest or to interact with an outside speaker. This proactive advising and community building allows PSM students to immediately feel at home in the mathematics department. Kalamazoo College has a similar program, funded by the NSF, for underrepresented students in all STEM fields (Figure 6).

It is worth noting that while the "sticker price" of attending a liberal arts college is generally high, many colleges provide generous financial aid for students. Some liberal arts colleges have "need-blind" admissions; that is, they accept students regardless of financial need. A few of these liberal arts colleges are committed to meeting demonstrated financial need with loans or scholarships.

So You Want to Work at a Liberal Arts College?

A premise of this article is that liberal arts colleges are something of an unknown quantity for people at research universities. Graduate students and recent PhDs may be unaware of the career opportunities at liberal arts colleges and how best to prepare for them.

While some liberal arts colleges may consider new PhDs for tenure-track positions, they all prefer candidates who have led their own classes, who have taught courses beyond calculus, and who have developed an independent research trajectory. Most universities provide programs to help graduate students develop their teaching. Whether these are run by the mathematics department or by the university, they likely provide instruction on best practices as well as opportunities for critiques of teaching. Many departments provide opportunities for interested graduate students to take on more significant teaching responsibilities, such as teaching an advanced course. "Significant" here refers to the quality of teaching experience rather than the quantity.

In addition, liberal arts colleges look for some indication that an applicant knows the nature of the position. The cover letter will be read very carefully; it should be more than a perfunctory announcement of the fact that the applicant is applying for the position. Instead, the applicant should address his or her interest in a professional atmosphere as described in this article. A cover letter which also conveys familiarity with some of the features of the particular institution will be noticed. The teaching statement will be read carefully, and the research statement will be read with an eye toward assessing how well an applicant communicates with a nonspecialist.

The balanced nature of … professional expectations is the centerpiece of working at a liberal arts college.

Graduate students might find a recent trend at liberal arts colleges interesting: the creation of postdoctoral positions which carry a reduced teaching load. These provide an opportunity to develop one's teaching while protecting some time for research. Liberal arts colleges view these as a means of recruitment and as a method of providing potential candidates for tenure-track positions time to hone their skills beyond the structure of graduate school.

Although teaching at a liberal arts college is not for everyone, liberal arts colleges provide a stimulating environment in which both teaching and research are appreciated and supported throughout one's career. Indeed, the balanced nature of those professional expectations is the centerpiece of working at a liberal arts college. This has been true for the authors, who are at different stages of their careers and at substantially different liberal arts colleges; it is also true for many of our colleagues.

We hope some of our readers will consider joining us and that all our readers can appreciate the attractions of careers at liberal arts colleges.

Biographical Sketches

David Damiano has been at the College of the Holy Cross since 1984. He was trained as a topologist, and in the past decade he has pursued projects in applied mathematics, including immunological modeling and computational topology. He has been extensively involved in undergraduate research and the mentoring of majors as well as mentoring minority students. He was the recipient of the college's Distinguished Teacher of the Year Award in 2011.

Stephan Ramon Garcia has been at Pomona College since 2006. He is the author of two books and over sixty research articles in operator theory, complex analysis, matrix analysis, number theory, discrete geometry, and other fields. Many of these publications are with undergraduates. He received three NSF research grants and five teaching awards from three different institutions.

Michele Intermont has been at Kalamazoo College since 1998. She is an algebraic topologist. Her interests are in homotopy theory and, more recently, in topological data analysis. She has taught in the college's freshman seminar program and directed several undergraduate research projects. She has also served in the leadership of the Michigan section of the MAA.

Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World

Reviewed by Slava Gerovitch

Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World
Amir Alexander
Scientific American/Farrar, Straus and Giroux, April 2014
US$27.00/US$16.00, 368 pages
Hardcover ISBN-13: 978-0-37-417681-5
Paperback ISBN-13: 978-0-37-453499-8

From today's perspective, a mathematical technique that lacks rigor or leads to paradoxes is a contradiction in terms. It must be expelled from mathematics, lest it discredit the profession, sow chaos, and put a large stain on the shining surface of eternal truth. This is precisely what the leading mathematicians of the most learned Catholic order, the Jesuits, said about the "method of indivisibles", a dubious procedure of calculating areas and volumes by representing plane figures or solids as a composition of indivisible lines or planes, "infinitesimals". While the method often produced correct results, in some cases it led to spectacular failures generating glaring contradictions.

This kind of imprecise reasoning seems to undermine the very ideals of rationality and certainty often associated with mathematics. A rejection of infinitesimals might look like a natural step in the progress of mathematical thinking, from the chaos of imprecise analogies to the order of disciplined reasoning. Yet, as Amir Alexander argues in his fascinating book *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World*, it was precisely the champions of offensive infinitesimals who propelled mathematics forward, while the rational critics slowed the development of mathematical thought. Moreover, the debate over infinitesimals reflected a larger clash in European culture between religious dogma and intellectual pluralism and between the proponents of traditional order and the defenders of new liberties.

Known since antiquity, the concept of indivisibles gave rise to Zeno's paradoxes, including the famous "Achilles and the Tortoise" conundrum, and was subjected to scathing philosophical critique by Plato and Aristotle. Archimedes used the method of indivisibles with considerable success, but even he, once a desired volume was calculated, preferred proving the result with a respectable geometrical method of exhaustion. Infinitesimals were revived in the works of the Flemish mathematician Simon Stevin, the Englishman Thomas Harriot, and the Italians Bonaventura Cavalieri and Evangelista Torricelli in the late sixteenth to the early seventeenth century. The method of indivisibles was appealing not only because it helped solve difficult problems but also because it gave an insight into the structure of geometrical figures. Cavalieri showed, for example, that the area enclosed within an Archimedean spiral was equal to one-third of its enclosing circle because the indivisible lines comprising this area could be rearranged into a parabola. Torricelli, in order to demonstrate the power and flexibility of the new method, published a remark-

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Bonaventura Cavalieri (1598–1647).

Evangelista Torricelli (1608–1647), by Lorenzo Lippi (circa 1647, Galleria Silvano Lodi & Due).

able treatise with twenty-one different proofs of an already-known result (the area inside a parabola). In ten different proofs, Torricelli used indivisibles, producing effective explanatory arguments instead of cumbersome Euclidean constructions. He called the method of indivisibles "the Royal Road through the mathematical thicket," while the traditional Euclidean approach deserved "only pity" (p. 110). Due to their calculating power and explanatory appeal, infinitesimals quickly gained popularity until they faced stern opposition from the Jesuits.

The method of indivisibles had glaring flaws. Comparing the infinitesimals composing one figure to the infinitesimals composing another could produce different results, depending on the procedure used. For example, if one drew a diagonal in a rectangle with a greater horizontal side, it would split into two equal triangles, as in Figure 1 (below). Using the method of indivisibles, however, one could argue that each horizontal segment in the upper triangle was greater than the vertical segment drawn through the same point on the diagonal in the lower triangle, and therefore the two triangles differed in size. Torricelli found a way out of this conundrum by claiming that the indivisible lines of the lower triangle were "wider" than the lines of the upper and built a whole mathematical apparatus around the concept of indivisibles of different width. However ingenious, this explanation did not fly with the Jesuits.

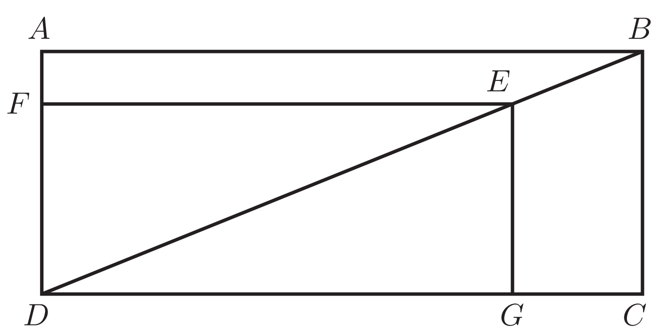


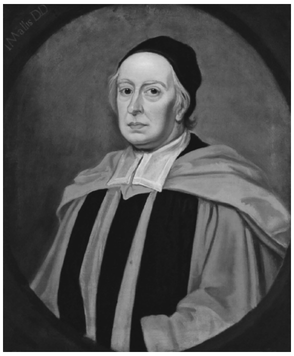
Figure 1. The method of indivisibles can lead to contradictions. Here each horizontal segment in the upper triangle is longer than the corresponding vertical segment in the lower triangle, wrongly implying that the upper triangle has greater area.

The Jesuits were largely responsible for raising the status of mathematics in Italy from a lowly discipline to a paragon of truth and a model for social and political order. The Gregorian reform of the calendar of 1582, widely accepted in Europe across the religious divide, had very favorable political ramifications for the Pope, and this project endeared mathematics to the hearts of Catholics. In an age of religious strife and political disputes, the Jesuits hailed mathematics in general, and Euclidean geometry in particular, as an exemplar of resolving arguments with unassailable certainty through clear definitions and careful logical reasoning. They lifted mathematics from its subservient role well below philosophy and theology in the medieval tree of knowledge and made it the centerpiece of their college curriculum as an indispensable tool for training the mind to think in an orderly and correct way.

The Jesuits viewed the notion of infinitesimals as a dangerous idea.

The new, enviable position of mathematics in the Jesuits' epistemological hierarchy came with a set of strings attached. Mathematics now had a new responsibility to publicly symbolize the ideals of certainty and order. Various dubious innovations, such as the method of indivisibles, with their inexplicable paradoxes, undermined this image. The Jesuits therefore viewed the notion of infinitesimals as a dangerous idea and wanted to expunge it from mathematics. In their view, infinitesimals not only tainted mathematics but also opened the door to subversive ideas in other areas, undermining the established social and political order. The Jesuits never aspired to mathematical originality. Their education was oriented toward an unquestioning study of established truths, and it discouraged open-ended intellectual explorations. In the first decades of the seventeenth century the Revisors General in Rome issued a series of injunctions against infinitesimals, forbidding their use in Jesuit colleges. Jesuit mathematicians called the indivisibles "hallucinations" and argued that "[t]hings that do not exist, nor could they exist, cannot be compared" (pp. 154, 159).

The champions of infinitesimals chose different strategies to deal with the Jesuit onslaught. In 1635 Cavalieri expounded the method of indivisibles in a heavy volume, filled with impenetrable prose, which even the best mathematicians of the day found hard to get through. He dismissed the paradoxes generated by his method with long and convoluted explanations aimed to intimidate more than persuade. Later on, when asked about the paradoxes, the defenders of infinitesimals often simply gestured toward Cavalieri's volumes, claiming that he had already resolved all of them. Neither the critics nor the support-



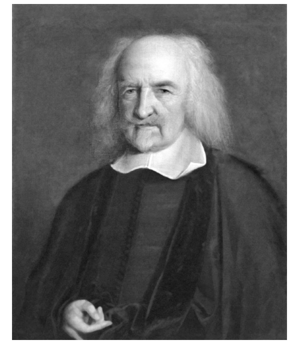
John Wallis (1616–1703), after Sir Godfrey Kneller, Bt oil on canvas, feigned oval, (1701), NPG 578.

ers of the indivisibles dared to penetrate Cavalieri's obscure fortress. Evangelista Torricelli, by contrast, found the paradoxes the most fascinating part of the topic and published several detailed lists of them, believing that a study of such paradoxes was the best way to understand the structure of the continuum. For him, the study of paradoxes was akin to an experiment, for it pushed a phenomenon to its extreme in order to reveal its true nature.

The dispute over infinitesimals was at the same time a dispute over the nature of mathematics. Do mathematical proofs have to show only the correctness of a theorem (as in Euclidean geometry) or to explain why it is true (as in the method of indivisibles)? Should one pursue a top-down approach, starting with universal first principles, put mathematical objects in order, and then impose this order on the world, or should one build mathematics from the bottom up, beginning with one's intuition about the world and moving up to higher and higher abstractions? The latter question pitted the Jesuits against Galileo, which led to his eventual condemnation and lifetime house arrest. Harsh administrative measures were also taken against the remaining stalwarts of infinitesimals, who lost their jobs and were forbidden to teach or publish. The Jesuits even went so far as to engineer the dissolution of a small monastic order, the Jesuats, which had sheltered Cavalieri and Stefano degli Angeli, the leading promoters of the method of indivisibles. The Jesuit mathematicians saw their mission in preserving the eternal truths of Euclidean geometry and in suppressing any threat of potentially disruptive innovation. This led, Alexander argues, to "the slow suffocation and ultimate death of a brilliant Italian mathematical tradition" (p. 165).

...a clash between two visions of modernity.

The battle over the method of indivisibles played out differently in England, where the Royal Society proved capable of sustaining an open intellectual debate. One of the most prominent critics of infinitesimals in England was philosopher and amateur mathematician Thomas Hobbes. A sworn enemy of the Catholic Church, he nevertheless shared with the Jesuits a fundamental commitment to hierarchical order in society. He believed that only a single-purpose organic unity of a nation, symbolized by the image of Leviathan, could save England from the chaos and strife sowed by the civil war. In the 1650s–70s his famously acrimonious dispute with John Wallis, the Savil-ian Professor of Geometry at Oxford and a leading propo-nent of the method of indivisibles, again pitted a champion of social order against an advocate of intellectual freedom.



Thomas Hobbes (1588–1679), by John Michael Wright, oil on canvas, circa 1669–1670, NPG 225.

The Royal Society was initially suspicious of mathematics. Society fellows prized experimental science, public demonstrations, and open intellectual debate as a model for peaceful resolution of societal tensions. Mathematics, with its reputation as a solitary, private pursuit, its claims for incontrovertible truth, its reliance on obscure professional language, and its inaccessibility to laymen, seemed like a poor match for the liberal ideals of the society. Wallis, the only mathematician among the founders, took upon himself the task of reconciling mathematics with the spirit of the society's ideals. Claiming that "[m]athematical entities exist not in the imagination but in reality" (p. 263), he put forward a new, "experimental" mathematics. In contrast to the Euclidean approach of constructing geometrical objects from the first principles, Wallis assumed that geometrical figures already existed in the world. Modifying the method of indivisibles, he viewed a triangle as actually composed not of lines but of infinitely thin trapezoids, two-dimensional objects making up the original triangle, just like mountains formed by geological strata. Studying geometrical objects for him was akin to the work of a scientist probing geological formations. His method relied on induction, was open to discussion, and aimed to persuade the reader by examining a series of particular cases, much like the laboratory experiments that became the hallmark of the Royal Society's approach to studying nature. In the eyes of its fellows, this kind of mathematics was aligned with the society's epistemological ideals, and its legitimation paved the way for the later transformation of the method of infinitesimals into calculus in the hands of Isaac Newton.

Alexander persuasively argues that the fight over infinitesimals was a reflection of a more fundamental clash between what he calls two "visions of modernity." While the Jesuits and Hobbes embodied the desire to achieve a societal unity through the imposition of a single truth and suppression of debate, the champions of infinitesimals valued the freedom of discussion and investigation and a pluralism of opinions. Their opponents feared that intellectual pluralism might lead to political and religious pluralism and wanted to squash the seeds of instability before they produced full chaos. Following the Jesuits' purge of creative mathematicians, not only Italy's mathematical tradition declined but the country itself became unreceptive to innovation and began falling behind. In England, by contrast, the support of mathematical novelty by the Royal Society was part of greater openness in intellectual and social debates and resulted in rapid scientific and technological development, leading up to the Industrial Revolution. The author implies that the different fate of infinitesimals in different countries shaped the fortunes of these nations in the long run.

Alexander clearly outlines a cultural split between political conservatives and "liberalizers" with respect to the method of indivisibles. His own discussion of Hobbes's early fascination with infinitesimals, however, somewhat challenges this overly neat separation. Despite his royalist and traditionalist convictions, Hobbes carefully read and absorbed Cavalieri's subversive mathematical treatises. Reinterpreting the indivisibles as material objects, he developed an unconventional geometry in which mathematical objects were generated by the motion of simpler objects—lines by motion of points, surfaces by motion of lines, and solids by motion of surfaces—before he turned against infinitesimals in his personal vendetta against Wallis. Well, good history of mathematics, like good mathematics, might occasionally benefit from a paradox or two.

Innovation often grows out of outlandish ideas.

In the 1960s, three hundred years after the Jesuits' ban, infinitesimals eventually earned a rightful place in mathematics by acquiring a rigorous foundation in Abraham Robinson's work on nonstandard analysis. They had played their most important role, however, back in the days when the method of indivisibles lacked rigor and was fraught with paradoxes. Perhaps it should not come as a surprise that today's mathematics also borrows extremely fruitful ideas from nonrigorous fields, such as supersymmetric quantum field theory and string theory.

Alexander's book meaningfully points to a fundamental tension between the popular image of mathematics as a collection of eternal truths which never changes and knows no debate and its actual practice, filled with uncertainty, frustration, failure, and rare glimpses of profound insight. If, as in the case of the Jesuits, maintaining the appearance of infallibility becomes more important than exploration of new ideas, mathematics loses its creative spirit and turns into a storage of theorems. Innovation often grows out of outlandish ideas, but to make them acceptable one needs a different cultural image of mathematics—not a perfectly polished pyramid of knowledge, but a freely growing tree with tangled branches.

About the Author

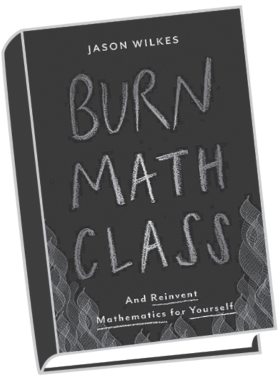

Photo courtesy of Slava Gerovitch.

Slava Gerovitch teaches cultural history of mathematics at MIT. His research interests include history of twentieth-century mathematics, cybernetics, astronautics, and computing. His current project explores how the Soviet mathematics community creatively adapted to various political, institutional, and cultural pressures. He is the author of *From Newspeak to Cyberspeak: A History of Soviet Cybernetics, Voices of the Soviet Space Program,* and *Soviet Space Mythologies.* A poet and a translator, he has also published *Wordplay: A Book of Russian and English Poetry.*

BOOKSHELF

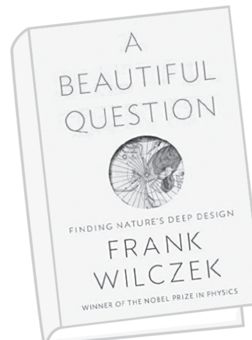
A man is known by the books he reads. —Emerson

New and Noteworthy Titles on Our Bookshelf
May 2016

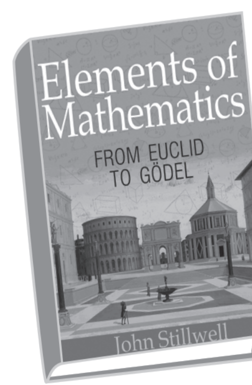


Burn Math Class and Reinvent Mathematics for Yourself, by Jason Wilkes (Perseus Books, March 2016). Jason Wilkes is on a mission to liberate mathematics from its position as an arcane set of rules that must be memorized and to restore the field to its proper status as a living subject that everyone can explore, enjoy, and use. After a preface, followed by a "prefacer", the book begins in earnest with these sentences: "Forget everything you've been told about math. Forget all those silly formulas you've been told to memorize. Make a little room in your head with clean white walls and no math. Without leaving that room, let's reinvent math for ourselves." Wilkes believes people should learn mathematics by creating concepts and ideas on their own through experimentation and discovery rather than by memorizing facts handed down from authority figures. Using this approach, he takes readers from the basics of arithmetic to calculus. One of his main strategies is to replace standard terminology that doesn't capture concepts very well, such as "chain rule", and replace it with more evocative terms. In this book the chain rule is one of several types of "hammers" for taking derivatives. He tackles ideas that can be mystifying through conversational and often humorous explanations: "What does it mean for two points to be infinitely close to each other? I don't know! Let's decide. Let's write tiny to stand for a number that's infinitely small. It's not zero, but it's also smaller than any positive number…." He tries to reveal the big ideas, as in this summary of what calculus is all about: "If we zoom in on curvy stuff, it starts to look more and more straight." But the book is not filled only with entertaining prose: once Wilkes has presented the concepts in narrative form, he translates them into mathematical equations and formulas. The writing style will no doubt have its devotees and detractors, but the author's whimsy and subversive humor are complemented by a sincere desire to get people to experience the beauty of mathematics for themselves.

A Beautiful Question: Finding Nature's Deep Design, by Frank Wilczek (Penguin Press, July 2015). Right after the "User's Manual", which substitutes for a preface for this book, Nobel laureate Frank Wilczek reveals the question referred to in the title: Does the world embody beautiful ideas? Noting that what artists do is embody ideas, he poses a closely related question: Is the world a work of art? He notes that in art one is accustomed to the idea that old styles that have been superseded by newer ones can nevertheless be enjoyed and appreciated. In the book he similarly approaches his question through the history of science, an approach that compels one to proceed from the simpler to the more complex and shows how great thinkers struggled with new ideas. A central theme of the book is the Standard Model of physics, which Wilczek suggests should instead be called the Core Theory. He presents the Core Theory as geometry, adapting for a wide audience his work in fundamental physics. In an original twist, he uses human color perception as a means for thinking about extra dimensions and for opening new ways of understanding local symmetry. While the book is really about physics, it contains much that would appeal to readers with a mathematical bent.



Elements of Mathematics: From Euclid to Gödel, by John Stillwell (Princeton University Press, May 2016). This new book by the author of *Roads to Infinity* (A K Peters, 2010) traces the history of mathematical topics that are today considered as "elementary" and shows that they were anything but at the time they were first discovered. Stillwell argues that it is the notion of infinity that leads us from the elementary to the advanced in mathematics. Stillwell has written a few articles for the *Notices,* including "The Story of the 120-Cell" (January 2001, `www.ams.org/notices/200101/fea-stillwell.pdf`) and a review of Jeremy Gray's scientific biography of Poincaré (April 2014, `www.ams.org/notices/201404/rnoti-p378.pdf`). Stillwell's earlier book *Yearning for the Impossible* (A K Peters, 2006) was reviewed by Daniel Biss in the June/July 2007 issue of the *Notices* (`www.ams.org/notices/200706/tx070600722p.pdf`).



Inside the AMS

AMS Holds Department Chairs Workshop

The annual AMS Department Chairs Workshop was held on January 5, 2016, just prior to the Joint Mathematics Meetings in Seattle, Washington. Forty-five department leaders from across the country participated in the workshop to discuss diversity in mathematics, evaluation of teaching, and issues in entry-level mathematics.

The workshop was led by Matthew Ando, University of Illinois at Urbana-Champaign; William (Bus) Jaco, Oklahoma State University; Krista Maxson, University for Science and Arts of Oklahoma; and Judy Walker, University of Nebraska–Lincoln.

The workshop format helps to stimulate discussion among attending chairs and workshop leaders.

Photo courtesy of Anita Benjamin.

AMS Department Chairs Workshop, 2016.

Sharing ideas and experiences with peers provides a form of "department chair therapy", creating an environment that enables attending chairs to address departmental matters from new perspectives.

—AMS Washington Office

From the AMS Public Awareness Office

A Sampling from AMS Blogs. "Keep Good Company", by Edward Dunne, on Fields Medalists, Abel Prize winners, and Nobel Prize winners who were also active reviewers; "Collaboration, Bias, and Tenure", by Beth Malmskog; "Preparing the Next Generation of Students in the Mathematical Sciences to 'Think with Data'", by Johanna Hardin and Nicholas J. Horton; "Counting Aloud and Other Saucy Things", by Anna Haensch; "Look Around You: Spherical Videos and Möbius Transformations", by Evelyn Lamb; "Are You Thinking of Working in Industry?", by Sharon Arroyo and Les Button; "27 Lines on a Cubic Surface", by John Baez; and "The Benefits of Tutoring: More Than Extra Income", by Natalie Coston.

Browse the AMS Blogs at `blogs.ams.org` and subscribe to get email notification of new posts.

Photo courtesy of the AMS Public Awareness Office.

Pi Day 2016's WWtBaM contestants with AMS Public Awareness Officer Mike Breen.

Pi Day Celebration. One of the most popular events annually is the Pi Day (3/14) game that takes place at Providence College. Read about this year's Who Wants to Be a Mathematician at `www.ams.org/programs/students/wwtbam/pi-day-2016`.

—Annette Emerson and Mike Breen
AMS Public Awareness Officers

Deaths of AMS Members

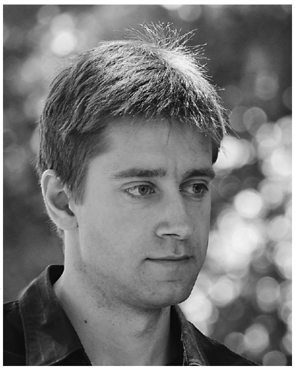
Florencio Gonzales Asenjo, of Aspinwall, Pennsylvania, died on June 10, 2013. Born on September 28, 1926, he was a member of the Society for 54 years.

Lalgudy J. Balasundaram, of Quincy, Massachusetts, died on December 2, 2014. Born on May 24, 1932, he was a member of the Society for 7 years.

Donald A. Darling, of Newport Beach, California, died on June 24, 2014. Born on May 4, 1915, he was a member of the Society for 68 years.

Mathematics People

Pestun Awarded Weyl Prize


Credit: MCV/IHES.

Vasily Pestun

Vasily Pestun of the Institut des Hautes Études Scientifiques has been awarded the 2016 Hermann Weyl Prize for his groundbreaking results in the study of supersymmetric gauge theories, such as his ingenious computation of partition functions that led to the discovery of rich connections between four-dimensional and two-dimensional quantum field theories.

The chair of the Selection Committee, Edward Frenkel of the University of California Berkeley, said: "Vasily Pestun's original contributions opened new opportunities for fruitful interaction between mathematics and quantum physics. It is quite fitting that his work is honored by the prize named after Hermann Weyl, a pioneer in both of these fields who used to say that in his research he always tried to unite the true and the beautiful."

The Hermann Weyl Prize was established by the Standing Committee of the International Colloquium on Group Theoretical Methods in Physics in 2002 and is awarded every two years to recognize young scientists who have performed original work of significant scientific quality in the area of understanding physics through symmetries. The International Colloquium on Group Theoretical Methods in Physics takes place every two years. In 2016 it will be held in Rio de Janeiro, Brazil.

—Edward Frenkel

2016 Clay Research Fellows Chosen


Photo by Eduard Duryev.

Simion Filip

Simion Filip of the University of Chicago and Tony Yue Yu of the Université Paris Diderot have been appointed Clay Research Fellows for 2016 by the Clay Mathematics Institute (CMI).

Filip will receive his PhD in June 2016 from the University of Chicago under the supervision of Alex Eskin. He is interested in the connections between dynamical systems and algebraic geometry, in particular between Teichmüller dynamics and Hodge theory. His recent interests also involve K3 surfaces and their special geometric properties. He has been appointed as a Clay Research Fellow for a term of five years beginning July 1, 2016.

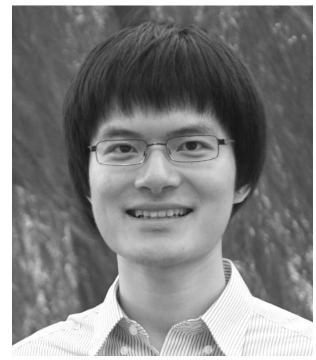


Photo by Jindie Mi.

Tony Yue Yu

Yu received his PhD in 2016 from Université Paris Diderot under the supervision of Maxim Kontsevich and Antoine Chambert-Loir. He works on nonarchimedean geometry, tropical geometry, and mirror symmetry. He aims to build a theory of enumerative geometry in the setting of Berkovich spaces. Such a theory will give us a new understanding of the enumerative geometry of Calabi-Yau manifolds, as well as the structure of their mirrors. It is also intimately related to the theory of cluster algebras and wall-crossing structures. He has been appointed as a Clay Research Fellow for a term of five years beginning September 1, 2016.

—From a CMI announcement

Salisbury Receives Graham Wright Award


Photo by Faculty of Science, York University.

Thomas Salisbury

Thomas Salisbury of York University has been named the recipient of the 2015 Graham Wright Award for Distinguished Service of the Canadian Mathematical Society (CMS). His service to the mathematics community includes terms as president of the CMS (2006–2008), as well as deputy director of the Fields Institute (2003–2006), editor in chief of the *Canadian Mathematical Bulletin* and associate editor of *Probability Theory and Related Fields, Potential Analysis*, and the *Canadian Journal of Statistics*. He has served on numerous CMS committees and organizing committees for CMS meetings and training camps. He is a Fellow of the Institute of Mathematical Statistics and of the Fields Institute. He and his wife, Kathy, have three grown children and enjoy hiking, music, books, and their cottage near Minden, Ontario.

—From a CMS announcement

An incorrect version of this announcement was printed in the December 2015 Notices. Following is the official announcement from the Royal Spanish Mathematical Society. Notices regrets the error:

Freitas Awarded Rubio de Francia Prize

Nuno Freitas of the Max Planck Institute, Bonn, has been awarded the eleventh Rubio de Francia Prize of the Royal Spanish Mathematical Society (RSME). Freitas's contributions are in the fields of arithmetic and number theory. Freitas and his collaborators, B. V. Le Hung and S. Siksek, have proven that elliptic curves defined over real quadratic fields are modular, extending the pioneering work on Fermat's last theorem by Wiles and Taylor, who proved the same result for elliptic curves defined over the rational numbers. This result is a crucial step toward the general modularity conjecture. The jury also praised Freitas's recent work with S. Siksek, in which they prove that Fermat's equation $x^n + y^n = z^n$ has no solution over a real quadratic field K once n is large enough, for an infinite and rather explicit set of real quadratic fields, which is then shown to have density at least 5/6. Freitas received his PhD in Mathematics from the University of Barcelona in 2012.

The prize honors the memory of renowned Spanish analyst J. L. Rubio de Francia (1949-1988). The RSME awards the prize annually to a mathematician from Spain or who has received a Ph.D. from a university in Spain and who is at most thirty-two years of age. The prize is awarded for high-caliber contributions to any area of pure or applied mathematics. This year a three-year fellowship provided by the BBVA Foundation will also be awarded to the recipient, together with the prize.

The Rubio de Francia Prize is awarded by an international jury covering a range of mathematical areas. This year the prize committee was chaired by Jesús Bastero (Universidad de Zaragoza) and consisted of Ingrid Daubechies (Duke University), Timothy Gowers (University of Cambridge), Subhash Khot (Courant Institute, New York University), Marco A. López Cerdá (Universidad de Alicante), Álvaro Pelayo (University of California San Diego), and Claire Voisin (École Polytechnique). Recent prize winners include Angel Castro (2013), María Pe (2012), Alberto Enciso (2011), Carlos Beltran (2010), Álvaro Pelayo (2009), and Francisco Gancedo (2008).

—From an RSME announcement

2016 MAA Awards

The Mathematical Association of America (MAA) awarded a number of prizes at the Joint Mathematics Meetings in Seattle, Washington, in January 2016.

George Berzsenyi of Rose-Hulman Institute of Technology was awarded the Gung and Hu Award for Distinguished Service to Mathematics "for his remarkable career empowering generations of high school students to pursue their mathematical and scientific passions by promoting the art of problem solving, creating national and international mathematical talent searches, and supporting mathematical competitions."

Susan Marshall and Donald R. Smith, both of Monmouth University, were awarded the Chauvenet Prize for their article "Feedback, Control, and the Distribution of Prime Numbers", *Mathematics Magazine* **86** (2013), no. 3.

Jordan Ellenberg of the University of Wisconsin–Madison received the Euler Book Prize for his book *How Not to Be Wrong: The Power of Mathematical Thinking* (Penguin Press, New York, 2014).

The Deborah and Franklin Tepper Haimo Awards for Distinguished College or University Teaching of Mathematics were awarded to Satyan Davadoss (Williams College), Tyler Jarvis (Brigham Young University), and Glen Van Brummelen (Quest University, British Columbia).

—From MAA announcements

2016 AWM Awards

The Association for Women in Mathematics (AWM) presented a number of awards at the Joint Mathematics Meetings in Seattle, Washington, in January 2016.

Photo by Carl Mueller.

Naomi Jochnowitz

Naomi Jochnowitz of the University of Rochester was honored with the M. Gweneth Humphreys Award for Mentorship of Undergraduate Women in Mathematics for her devotion "to the development and support of undergraduate students of mathematics, in addition to her activities with math graduate students and postdocs, with a particular impact on scores of women students." Jochnowitz tells the *Notices*: "In addition to my mathematics, I have an interest in Talmudic studies, which despite significant progress in recent years remains a field of study that is to a large extent closed to women, even more so than math."

Photo by UNL Communications.

Judy Walker

Judy Walker of the University of Nebraska was awarded the Louise Hay Award for Contributions to Mathematics Education for "creating and adapting innovative courses at all levels", from high school through graduate school, including practicing teachers.

Mackenzie Simper of the University of Utah was awarded the Alice T. Schafer Prize for Excellence in Mathematics by an Undergraduate Woman for her "stellar academic track record, proven ability to do original mathematical research," and her commitment to "excelling in her mathematical career."

Bettye Anne Case of Florida State University was honored with the AWM Life Time Service Award for her many services, including her decades-long role as meetings coordinator, as well as being on the executive committee from 1978 through 2015. Heather Lewis of the University of Richmond, Heather Russell of Nazareth College, and Rebecca Segal of Virginia Commonwealth University also received service awards for their involvement in AWM programs and activities.

—From AWM announcements

Sloan Research Fellows Announced

The Alfred P. Sloan Foundation has announced the names of the recipients of the 2016 Sloan Research Fellowships. Each year the foundation awards fellowships in the fields of mathematics, chemistry, computational and evolutionary molecular biology, computer science, economics, neuroscience, physics, and ocean sciences. Grants of US$55,000 for a two-year period are administered by each Fellow's institution. Once chosen, Fellows are free to pursue whatever lines of inquiry most interest them, and they are permitted to employ the Fellowship funds in a wide variety of ways to further their research aims.

Following are the names and institutions of the 2016 awardees in mathematics.

Stefanos Aretakis, Princeton University
Rina Foygel Barber, University of Chicago
Venkat Chandrasekaran, California Institute of Technology
Artem Chernikov, University of California, Los Angeles
Thomas Church, Stanford University
Jeffrey Danciger, University of Texas, Austin
Benjamin Elias, University of Oregon
Elena Fuchs, University of Illinois, Urbana-Champaign
Adrianna Gillman, Rice University
Vadim Gorin, Massachusetts Institute of Technology
Zaher Hani, Georgia Institute of Technology
Matthew J. Hirn, Michigan State University
Zongming Ma, University of Pennsylvania
Matthias Morzfeld, University of Arizona
Marcel Nutz, Columbia University
Wesley Pegden, Carnegie Mellon University
Claudiu Raicu, University of Notre Dame
Nikhil Srivastava, University of California Berkeley
Kevin Tucker, University of Illinois at Chicago
Lu Wang, University of Wisconsin, Madison.

—From a Sloan Foundation announcement

National Academy of Engineering Elections

The National Academy of Engineering (NAE) has elected a number of new members and foreign associates for 2016. Following are the new members whose work involves the mathematical sciences:

Dan Boneh of Stanford University for contributions to the theory and practice of cryptography and computer security.

Emily A. Carter of Princeton University for development of quantum chemistry computational methods for the design of molecules and materials for sustainable energy.

Gérard P. Cornuéjols of the Tepper School of Business, Carnegie Mellon University, for contributions to the theory, practice, and application of integer programming.

David S. Johnson of Columbia University for contributions to the theory and practice of optimization and approximation algorithms.

Charles E. Leiserson of the Massachusetts Institute of Technology for theoretically grounded approaches to digital design and parallel computer systems.

Elected as foreign members were:

Peter Stoica of Uppsala University for contributions to array signal processing in communications, sensing, and imaging.

Peter Whittle of the University of Cambridge for contributions to the mathematics of operations research and statistics.

—From an NAE announcement

Corrections

Notices regrets the following error that appeared in the 2016 March issue:

The Mathematics People section within the March issue had incorrect placements of Rahul Singh and Lesley Sibner's photos.

RAHUL SINGH of Yale University was awarded a Marshall Scholarship for 2015 and will study econometrics and mathematical economics at the London School of Economics, and computational statistics and machine learning at University College London.

LESLEY SIBNER unexpectedly passed away on September 11, 2013. Lesley was the Eastern Section Associate Secretary of the AMS from 1993 to 2009. She was also in the inaugural class of Fellows of the AMS.

Mathematics Opportunities

Participants in Women in Numbers 3, AWM, April 20–25, 2014.

AWM ADVANCE Grant for Research Networks for Women

The Association for Women in Mathematics (AWM) has received a five-year, US$750,000 ADVANCE grant for research networks for women in mathematics. Workshops at the annual Joint Mathematics Meetings and SIAM Meetings will bring together women from one of the Research Collaboration Networks to showcase their work and encourage continued collaboration and mentoring. AWM will also organize biennial Research Symposia, with high-profile plenary speakers and special sessions in research areas linked to the Research Collaboration Networks. Proposals for new Research Networks for Women may be submitted to Magnhild Lien by July 1. For more information see `awmadvance.org`.

Most Research Collaboration Networks for Women conferences so far have taken place at Banff International Research Station.

—AWM announcement

Call for Nominations for Clifford Prize

The W. K. Clifford Prize is an international scientific prize for young researchers which aims to encourage them to compete for excellence in theoretical and applied Clifford algebras, their analysis and geometry. The deadline for nominations is **September 30, 2016**. Nominations should be sent to `secretary@wkcliffordprize.org`. For details see `www.wkcliffordprize.org`.

—Hendrik De Bie
Ghent University

NSF Postdoctoral Research Fellowships

The National Science Foundation (NSF) awards Mathematical Sciences Postdoctoral Research Fellowships (MSPRF) in areas of the mathematical sciences, including applications to other disciplines. Awards are either Research Fellowships or Instructorships. The Research Fellowship provides full-time support for any eighteen academic-year months in a three-year period. The Research Instructorship provides either two academic years of full-time support or one academic year of full-time and two academic years of half-time support. The deadline for proposals is **October 19, 2016**. See www.nsf.gov/funding/pgm_summ.jsp?pims_id=5301&org=NSF.

—NSF announcement

Research Training Groups in the Mathematical Sciences

The National Science Foundation (NSF) Research Training Groups in the Mathematical Sciences (RTG) program provides funds for the training of US students and postdoctoral researchers. The deadline for full proposals is **June 7, 2016**. See www.nsf.gov/funding/pgm_summ.jsp?pims_id=5732.

—NSF announcement

International Mathematics Competition for University Students

The Twenty-Third International Mathematics Competition for University Students will be July 25–31, 2016, at American University in Blagoevgrad, Bulgaria. Students completing their first, second, third, or fourth years of university education are eligible. See www.imc-math.org.uk.

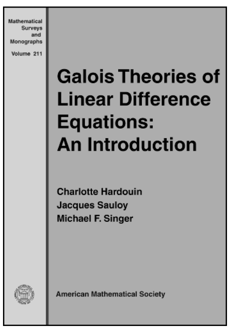
—John Jayne, University College London

The most up-to-date listing of NSF funding opportunities from the Division of Mathematical Sciences can be found online at: www.nsf.gov/dms and for the Directorate of Education and Human Resources at www.nsf.gov/dir/index.jsp?org=ehr. To receive periodic updates, subscribe to the DMSNEWS listserv by following the directions at www.nsf.gov/mps/dms/about.jsp.

New Publications Offered by the AMS

To subscribe to email notification of new AMS publications, please go to www.ams.org/bookstore-email.

Algebra and Algebraic Geometry



Galois Theories of Linear Difference Equations: An Introduction

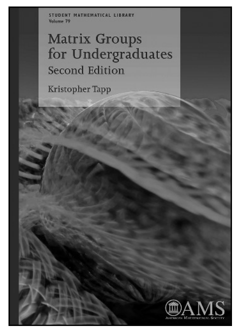
Charlotte Hardouin and **Jacques Sauloy**, *Institut de Mathématiques de Toulouse, France*, and **Michael F. Singer**, *North Carolina State University, Raleigh, NC*

This book is a collection of three introductory tutorials coming out of three courses given at the CIMPA Research School "Galois Theory of Difference Equations" in Santa Marta, Columbia, July 23–August 1, 2012. The aim of these tutorials is to introduce the reader to three Galois theories of linear difference equations and their interrelations. Each of the three articles addresses a different galoisian aspect of linear difference equations. The authors motivate and give elementary examples of the basic ideas and techniques, providing the reader with an entry to current research. In addition each article contains an extensive bibliography that includes recent papers; the authors have provided pointers to these articles allowing the interested reader to explore further.

Contents: M. F. Singer, Algebraic and algorithmic aspects of linear difference equations; **C. Hardouin**, Galoisian approach to differential transcendence; **J. Sauloy**, Analytic study of q-difference equations.

Mathematical Surveys and Monographs, Volume 211

May 2016, 171 pages, Hardcover, ISBN: 978-1-4704-2655-2, LC 2015039225, 2010 *Mathematics Subject Classification:* 05A19, 12H05, 12H10, 20G15, 20H20, 39A06, 39A10, 39A13, **AMS members US$88**, List US$110, Order code SURV/211



Matrix Groups for Undergraduates

Second Edition

Kristopher Tapp, *Saint Joseph's University, Philadelphia, PA*

Matrix groups touch an enormous spectrum of the mathematical arena. This textbook brings them into the undergraduate curriculum. It makes an excellent one-semester course for students familiar with linear and abstract algebra and prepares them for a graduate course on Lie groups.

Matrix Groups for Undergraduates is concrete and example-driven, with geometric motivation and rigorous proofs. The story begins and ends with the rotations of a globe. In between, the author combines rigor and intuition to describe the basic objects of Lie theory: Lie algebras, matrix exponentiation, Lie brackets, maximal tori, homogeneous spaces, and roots.

This second edition includes two new chapters that allow for an easier transition to the general theory of Lie groups.

From reviews of the First Edition:

This book could be used as an excellent textbook for a one semester course at university and it will prepare students for a graduate course on Lie groups, Lie algebras, etc. … The book combines an intuitive style of writing with rigorous definitions and proofs, giving examples from fields of mathematics, physics, and other sciences where matrices are successfully applied. The book will surely be interesting and helpful for students in algebra and their teachers.

—***European Mathematical Society Newsletters***

This is an excellent, well-written textbook which is strongly recommended to a wide audience of readers interested in mathematics and its applications. The book is suitable for a one semester undergraduate lecture course in matrix groups, and would also be useful supplementary reading for more general group theory courses.

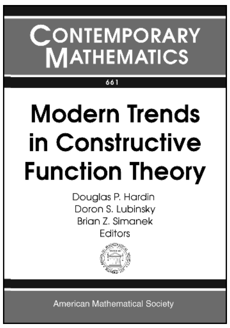
—***MathSciNet (or Mathematical Reviews)***

Contents: Why study matrix groups?; Matrices; All matrix groups are real matrix groups; The orthogonal groups; The topology of matrix groups; Lie algebras; Matrix exponentiation; Matrix groups are manifolds; The Lie bracket; Maximal tori; Homogeneous manifolds; Roots; Bibliography; Index.

Student Mathematical Library, Volume 79

May 2016, 239 pages, Softcover, ISBN: 978-1-4704-2722-1, LC 2015038141, 2010 *Mathematics Subject Classification:* 20-02, 20G20; 20C05, 22E15, **All Individuals US$39.20**, List US$49, Institutional member US$39.20, Order code STML/79

Analysis



Modern Trends in Constructive Function Theory

Douglas P. Hardin, *Vanderbilt University, Nashville, TN*, **Doron S. Lubinsky**, *Georgia Institute of Technology, Atlanta, GA*, and **Brian Z. Simanek**, *Vanderbilt University, Nashville, TN*, Editors

This volume contains the proceedings of the conference Constructive Functions 2014, held from May 26–30, 2014, at Vanderbilt University, Nashville, TN, in honor of Ed Saff's 70th birthday.

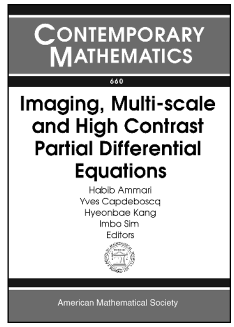
The papers in this volume contain results on polynomial approximation, rational approximation, Log-optimal configurations on the sphere, random continued fractions, ratio asymptotics for multiple orthogonal polynomials, the bivariate trigonometric moment problem, minimal Riesz energy, random polynomials, Padé and Hermite-Padé approximation, orthogonal expansions, hyperbolic differential equations, Bergman polynomials, the Meijer G-function, polynomial ensembles, and integer lattice points.

Contents: **D. S. Lubinsky**, Ed Saff at three score and ten; **V. Totik**, The tale of a formula; **P. D. Dragnev**, Log-optimal configurations on the sphere; **L. Lorentzen**, Convergence of random continued fractions and random iterations of Möbius transformations; **W. Van Assche**, Ratio asymptotics for multiple orthogonal polynomials; **J. S. Geronimo** and **A. Pangia**, Study of a parametrization of the bivariate trigonometric moment problem; **J. S. Brauchart**, Explicit formulas for the Riesz energy of the Nth roots of unity; **I. E. Pritsker**, Asymptotic zero distribution of random polynomials spanned by general bases; **G. López Lagomasino**, On row sequences of Padé and Hermite-Padé approximation; **Y. Xu**, Orthogonal expansions for generalized Gegenbauer weight function on the unit ball; **A. I. Aptekarev**, The Mhaskar-Saff variational principle and location of the shocks of certain hyperbolic equations; **N. Stylianopoulos**, Boundary estimates for Bergman polynomials in domains with corners; **A. Martínez-Finkelshtein**, **E. A. Rakhmanov**, and **S. P. Suetin**, Asymptotics of type I Hermite–Padé polynomials for semiclassical functions; **A. Cuyt** and **W.-S. Lee**, Sparse interpolation and rational approximation; **Y. Lin** and **R. Wong**, Asymptotics of the Meijer G-functions; **A. B. J. Kuijlaars**, Transformations of polynomial ensembles; **J. Bourgain**, **P. Sarnak**, and **Z. Rudnick**, Local statistics of lattice points on the sphere; **M. Budišić** and **M. Putinar**, Conditioning moments of singular measures for entropy maximization II: Numerical examples.

Contemporary Mathematics, Volume 661

April 2016, 297 pages, Softcover, ISBN: 978-1-4704-2534-0, LC 2015035790, 2010 *Mathematics Subject Classification:* 11P21, 26C10, 31A15, 33C50, 35L67, 40A15, 41A21, 42C05, 60B20, **AMS members US$86.40**, List US$108, Order code CONM/661

Differential Equations



Imaging, Multi-scale and High Contrast Partial Differential Equations

Habib Ammari, *Ecole Normale Supérieure, Paris, France*, **Yves Capdeboscq**, *Mathematical Institute, Oxford, United Kingdom*, **Hyeonbae Kang**, *Inha University, Incheon, Korea*, and **Imbo Sim**, *National Institute of Mathematical Sciences, Daejeon, Korea*, Editors

This volume contains the proceedings of the Seoul ICM 2014 Satellite Conference on Imaging, Multi-scale and High-Contrast PDEs, held from August 7–9, 2014, in Daejeon, Korea.

The mathematical analysis of partial differential equations modelling materials, or tissues, presenting multiple scales has been a very active area of research. The study of the corresponding imaging or reconstruction problem is a more recent area. If the material parameters of the partial differential equation present high contrast ratio, then the solution to the PDE becomes particularly challenging to analyze and compute. On the other hand, imaging in highly heterogeneous media poses significant challenges to the mathematical community.

The focus of this volume is on recent progress towards complete understanding of the direct problem with high contrast or high frequencies, and unified approaches to the inverse and imaging problems for both small and large contrast or frequencies. Of particular importance in imaging are shape representation techniques and regularization approaches. Special attention is devoted to new models and problems coming from physics leading to innovative imaging and signal processing methods.

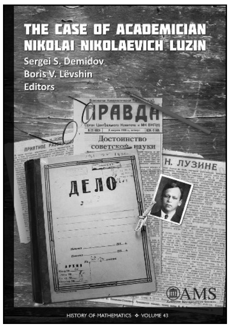
This item will also be of interest to those working in applications.

Contents: **H. Ammari**, **S. Mallat**, **I. Waldspurger**, and **H. Wang**, Wavelet methods for shape perception in electro-sensing; **H. Ammari** and **H. Wang**, Time-domain multiscale shape identification in electro-sensing; **H. Kang** and **E. Kim**, Estimation of stress in the presence of closely located elastic inclusions: A numerical study; **H. Lee** and **J. Lee**, Array dependence of effective parameters of dilute periodic elastic composite; **M. Lim**, A review on the enhancement of near-cloaking using the multilayer structure; **G. S. Alberti** and **Y. Capdeboscq**, On local non-zero constraints in PDE with analytic coefficients; **J. Garnier**, Daylight imaging for virtual reflection seismology; **M. Afzal**, **M. Ayub**, **R. Nawaz**, and **A. Wahab**, Mode-matching solution of a scattering problem in flexible waveguide with abrupt geometric changes; **D. P. Challa**, Direct scattering by a sound hard small body; **G. Ciraolo**, Helmholtz equation in unbounded domains: Some convergence results for a constrained optimization problem.

Contemporary Mathematics, Volume 660

April 2016, 148 pages, Softcover, ISBN: 978-1-4704-1923-3, LC 2015037186, 2010 *Mathematics Subject Classification:* 35B10, 35B30, 35J05, 35J25, 35R30, 35R60, 65N15, 65T60, 86A15, **AMS members US$86.40**, List US$108, Order code CONM/660

General Interest



The Case of Academician Nikolai Nikolaevich Luzin

Sergei S. Demidov, *Russian Academy of Sciences, Moscow, Russia*, and **Boris V. Lëvshin**, Editors
Translated by Roger Cooke

The Soviet school, one of the glories of twentieth-century mathematics, faced a serious crisis in the summer of 1936. It was suffering from internal strains due to generational conflicts between the young talents and the old establishment. At the same time, Soviet leaders (including Stalin himself) were bent on "Sovietizing" all of science in the USSR by requiring scholars to publish their works in Russian in the Soviet Union, ending the nearly universal practice of publishing in the West. A campaign to "Sovietize" mathematics in the USSR was launched with an attack on Nikolai Nikolaevich Luzin, the leader of the Soviet school of mathematics, in Pravda. Luzin was fortunate in that only a few of the most ardent ideologues wanted to destroy him utterly. As a result, Luzin, though humiliated and frightened, was allowed to make a statement of public repentance and then let off with a relatively mild reprimand. A major factor in his narrow escape was the very abstractness of his research area (descriptive set theory), which was difficult to incorporate into a propaganda campaign aimed at the broader public.

The present book contains the transcripts of five meetings of the Academy of Sciences commission charged with investigating the accusations against Luzin, meetings held in July of 1936. Ancillary material from the Soviet press of the time is included to place these meetings in context.

It is wonderful to have this book available in English translation. "The Case of Academician Luzin" is a highly significant event in the history of Soviet mathematics; with its presentation of original sources, together with ample commentary, this book will now convey the full import of this event to a new readership.

—Christopher Hollings, Oxford University, author of "Mathematics across the Iron Curtain"

... an important contribution toward the understanding of the fate of a great mathematician in Stalin's time. We learn here the details of how he was judged in a political trial. I would like to immodestly suggest that reading this source together with Jean-Michel Kantor's and my recent book "Naming Infinity" will clarify an episode in both the history of mathematics and of the Soviet Union that has long mystified observers.

—Loren Graham, professor emeritus of the history of science, MIT and Harvard

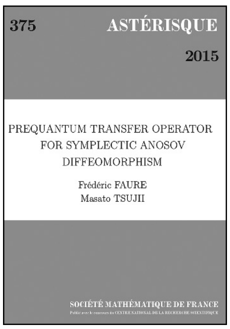
Contents:

History of Mathematics, Volume 43

May 2016, approximately 386 pages, Hardcover, ISBN: 978-1-4704-2608-8, 2010 *Mathematics Subject Classification:* 01A70, 01A72, 01A60, **AMS members US$47.20**, List US$59, Order code HMATH/43

New AMS-Distributed Publications

Analysis



Prequantum Transfer Operator for Symplectic Anosov Diffeomorphism

Frédéric Faure, *Institut Fourier, St. Martin d'Hères, France*, and **Masato Tsujii**, *Kyushu University, Fukuoka, Japan*

The authors define the prequantization of a symplectic and Anosov diffeomorphism. They study the spectral properties of the transfer operator associated to a smooth potential. After restriction to the N-th Fourier mode and letting N tend to infinity, a structure of concentric bands appear. They study the eigenvalue repartition in the most external band and they show that this repartition follows a Weyl law. The authors give a physical interpretation of the results and they compare them to the geometric quantization which appears in quantum chaos theory.

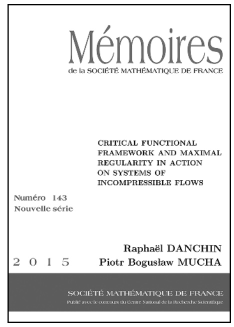
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: Go to www.ams.org/bookstore.

Astérisque, Number 375

December 2015, 222 pages, Softcover, ISBN: 978-2-85629-823-7, 2010 *Mathematics Subject Classification:* 37D20, 37D35, 37C30, 81Q20, 81Q50, **AMS members US$53.60**, List US$67, Order code AST/375

Differential Equations



Critical Functional Framework and Maximal Regularity in Action on Systems of Incompressible Flows

Raphaël Danchin, *Université Paris-Est, Créteil, France*, and **Piotr Bogusław Mucha**, *Uniwersytet Warszawski, Poland*

This memoir is devoted to endpoint maximal regularity in Besov spaces for the evolutionary Stokes system in bounded or exterior domains. The authors get estimates with global-in-time integrability in intersection of Besov spaces. They apply them in particular to solve locally for large data or globally for small data the slightly inhomogeneous Navier–Stokes equations in critical Besov spaces in an exterior domain.

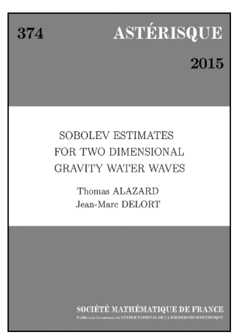
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: Go to www.ams.org/bookstore.

Mémoires de la Société Mathématique de France, Number 143

December 2015, 151 pages, Softcover, ISBN: 978-2-85629-824-4, 2010 *Mathematics Subject Classification:* 35B33, 35B45, 35B65, 35Q30, 35Q35, 76D03, 76N10, **AMS members US$41.60**, List US$52, Order code SMFMEM/143

Geometry and Topology



Sobolev Estimates for Two Dimensional Gravity Water Waves

Thomas Alazard, *École Normale Supérieure et CNRS UMR, Paris, France*, and **Jean-Marc Delort**, *Université Paris 13, Villetaneuse, France*

The authors' goal in this volume is to apply a normal forms method to estimate the Sobolev norms of the solutions of the water waves equation. They construct a paradifferential change of unknown, without derivatives losses, which eliminates the part of the quadratic terms that bring non zero contributions in a Sobolev energy inequality. The authors' approach is purely Eulerian: they work on the Craig-Sulem-Zakharov formulation of the water waves equation.

In addition to these Sobolev estimates, the authors also prove L^2-estimates for the $\partial_x^\alpha Z^\beta$-derivatives of the solutions of the water waves equation, where Z is the Klainerman vector field $t\partial_t + 2x\partial_x$. These estimates are used in one of the book's references. In that reference, the authors prove a global existence result for the water waves equation with smooth, small, and decaying at infinity Cauchy data, and they obtain an asymptotic description in physical coordinates of the solution, which shows that modified scattering holds. The proof of this global in time existence result relies on the simultaneous bootstrap of some Hölder and Sobolev a priori estimates for the action of iterated Klainerman vector fields on the solutions of the water waves equation. The present volume contains the proof of the Sobolev part of that bootstrap.

This item will also be of interest to those working in algebra and algebraic geometry.

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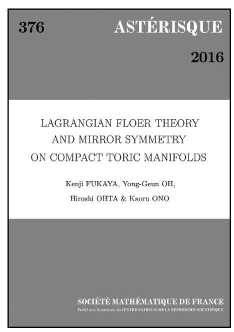
Contents: Go to www.ams.org/bookstore.

Astérisque, Number 374

November 2015, 241 pages, Softcover, ISBN: 978-2-85629-821-3, 2010 *Mathematics Subject Classification:* 53C24, 20F65, 53C23, 20E08, 20F67, 20F69, 19D35, **AMS members US$60**, List US$75, Order code AST/374



Lagrangian Floer Theory and Mirror Symmetry on Compact Toric Manifolds

Kenji Fukaya, *State University of New York, Stony Brook, NY*, **Yong-Geun Oh**, *Pohang University of Science and Technology, Korea*, **Hiroshi Ohta**, *Nagoya University, Japan*, and **Kaoru Ono**, *Kyoto University, Japan*

In this volume, the authors study Lagrangian Floer theory on toric manifolds from the point of view of mirror symmetry. They construct a natural isomorphism between the Frobenius manifold structures of the (big) quantum cohomology of the toric manifold and of Saito's theory of singularities of the potential function constructed in [Fukaya, Tohoku Math. J. **63** (2011)] via the Floer cohomology deformed by ambient cycles. Their proof of the isomorphism involves the open-closed Gromov–Witten theory of one-loop.

This item will also be of interest to those working in algebra and algebraic geometry.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: Go to www.ams.org/bookstore.

Astérisque, Number 376

February 2016, 340 pages, Softcover, ISBN: 978-2-85629-825-1, 2010 *Mathematics Subject Classification:* 53D37, 53D40, 53D45, 14B07, 14M25, 37J05, **AMS members US$65.60**, List US$82, Order code AST/376

MATHEMATICS CALENDAR

This section contains new announcements of worldwide meetings and conferences of interest to the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. New announcements only are published in the print Mathematics Calendar featured in each *Notices* issue.

An announcement will be published in the *Notices* if it contains a call for papers and specifies the place, date, subject (when applicable). A second announcement will be published only if there are changes or necessary additional information. Asterisks (*) mark those announcements containing revised information.

In general, print announcements of meetings and conferences carry only the date, title and location of the event.

The complete listing of the Mathematics Calendar is available at: `www.ams.org/meetings/calendar/mathcal`

All submissions to the Mathematics Calendar should be done online via: `www.ams.org/cgi-bin/mathcal/mathcal-submit.pl`

Any questions or difficulties may be directed to `mathcal@ams.org`.

April 2016

25 – 29 **Conference - Workshop on Nonsmooth Dynamics**
Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain.*
URL: `www.crm.cat/en/Activities/Curs_2015-2016/Pages/Workshop-on-Nonsmooth-Dynamics.aspx`

May 2016

2 – 7 **Quivers and Bipartite Graphs: Mathematics and Physics**
Location: *University of Notre Dame's London Global Gateway, 1 Suffolk St. London SW1Y 4HG, United Kingdom.*
URL: `www3.nd.edu/~conf/quivers/`

4 – 6 **Eighth Discrete Geometry and Algebraic Combinatorics Conference**
Location: *South Padre Island, TX.*
URL: `www.utrgv.edu/discgeo/`

9 – 13 **Advances in Geometric Representation Theory**
Location: *University of Michigan, Ann Arbor, MI.*
URL: `www-personal.umich.edu/~snkitche/Conference/`

12 – 15 **Computationally Assisted Mathematical Discovery and Experimental Mathematics**
Location: *University of Western Ontario, London, Ontario, Canada.*
URL: `www.acmes.org`

16 – 20 **Workshop: Categorification**
Location: *Mathematical Institute of the University of Bonn, Bonn, Germany.*
URL: `www.math.uni-bonn.de/people/dtubben/categorification2016.htmpl`

17 – 20 **Static Metrics and Bartnik's Quasi-Local Mass Conjecture 2016**
Location: *University of Tübingen, Germany.*
URL: `www.math.uni-tuebingen.de/static-metric-conference-2016`

18 – 20 **Spectra of Graphs and Applications (SGA 2016)**
Location: *Serbian Academy of Sciences and Arts, Belgrade, Serbia.*
URL: `www.mi.sanu.ac.rs/conferences/sga2016.htm`

18 – 20 **The 9th Asian Conference on Fixed Point Theory and Optimization 2016 (ACFPTO2016)**
Location: *Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok, Thailand.*
URL: `acfpto2016.kmutt.ac.th`

27 – 29 **Lehigh University Geometry/Topology Conference**
Location: *Lehigh University, Bethlehem, PA.*
URL: `www.lehigh.edu/~dlj0/geotop.html`

30 – June 3 **Constructive Approximation and Harmonic Analysis**
Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain.*
URL: `www.crm.cat/en/Activities/Curs_2015-2016/Pages/Constructive-Approximation.aspx`

June 2016

1 – 2 **The First Indonesian GeoGebra Institutes Conference 2016**
Location: *Universitas Sultan Ageng Tirtayasa, Serang, Indonesia.*
URL: `www.geogebra.org`

6 – 8 **VI International Conference in Optimization Theory and its Applications (ALEL)**
Location: *Cartagena, Spain.*
URL: `www.um.es/beca/alel2016`

6 – 10 **School/Workshop on Applicable Theory of Switched Systems**
Location: *University of Texas at Dallas, Richardson, TX.*
URL: `www.utdallas.edu/sw16/`

6 – 10 **Conference on Harmonic Analysis and Approximation Theory (HAAT 2016)**
Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain.*
URL: `www.crm.cat/en/Activities/Curs_2015-2016/Pages/HAAT2016.aspx`

9 – 10 **IV Workshop on Functional Analysis**
Location: *Cartagena, Spain.*
URL: www.um.es/beca/workshop2016

13 – 17 **Nilpotent Orbits and Representation Theory**
Location: *Centro di Ricerca Matematica Ennio De Giorgi, Collegio Puteano, Piazza dei Cavalieri 3, 56100 PISA, Italy.*
URL: http://www.crm.sns.it/event/367/

13 – 17 **Leuca2016 Celebrating Michel Waldschmidt's 70th birthday**
Location: *Hotel Monte Callini, Marina di San Gregorio, Patu' (Lecce), Italy.*
URL: www.mw70.eu

13 – 17 **Conference on MURPHYS-HSFS 2016 WORKSHOP**
Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain.*
URL: www.crm.cat/en/Activities/Curs_2015-2016/Pages/MURPHYS.aspx

16 – 17 **Conference on BARCCSYN 2016**
Location: *Institut d'Estudis Catalans, Barcelona, Spain.*
URL: www.crm.cat/en/Activities/Curs_2015-2016/Pages/BARCCSYN-2016.aspx

20 – 22 **Conference on Probability and Statistics in High Dimensions: A Scientific Tribute to Evarist Giné**
Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain.*
URL: www.crm.cat/en/Activities/Curs_2015-2016/Pages/MEG.aspx

20 – 24 **International Conference on Complex Analysis and Related Topics - The 14th Romanian-Finnish Seminar**
Location: *Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.*
URL: imar.ro/RoFinSem2016/conf.php

20 – 24 **10th International Summer School on Geometry, Mechanics, and Control**
Location: *La Cristalera, Madrid, Spain.*
URL: gmcnet.webs.ull.es/?q=activity-detaill/1656

20 – 24 **Short Term Course On Reliability and Safety Analysis**
Location: *Indian School of Mines, Dhanbad, India.*
URL: www.ismdhanbad.ac.in/short-course/

21 – 24 **Workshop on Algorithms for Modern Massive Data Sets (MMDS 2016)**
Location: *University of California, Berkeley.*
URL: mmds-data.org/

26 – July 9 **Integrable Systems and Geometry at the XXXVth Workshop on Geometric Methods in Physics (WGMP)**
Location: *Bialowieza, Poland.*
URL: wgmp.uwb.edu.pl

27 – 29 **Recent Trends in Differential Equations (RTDE2016)**
Location: *University of Aveiro, Aveiro, Portugal.*
URL: rtde2016.weebly.com

27 – July 1 **This meeting is one of the ESGI and will bring together several academics with a large experience in industrial mathematics to tackle problems that the leading industries are facing.**
Location: *Porto Design Factory, Porto, Portugal.*
URL: www2.estgf.ipp.pt/esgi/

27 – July 1 **3rd Barcelona Summer School on Stochastic Analysis**
Location: *Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain.*
URL:
www.crm.cat/en/Activities/Curs_2015-2016/Pages/3rd-BCN-Summer-School-on-Stochastic-Analysis.aspx

27 – July 2 **CNRS-PAN Mathematics Summer Institute, Cracow, Poland, 27 June - 2 July, 2016**
Location: *Cracow, Poland.*
URL: www.impan.pl/~peszat/

July 2016

4 – 7 **VII Jaen Conference on Approximation Theory, Computer Aided Geometric Design, Numerical Methods and Applications**
Location: *Ubeda, Jaen, Spain.*
URL: www.ujaen.es/revista/jja/jca/index.php

5 – 8 **5th International Conference on Uniform Distribution Theory**
Location: *Sopron, Hungary.*
URL: udt2016.inf.unideb.hu/

5 – 9 **New Methods in Finsler Geometry**
Location: *Department of Mathematics, Leipzig University Paulinum, Augustusplatz 10, D-04109 Leipzig Germany.*
URL: www.math.uni-leipzig.de/Finsler2016/

7 – 8 **1st Workshop on Dynamical Systems in the Real Life. RDS 2016**
Location: *IMAC. Instituto de Matemáticas y Aplicaciones de Castellón. Universitat Jaume I. Castellón, Spain.*
URL: www.rds2016.uji.es/

10 – 15 **Differential Geometry and its Applications**
Location: *Masaryk University, Brno, Czech Republic.*
URL: web.math.muni.cz/dga2016/

17 – 23 **Knots in Hellas 2016 International Conference on Knots, Low-Dimensional Topology and Applications**
Location: *International Olympic Academy Ancient Olympia, Greece.*
URL: www.math.ntua.gr/~sofia/KnotsinHellas2016/index.html

18 – 22 **Summer School on Surgery and the Classification of Manifolds**
Location: *University of Calgary, Calgary, Alberta, Canada.*
URL: www.pims.math.ca/scientific-event/160718-ssscm

25 – 29 **ATMCS7: Algebraic Topology: Methods, Computation, & Science**
Location: *Polytechnic University of Turin, Turin, Italy.*
URL: atmcs7.appliedtopology.org/

August 2016

8 – 14 **Connecticut Summer School in Number Theory**
Location: *University of Connecticut, Storrs, CT.*
URL: ctnt-summer.math.uconn.edu/

12 – 14 **Conference on Elliptic Curves, Modular Forms, and related topics**
Location: *University of Connecticut, Storrs, CT.*
URL: ctnt-summer.math.uconn.edu/

21 – 24 **Mathematics: Applied, an International Conference**
Location: *Congress Centre of Ss. Cyril and Methodius University, Ohrid, Republic of Macedonia.*
URL: www.research-publication.com/index.php/ma-2016

22 – 26 **24th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications (24ICFIDCAA)**
Location: *Anand International College of Engineering, Jaipur, Rajasthan, India.*
URL: anandice.ac.in/24icfidcaa-2016/

22 – 26 **International Conference Waves in Science and Engineering 2016**
Location: *Center for Research and Advanced Studies of the National Polytechnic Institute, Queretaro, Mexico.*
URL: qro.cinvestav.mx/wise2016

25 – 26 **Caucasian Mathematics Conference (CMC-II)**
Location: *Department of Mathematics, Faculty of Sciences, Yuzuncu Yil University, Van, Turkey.*
URL: www.euro-math-soc.eu/cmc/

25 – 28 **Geometric Function Theory and Applications - GFTA 2016**
Location: *1 Decembrie 1918 University of Alba Iulia, Alba Iulia, Romania.*
URL: gfta2016.uab.ro

September 2016

5 – 9 **Combinatorics and Operators in Quantum Information Theory**
Location: *Queen's University Belfast, Belfast, United Kingdom.*
URL: www.qciao.org

7 – 10 **The Organizing Committee of ICAAM and Institute of Mathematics and Mathematical Modelling are pleased to invite you to the Third International Conference on Analysis and Applied Mathematics.**
Location: *Institute of Mathematics and Mathematical Modelling, Almaty, Kazakhstan.*
URL: www.icaam-online.org/

19 – 23 **AIM Workshop: Soft Packings, Nested Clusters, and Condensed Matter**
Location: *American Institute of Mathematics, San Jose, CA.*
URL: aimath.org/workshops/upcoming/softpack

28 – 30 **Third Conference on Recent Trends in Nonlinear Phenomena**
Location: *Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Italy.*
URL: www.sti.uniurb.it/servadei/ConferencePerugia2016

October 2016

10 – 14 **AIM Workshop: Boundaries of Groups**
Location: *American Institute of Mathematics, San Jose, CA.*
URL: aimath.org/workshops/upcoming/groupbdy

14 – 16 **9th International Symposium on Symposium on Biomathematics and Ecology: Education and Research (B.E.E.R.-2016)**
Location: *College of Charleston, Charleston, SC.*
URL: symposium.beer

19 – 21 **International Conference on Modeling, Simulation and Control 2016**
Location: *UC Berkeley, San Francisco Bay Area.*
URL: www.iaeng.org/WCECS2016/ICMSC2016.html

24 – 28 **AIM Workshop: Rational Subvarieties in Positive Characteristic**
Location: *American Institute of Mathematics, San Jose, CA.*
URL: aimath.org/workshops/upcoming/ratsubvarpos

November 2016

5 – 6 **36th Southeastern-Atlantic Regional Conference on Differential Equations (SEARCDE)**
Location: *Florida Gulf Coast University, Fort Myers, FL USA.*
URL: lebesgue.fgcu.edu/SEARCDE2016/

28 – December 2 **International Conference on Mathematical Analysis and its Applications, ICMAA - 2016**
Location: *Indian Institute of Technology Roorkee (I.I.T. Roorkee), Roorkee, India.*
URL: www.iitr.ac.in/icmaa/2016/index.html

December 2016

1 – 2 **Mycrypt 2016**
Location: *KL, Malaysia.*
URL: `https://foe.mmu.edu.my/mycrypt2016`

5 – 9 **AIM Workshop: Global Langlands Correspondence**
Location: *American Institute of Mathematics, San Jose, CA.*
URL: `aimath.org/workshops/upcoming/globlanglands`

9 – 11 **International Conference on Applications of Mathematics in Topological Dynamics, Physical, Biological and Chemical Systems (ICAMTPBCS-2016)**
Location: *Calcutta Mathematical Society Asutosh Bhavan AE-374, Sector-I, Salt Lake City, Kolkata-700064, West Bengal, India.*
URL: `www.calmathsoc.org`

16 – 18 **SPACE 2016 – Sixth International Conference on Security, Privacy and Applied Cryptographic Engineering**
Location: *C.R.Rao Advanced Institute of Mathematics Statistics and Computer Science, Hyderabad-India (AIMSCS).*
URL: `www.math.umn.edu/~math-sa-sara0050/space16/`

19 – 23 **International Conference "Anosov Systems and Modern Dynamics"**
Location: *Steklov Mathematical Institute of Russian Academy of Sciences and Department of Mathematics of National Research University Higher School of Economics, Moscow, Russia.*
URL: `anosov80.mi.ras.ru`

January 2017

17 – 21 **The Third International Conference on Mathematics and Computing (ICMC 2017)**
Location: *Haldia Institute of Technology, Haldia, West-Bengal, India.*
URL: `www.hithaldia.co.in/icmc2017/`

17 – May 26 **Analytic Number Theory (ANT2)**
Location: *Mathematical Sciences Research Institute, Berkeley, CA.*
URL: `www.msri.org/programs/297`

17 – May 26 **Harmonic Analysis (HA2)**
Location: *Mathematical Sciences Research Institute, Berkeley, CA.*
URL: `www.msri.org/programs/300`

30 – February 3 **AIM Workshop: Zero Forcing and its Applications**
Location: *American Institute of Mathematics, San Jose, CA.*
URL: `aimath.org/workshops/upcoming/zeroforcing`

March 2017

20 – 24 **AIM Workshop: Trisections and Low-Dimensional Topology**
Location: *American Institute of Mathematics, San Jose, CA.*
URL: `aimath.org/workshops/upcoming/trisections`

27 – 31 **AIM Workshop: Fisher-Hartwig Asymptotics, Szego Expansions, and Applications to Statistical Physics**
Location: *American Institute of Mathematics, San Jose, CA.*
URL: `aimath.org/workshops/upcoming/fhszego`

April 2017

10 – 14 **AIM Workshop: Foundations of Tropical Schemes**
Location: *American Institute of Mathematics, San Jose, CA.*
URL: `aimath.org/workshops/upcoming/tropschemes`

17 – 21 **AIM Workshop: Engel structures**
Location: *American Institute of Mathematics, San Jose, CA.*
URL: `aimath.org/workshops/upcoming/engelstr`

June 2017

26 – 30 **The Second Malta Conference in Graph Theory and Combinatorics (2MCGTC 2017)**
Location: *Qawra, St. Paul's Bay, Malta.*
URL: `www.um.edu.mt/events/2mcgtc2017/`

July 2017

3 – 7 **Banach Spaces and Operator Theory with Applications. A Conference on the Occasion of the 60th birthday of Mieczysław Mastyło**
Location: *Adam Mickiewicz University in Poznań, Poznań, Poland.*
URL: `banachspacetheory.wmi.amu.edu.pl/`

10 – 15 **Computational Methods and Function Theory 2017**
Location: *Maria Curie-Skłodowska University, Lublin, Poland.*
URL: `www.cmft2017.umcs.lublin.pl/index.html`

August 2017

14 – December 15 **Geometric Functional Analysis and Applications**
Location: *Mathematical Sciences Research Institute, Berkeley, California.*
URL: `www.msri.org/programs/298`

January 2018

16 – May 25 **Group Representation Theory and Applications**
Location: *Mathematical Sciences Research Institute, Berkeley, California.*
URL: `www.msri.org/programs/293`

16 – May 25 **Enumerative Geometry Beyond Numbers**
Location: *Mathematical Sciences Research Institute, Berkeley, California.*
URL: `www.msri.org/programs/295`

January 2019

22 – May 24 **Derived Algebraic Geometry**
Location: *Mathematical Sciences Research Institute, Berkeley, California.*
URL: `www.msri.org/programs/306`

22 – May 24 **Birational Geometry and Moduli Spaces**
Location: *Mathematical Sciences Research Institute, Berkeley, California.*
URL: `www.msri.org/programs/311`

MEETINGS & CONFERENCES OF THE AMS

MAY TABLE OF CONTENTS

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at: **www.ams.org/meetings/**.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to page 88 in the January 2016 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LaTeX. Visit **www.ams.org/cgi-bin/abstracts/abstract.pl**. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

MEETINGS IN THIS ISSUE

Conferences in Cooperation with the AMS

Indian Mathematics Consortium

December 14–17, 2016

Banaras Hindu University

Varanasi, India

See **www.ams.org/meetings/** *for the most up-to-date information on these conferences.*

ASSOCIATE SECRETARIES OF THE AMS

Central Section: Georgia Benkart, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; e-mail: steve.weintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403, e-mail: brian@math.uga.edu; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: lapidus@math.ucr.edu; telephone: 951-827-5910.

Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See www.ams.org/meetings/.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL .

Brunswick, Maine

Bowdoin College

September 24–25, 2016

Saturday – Sunday

Meeting #1121

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: June 2016
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 37, Issue 3

Deadlines

For organizers: Expired
For abstracts: July 19, 2016

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Tim Austin, New York University, *Title to be announced.*

Moon Duchin, Tufts University, *Title to be announced.*

Thomas Lam, University of Michigan, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic and Enumerative Combinatorics (Code: SS 12A), **Thomas Lam**, University of Michigan.

Autonomous and Non-autonomous Discrete Dynamical Systems with Applications (Code: SS 2A), **M.R.S. Kulenović** and **O. Merino**, University of Rhode Island.

Combinatorial Aspects of Nilpotent Orbits (Code: SS 18A), **Anthony Iarrobino**, Northeastern University, **Leila Khatami**, Union College, and **Julianna Tymoczko**, Smith College.

Combinatorics, at the Crossroads of Algebra, Geometry, and Topology (Code: SS 11A), **Ivan Martino**, University of Fribourg (Switzerland), and **Alexander I. Suciu**, Northeastern University.

Convex Cocompactness (Code: SS 14A), **Tarik Aougab** and **Sara Maloni**, Brown University.

Decomposing 3-manifolds (Code: SS 8A), **Tao Li**, Boston College, and **Scott Taylor**, Colby College.

Financial Mathematics (Code: SS 13A), **Maxim Bichuch**, Johns Hopkins University, and **Stephan Strum** and **Xuwei Yang**, Worcester Polytechnic Institute.

Geometric Aspects of Harmonic Analysis (Code: SS 6A), **Matthew Badger** and **Vasileios Chousionis**, University of Connecticut.

Geometric Group Theory (Code: SS 4A), **Charles Cunningham**, Bowdoin College, **Moon Duchin**, Tufts University, and **Jennifer Taback**, Bowdoin College.

Geometry of Nilpotent Groups (Code: SS 5A), **Moon Duchin**, Tufts University, **Jennifer Taback**, Bowdoin College, and **Peter Wong**, Bates College.

Mathematics and Statistics Applied to Biology and Related Fields (Code: SS 7A), **Meredith L. Greer**, Bates College.

New Developments in Graphs and Hypergraphs (Code: SS 16A), **Deepak Bal** and **Jonathan Cutler**, Montclair State University, and **Jozef Skokan**, London School of Economics.

Noncommutative Ring Theory and Noncommutative Algebra (Code: SS 1A), **Jason Gaddis**, Wake Forest University, and **Manuel Reyes**, Bowdoin College.

Nonlinear Partial Differential Equations in Material Science and Mathematical Biology (Code: SS 3A), **Leonid Berlyand**, Pennsylvania State University, **Dmitry Golovaty**, University of Akron, and **Alex Misiats**, New York University.

Nonlinear Waves in Partial and Lattice Differential Equations (Code: SS 9A), **Christopher Chong**, Bowdoin College.

Plethysm and Kronecker Products in Representation Theory (Code: SS 17A), **Susanna Fishel**, Arizona State University, and **Sheila Sundaram**, Pierrepont School.

Topological Phases of Matter and Quantum Computation (Code: SS 15A), **Paul Bruillard** and **Carlos Ortiz**, Pacific Northwest National Laboratory, and **Julia Plavnik**, Texas A&M University.

Undergraduate Research (Code: SS 10A), **Christopher Chong** and **Adam Levy**, Bowdoin College.

Denver, Colorado

University of Denver

October 8–9, 2016

Saturday – Sunday

Meeting #1122

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: August 2016
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 37, Issue 3

Deadlines

For organizers: Expired
For abstracts: August 16, 2016

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Henry Cohn, Microsoft Research, New England, *Title to be announced*.

Ronny Hadani, University of Texas, Austin, *Title to be announced*.

Chelsea Walton, Temple University, Philadelphia, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Above and Beyond Fluid Flow studies: In celebration of the 60th birthday of Prof. William Layton (Code: SS 12A), **Traian Iliescu**, Virginia Polytechnic Institute and State University, **Alexander Labovsky**, Michigan Technological University, **Monika Neda**, University of Nevada, Las Vegas, and **Leo Rebholz**, Clemson University.

Algebraic Combinatorics (Code: SS 23A), **Anton Betten**, Colorado State University, **Jason Williford**, University of Wyoming, and **Bangteng Xu**, Eastern Kentucky University.

Algebraic Logic (Code: SS 1A), **Nick Galatos**, University of Denver, and **Peter Jipsen**, Chapman University.

Analysis on Graphs and Spectral Graph Theory (Code: SS 2A), **Paul Horn** and **Mei Yin**, University of Denver.

Aspects of PDE Arising from Modeling of the Flows in Porous Media (Code: SS 19A), **Akif Ibraguimov**, Texas Tech University, **Viktoria Savatorova**, University of Nevada, Las Vegas, and **Aleksey Telyakovskiy**, University of Nevada, Reno.

Discontinuous Galerkin methods for partial differential equations: Theory and applications (Code: SS 15A), **Mahboub Baccouch**, University of Nebraska at Omaha.

Floer Theoretic Invariants of 3-manifolds and Knots (Code: SS 22A), **Jonathan Hanselman**, University of Texas at Austin, and **Kristen Hendricks**, University of California, Los Angeles.

Foundations of Numerical Algebraic Geometry (Code: SS 14A), **Abraham Martin del Campo**, CIMAT, Guanajuato, Mexico, and **Frank Sottile**, Texas A&M University.

Groups and Representation Theory (Code: SS 20A), **C. Ryan Vinroot**, College of William and Mary, **Julianne Rainbolt**, Saint Louis University, and **Amanda Schaeffer Fry**, Metropolitan State University of Denver.

Integrable Systems and Soliton Equations (Code: SS 17A), **Anton Dzhamay**, University of Northern Colorado, and **Patrick Shipman**, Colorado State University.

Nonassociative Algebra (Code: SS 3A), **Izabella Stuhl**, University of Debrecen and University of Denver, and **Petr Vojtěchovský**, University of Denver.

Noncommutative Geometry and Fundamental Applications (Code: SS 4A), **Frederic Latremoliere**, University of Denver.

Nonlinear Wave Equations and Applications (Code: SS 18A), **Mark J. Ablowitz**, University of Colorado Boulder, and **Barbara Prinari**, University of Colorado Colorado Springs.

Nonlinear and Stochastic Partial Differential Equations (Code: SS 13A), **Michele Coti Zelati**, University of Maryland, **Nathan Glatt-Holtz**, Virginia Polytechnic Institute and State University, and **Geordie Richards**, University of Rochester.

Operator Algebras and Applications (Code: SS 5A), **Alvaro Arias**, University of Denver.

Quantum Algebra (Code: SS 11A), **Chelsea Walton**, Temple University, **Ellen Kirkman**, Wake Forest University, and **James Zhang**, University of Washington, Seattle.

Random matrices, integrable systems, and applications (Code: SS 16A), **Sean D. O'Rourke**, University of Colorado Boulder, and **David Renfrew**, University of California, Los Angeles.

Recent Advances in Structural and Extremal Graph Theory (Code: SS 21A), **Michael Ferrara**, **Stephen Hartke**, **Michael Jacobson**, and **Florian Pfender**, University of Colorado Denver.

Recent Trends in Semigroup Theory (Code: SS 6A), **Michael Kinyon**, University of Denver, and **Ben Steinberg**, City College of New York.

Set Theory of the Continuum (Code: SS 7A), **Natasha Dobrinen** and **Daniel Hathaway**, University of Denver.

Unimodularity in Randomly Generated Graphs (Code: SS 8A), **Florian Sobieczky**, University of Denver.

Vertex Algebras and Geometry (Code: SS 9A), **Andrew Linshaw**, University of Denver, and **Thomas Creutzig** and **Nicolas Guay**, University of Alberta.

Zero Dimensional Dynamics (Code: SS 10A), **Nic Ormes** and **Ronnie Pavlov**, University of Denver.

Minneapolis, Minnesota

University of St. Thomas (Minneapolis campus)

October 28–30, 2016

Friday – Sunday

Meeting #1123

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: August 2016
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 37, Issue 4

Deadlines

For organizers: Expired
For abstracts: August 30, 2016

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Thomas Nevins, University of Illinois Urbana-Champaign, *Title to be announced*.

Charles Rezk, University of Illinois Urbana-Champaign, *Title to be announced*.

Christof Sparber, University of Illinois at Chicago, *Title to be announced*.

Samuel Stechmann, University of Wisconsin-Madison, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances in Algebraic Coding Theory (Code: SS 11A), **Sarah E. Anderson**, University of St. Thomas, and **Katie Haymaker**, Villanova University.

Chip-Firing and Divisors on Graphs and Complexes (Code: SS 3A), **Caroline Klivans**, Brown University, and **Gregg Musiker** and **Victor Reiner**, University of Minnesota.

Combinatorial Representation Theory (Code: SS 5A), **Michael Chmutov**, University of Minnesota, **Tom Halverson**, Macalester College, and **Travis Scrimshaw**, University of Minnesota.

Enumerative Combinatorics (Code: SS 4A), **Eric Egge**, Carleton College, and **Joel Brewster Lewis**, University of Minnesota.

Extremal and Probabilistic Combinatorics (Code: SS 13A), **Andrew Beveridge**, University of Nebraska—Lincoln, **Jamie Radcliffe**, University of Minnesota, Twin Cities, and **Michael Young**, Iowa State University.

Geometric Flows, Integrable Systems and Moving Frames (Code: SS 2A), **Joseph Benson**, St. Olaf College, **Gloria Mari-Beffa**, University of Wisconsin-Madison, **Peter Olver**, University of Minnesota, and **Rob Thompson**, Carleton College.

Integrable Systems and Related Areas (Code: SS 8A), **Sam Evens**, University of Notre Dame, **Luen-Chau Li**, University of Minnesota, and **Zhaohu Nie**, Utah State University.

Modeling and Predicting the Atmosphere, Oceans, and Climate (Code: SS 1A), **Sam Stechmann**, University of Wisconsin-Madison.

New Developments in the Analysis of Nonlocal Operators (Code: SS 6A), **Donatella Danielli** and **Arshak Petrosyan**, Purdue University, and **Camelia Pop**, University of Minnesota.

Representation Theory, Automorphic Forms and Related Topics (Code: SS 7A), **Kwangho Choiy**, Southern Illinois University, **Dihua Jiang**, University of Minnesota, and **Shuichiro Takeda**, University of Missouri.

Symplectic Geometry and Contact Geometry (Code: SS 9A), **Tian-Jun Li** and **Cheuk Yu Mak**, University of Minnesota, and **Ke Zhu**, Minnesota State University.

Topology and Arithmetic (Code: SS 10A), **Tyler Lawson** and **Craig Westerland**, University of Minnesota, Twin Cities.

Topology and Physics (Code: SS 12A), **Ralph Kaufmann**, Purdue University, and **Alexander Voronov**, University of Minnesota, Twin Cities.

Raleigh, North Carolina

North Carolina State University

November 12–13, 2016

Saturday – Sunday

Meeting #1124

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: September 2016
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 37, Issue 4

Deadlines

For organizers: Expired
For abstracts: September 13, 2016

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Ricardo Cortez, Tulane University, *Title to be announced.*

Jason Metcalfe, University of North Carolina at Chapel Hill, *Title to be announced.*

Agnes Szanto, North Carolina State University, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances in Numerical Methods for Partial Differential Equations (Code: SS 7A), **Andreas Aristotelous**, West Chester University, and **Thomas Lewis**, The University of North Carolina at Greensboro.

Algebraic Structures Motivated by and Applied to Knot Theory (Code: SS 6A), **Jozef H. Przytycki**, The George Washington University, and **Radmila Sazdanovic**, North Carolina State University.

Control, Optimization, and Differential Games (Code: SS 12A), **Lorena Bociu**, North Carolina State University, and **Tien Khai Nguyen**, Penn State University.

Difference Equations and Applications (Code: SS 2A), **Michael A. Radin**, Rochester Institute of Technology, and **Youssef Raffoul**, University of Dayton.

Geometry and Topology in Image and Shape Analysis (Code: SS 13A), **Irina Kogan**, North Carolina State University, and **Facundo Mémoli**, The Ohio State University.

Homological Methods in Commutative Algebra (Code: SS 1A), **Alina Iacob** and **Saeed Nasseh**, Georgia Southern University.

Low-dimensional Topology (Code: SS 9A), **Caitlin Leverson**, Georgia Tech, **Tye Lidman**, North Carolina State University, and **Leonard Ng**, Duke University.

Mathematical Modeling of Infectious Disease and Immunity (Code: SS 11A), **Lauren Childs**, Virginia Tech and Harvard Chan School of Public Health, and **Stanca Ciupe**, Virginia Tech.

Mathematical String Theory (Code: SS 3A), **Paul Aspinwall**, Duke University, **Ilarion Melnikov**, James Madison University, and **Eric Sharpe**, Virginia Tech.

Metric and Topological Oriented Fixed Point Theorems (Code: SS 5A), **Clement Boateng Ampadu**, Boston, MA, **Sartaj Ali**, National College of Business Administration and Economics, Lahore, Pakistan, **Xiaorong Liu**, University of Colorado at Boulder, and **Xavier Alexius Udo-Utun**, University of Uyo, Uyo, Nigeria.

Nonlinear Boundary Value Problems (Code: SS 10A), **Maya Chhetri**, UNC Greensboro, and **Stephen Robinson**, Wake Forest University.

Representations of Lie Algebras, Quantum Groups and Related Topics (Code: SS 8A), **Naihuan Jing**, and **Kailash C. Misra**, North Carolina State University.

Varieties, Their Fibrations and Automorphisms in Mathematical Physics and Arithmetic Geometry (Code: SS 4A), **Jimmy Dillies** and **Enka Lakuriqi**, Georgia Southern University, and **Tony Shaska**, Oakland University.

Atlanta, Georgia

Hyatt Regency Atlanta and Marriott Atlanta Marquis

January 4–7, 2017

Wednesday – Saturday

Meeting #1125

Joint Mathematics Meetings, including the 123rd Annual Meeting of the AMS, 100th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: October 2016
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 38, Issue 1

Deadlines

For organizers: Expired
For abstracts: To be announced

Charleston, South Carolina

College of Charleston

March 10–12, 2017

Friday – Sunday

Meeting #1126

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: November 10, 2016
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://www.ams.org/cgi-bin/abstracts/abstract.pl.

Commutative Algebra (Code: SS 1A), **Bethany Kubik**, University of Minnesota Duluth, **Saeed Nasseh**, Georgia Southern University, and **Sean Sather-Wagstaff**, Clemson University.

Bloomington, Indiana

Indiana University

April 1–2, 2017

Saturday – Sunday

Meeting #1127

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://www.ams.org/cgi-bin/abstracts/abstract.pl.

Randomness in Complex Geometry (Code: SS 1A), **Turgay Bayraktar**, Syracuse University, and **Norman Levenberg**, Indiana University.

Pullman, Washington

Washington State University

April 22–23, 2017

Saturday – Sunday

Meeting #1128

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://www.ams.org/cgi-bin/abstracts/abstract.pl.

Fixed Point Methods in Differential and Integral Equations (Code: SS 1A), **Theodore A. Burton**, Southern Illinois University in Carbondale.

New York, New York

Hunter College, City University of New York

May 6–7, 2017

Saturday – Sunday

Meeting #1129

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: September 14, 2016
For abstracts: March 21, 2017

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Jeremy Kahn, City University of New York, *Title to be announced.*

Fernando Coda Marques, Princeton University, *Title to be announced.*

James Maynard, Magdalen College, University of Oxford, *Title to be announced* (Erdős Memorial Lecture).

Kavita Ramanan, Brown University, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://www.ams.org/cgi-bin/abstracts/abstract.pl.

Commutative Algebra (Code: SS 1A), **Laura Ghezzi**, New York City College of Technology-CUNY, and **Jooyoun Hong**, Southern Connecticut State University.

Cryptography (Code: SS 3A), **Xiaowen Zhang**, College of Staten Island and Graduate Center-CUNY.

Infinite Permutation Groups, Totally Disconnected Locally Compact Groups, and Geometric Group Theory (Code: SS 4A), **Delaram Kahrobaei**, New York City College of Technology and Graduate Center-CUNY, and **Simon Smith**, New York City College of Technology-CUNY.

Recent Advances in Function Spaces, Operators and Nonlinear Differential Operators (Code: SS 2A), **David Cruz-Uribe**, University of Alabama, **Jan Lang**, The Ohio State University, and **Osvaldo Mendez**, University of Texas at El Paso.

Montréal, Quebec Canada

McGill University

July 24–28, 2017

Monday – Friday

Meeting #1130

The second Mathematical Congress of the Americas (MCA 2017) is being hosted by the Canadian Mathematical Society (CMS) in collaboration with the Pacific Institute for the Mathematical Sciences (PIMS), the Fields Institute (FIELDS), Le Centre de Recherches Mathématiques (CRM), and the Atlantic Association for Research in the Mathematical Sciences (AARMS).
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: July 31, 2016
For abstracts: To be announced

Denton, Texas

University of North Texas

September 9–10, 2017

Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Buffalo, New York

State University of New York at Buffalo

September 16–17, 2017

Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: February 14, 2017
For abstracts: To be announced

Orlando, Florida

University of Central Florida, Orlando

September 23–24, 2017

Saturday – Sunday

Meeting #1209

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: February 23, 2017
For abstracts: July 25, 2017

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Commutative Algebra: Interactions with Algebraic Geometry and Algebraic Topology (Code: SS 1A), **Joseph Brennan**, University of Central Florida, and **Alina Iacob** and **Saeed Nasseh**, Georgia Southern University.

Riverside, California

University of California, Riverside

November 4–5, 2017

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 10–13, 2018

Wednesday – Saturday
Joint Mathematics Meetings, including the 124th Annual Meeting of the AMS, 101st Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2017
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2017
For abstracts: To be announced

Portland, Oregon

Portland State University

April 14–15, 2018

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Baltimore, Maryland

Baltimore Convention Center, Hilton Baltimore, and Baltimore Marriott Inner Harbor Hotel

January 16–19, 2019

Wednesday – Saturday
Joint Mathematics Meetings, including the 125th Annual Meeting of the AMS, 102nd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2018
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 2, 2018
For abstracts: To be announced

Denver, Colorado

Colorado Convention Center

January 15–18, 2020

Wednesday – Saturday
Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: November 1, 2019
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2019
For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center

January 6–9, 2021

Wednesday – Saturday
Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: October 2020
Program first available on AMS website: November 1, 2020
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

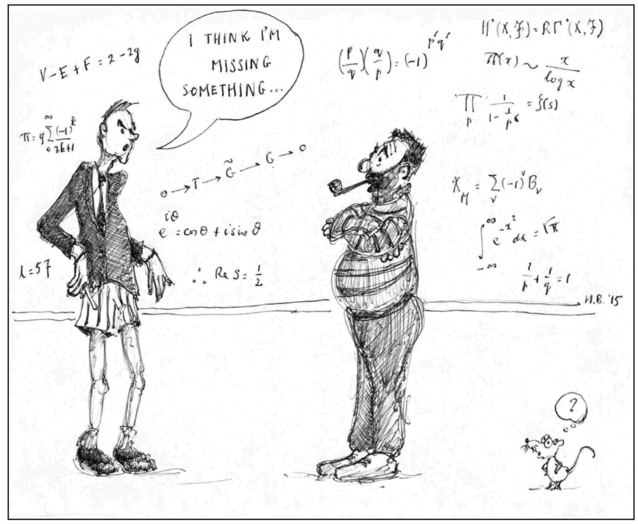
Deadlines

For organizers: April 1, 2020
For abstracts: To be announced

> "I am a little cautious ... when I try to attack some problem because the problem can attack back."
> —*John Forbes Nash Jr.*

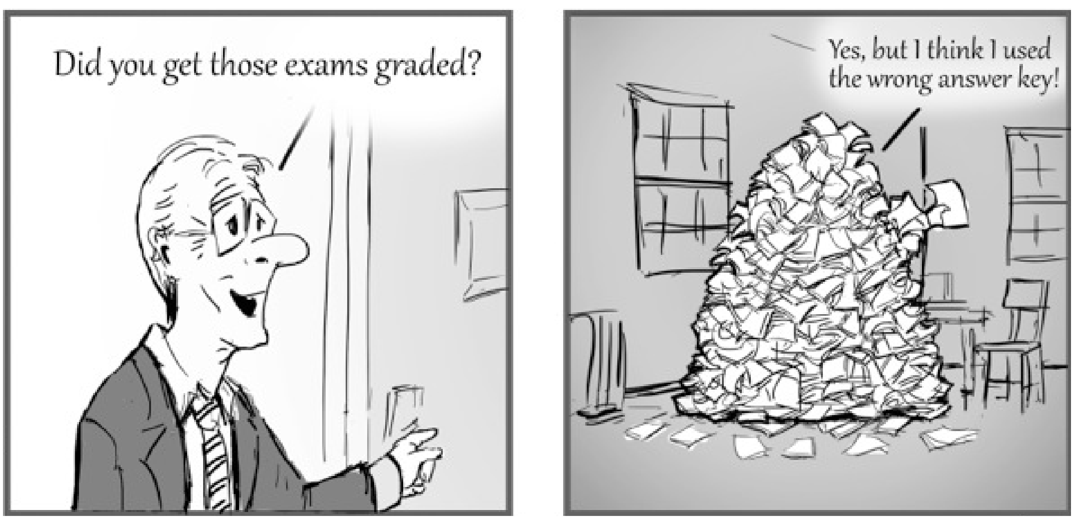
John Forbes Nash Jr.:

Number of publications: 26
Number of citations: 1650
Most cited paper: "Non-cooperative games," *Ann. of Math.*, 1951.
2nd most cited paper: "Continuity of solutions of parabolic and elliptic equations," *Amer. J. Math.*, 1958.



Artwork by Michael Berg.

My TA

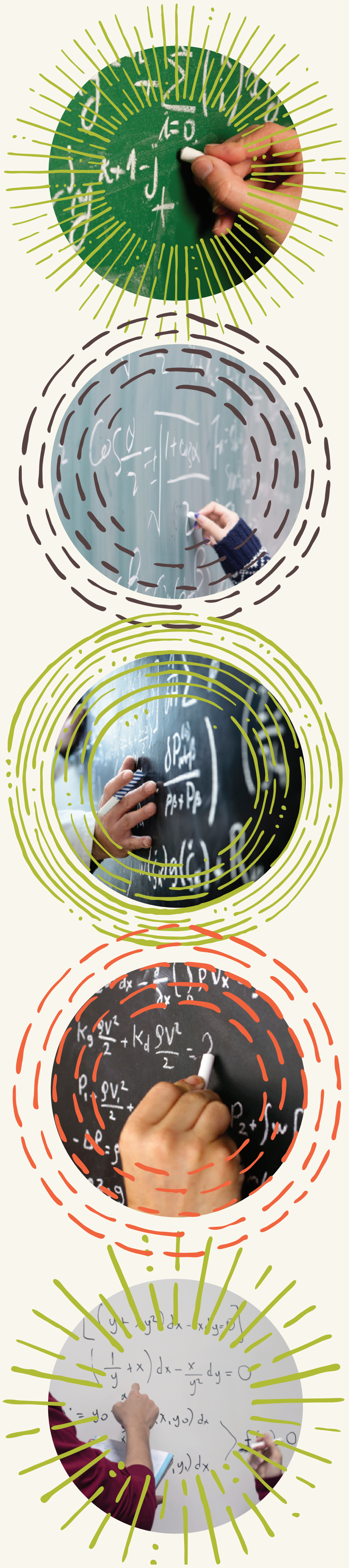


Artwork by Sam White.

QUESTIONABLE MATHEMATICS

Brett Weiner on `nytimes.com` (29 Dec '15) reports on an expert witness who couldn't rescale 3/16 inch by a factor of 20 feet per inch, even with a calculator.

What crazy things happen to you? Readers are invited to submit original short amusing stories, math jokes, cartoons, and other material to: `noti-backpage@ams.org`.



American Mathematical Society

AWARD FOR IMPACT ON THE TEACHING AND LEARNING OF MATHEMATICS

CALL FOR NOMINATIONS

The Award for Impact on the Teaching and Learning of Mathematics is given annually to a mathematician or group of mathematicians who have made significant contributions of lasting value to mathematics education. Priorities of the award include recognition of (a) accomplished mathematicians who have worked directly with precollege teachers to enhance teachers' impact on mathematics achievement for all students or (b) sustainable and replicable contributions by mathematicians to improving the mathematics education of students in the first two years of college.

The **$1,000** award is provided through an endowment fund established by a contribution from Kenneth I. and Mary Lou Gross in honor of their daughters Laura and Karen. The AMS Committee on Education selects the recipient.

Nominations with supporting information should be submitted online to **www.ams.org/profession/prizes-awards/ams-awards/impact**. Letters of nomination may be submitted by one or more individuals. The letter of nomination should describe the significant contributions made by the nominee(s) and provide evidence of the impact these contributions have made on the teaching and learning of mathematics. The letter of nomination should not exceed two pages and may include supporting documentation not to exceed three additional pages. A brief curriculum vitae for each nominee should also be included.

Deadline for nominations is September 15, 2016.

AMS
American Mathematical Society
www.ams.org