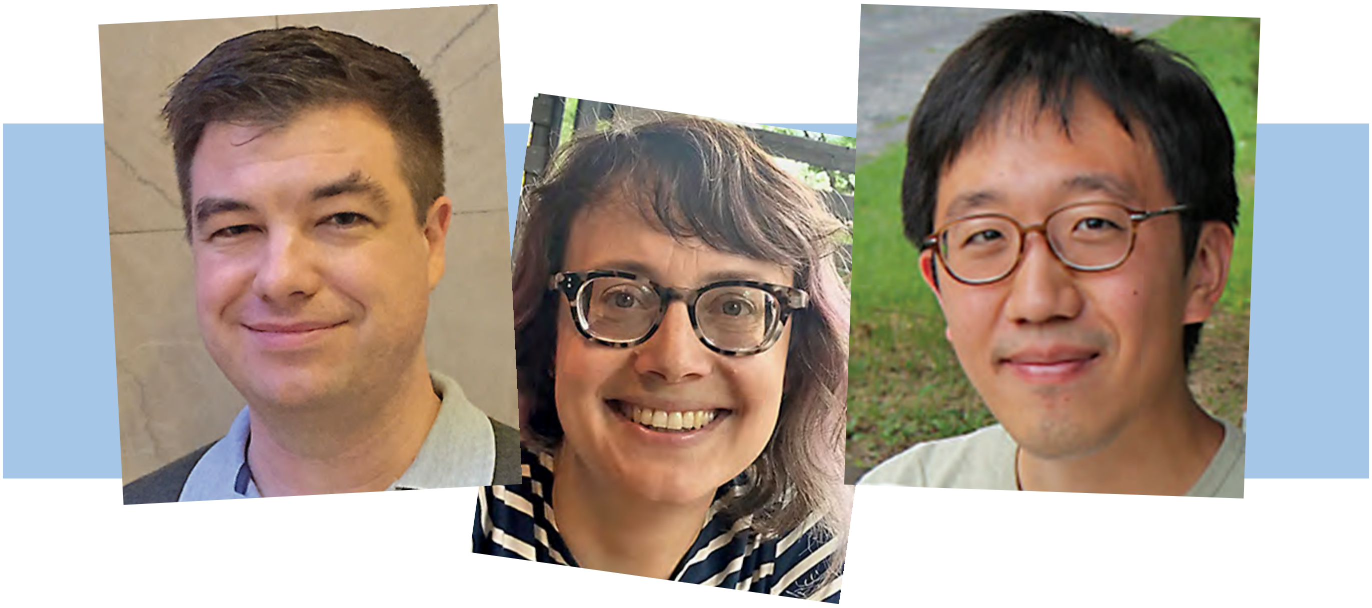
SPRING CENTRAL SECTIONAL SAMPLER



From left to right: Aaron Brown, Tullia Dymarz, and June Huh.

In this sampler, the speakers above have kindly provided introductions to their Invited Addresses for the upcoming AMS Spring Central Sectional Meeting.

March 17–18, 2018
(Saturday–Sunday)
The Ohio State University
Columbus, OH

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DOI: http://dx.doi.org/10.1090/noti1671

Aaron Brown

Recent Progress in the Zimmer Program

ABSTRACT. The Zimmer program refers to a number of questions and conjectures posed by Robert Zimmer in the 1980s concerning smooth actions of lattices in higher-rank Lie groups. We report on some recent progress by the author and collaborators.

Lattices in Higher-Rank Lie Groups

The primary objects in my talk are lattices Γ in higher-rank simple Lie groups G.

The simplest example of such a lattice is $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, the group of $n \times n$ integral matrices with determinant one. The group $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ is a discrete subgroup of $G = \mathrm{SL}(n, \mathbb{R})$, the group of $n \times n$ invertible matrices of determinant one. It is well known that the coset space G/Γ admits a finite-volume form which is invariant under the (left) action of G by translations on G/Γ. In general, a *lattice* in a Lie group G is a discrete subgroup $\Gamma \subset G$ such that the coset space G/Γ has a finite G-invariant volume. A lattice is *cocompact* if the coset space G/Γ is compact. The primary example of a lattice subgroup, $\mathrm{SL}(n, \mathbb{Z})$ in $\mathrm{SL}(n, \mathbb{R})$, is not cocompact; there do, however, exist cocompact lattices in $\mathrm{SL}(n, \mathbb{R})$.

The *rank* of the group $G = \mathrm{SL}(n, \mathbb{R})$ is $n - 1$; this is the dimension of the subgroup

$$A = \{\mathrm{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_n})\}$$

of diagonal matrices with positive entries. (Note that since we impose that the determinant is 1, we have $t_1 + t_2 + \cdots + t_n = 0$.) When $n \geq 3$, the rank of $G = \mathrm{SL}(n, \mathbb{R})$ is at least 2, and we say that G is *higher rank*.

Rigidity of Linear Representations

Given a discrete group Γ, a *linear representation* of Γ is a homomorphism $\pi\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ from Γ to a linear group $\mathrm{GL}(d, \mathbb{R})$. When Γ is a lattice in a higher-rank simple Lie group, it is well known that linear representations $\pi\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ exhibit a number of strong rigidity properties. Early rigidity results include the local rigidity results due to Selberg and Weil and the global rigidity results due to Mostow, Margulis, and Prasad.

The strongest rigidity results for linear representations are the superrigidity and arithmeticity theorems of Margulis. Margulis's superrigidity theorem roughly states that any linear representation $\pi\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ of a lattice Γ in a higher-rank simple Lie group G extends to a representation $\tilde{\pi}\colon G \to \mathrm{GL}(d, \mathbb{R})$ "up to a compact error." In particular, since representations $\tilde{\pi}\colon G \to \mathrm{GL}(d, \mathbb{R})$ are fully classified, the superrigidity theorem essentially classifies all linear representations of lattices in higher-rank Lie groups. As a motivating result for my talk, we have the following corollary of Margulis's superrigidity theorem:

Corollary. *For $n \geq 3$ and $d < n$, let Γ be a lattice in* $\mathrm{SL}(n, \mathbb{R})$. *Then the image of any linear representation*

$$\pi\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$$

is a finite group.

Actions on Manifolds and the Zimmer Program

Given a compact manifold M, let $\mathrm{Diff}(M)$ denote the group of smooth diffeomorphism of M. Given a fixed volume form on M, let $\mathrm{Diff}_{\mathrm{vol}}(M)$ be the group of volume-preserving diffeomorphisms of M. A *smooth action* of Γ on M (or a "nonlinear representation") is simply a homomorphism

$$\alpha\colon \Gamma \to \mathrm{Diff}(M).$$

Consider $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. We describe two prototype actions of Γ on low-dimensional compact manifolds; both examples are derived from the natural action of $\mathrm{SL}(n, \mathbb{Z})$ by linear transformations on $\mathbb{R}^n$. First, note that matrix multiplication preserves rays in $\mathbb{R}^n$. Thus, viewing the sphere S^{n-1} as a parametrization of all rays in $\mathbb{R}^n$, we obtain a natural action of $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ on S^{n-1}. Concretely, considering S^{n-1} as the set of unit vectors in $\mathbb{R}^{n-1}$, we define $\alpha\colon \Gamma \to \mathrm{Diff}(S^{n-1})$ by

$$\alpha(\gamma)(x) = \frac{\gamma \cdot x}{\|\gamma \cdot x\|}.$$

This action does not preserve any volume form on S^{n-1}. For the second action, since for every $\gamma \in \mathrm{SL}(n, \mathbb{Z})$ the coefficients of γ are integral and $\det \gamma = 1$, the action of $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{R}^n$ by linear transformations preserves the subgroup $\mathbb{Z}^n \subset \mathbb{R}^n$; this induces an action of $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Concretely, we define $\alpha\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^n)$ by

$$\alpha(\gamma)(x + \mathbb{Z}^n) = \gamma x + \mathbb{Z}^n.$$

This action of $\mathrm{SL}(n, \mathbb{Z})$ preserves the natural Lebesgue volume form on $\mathbb{T}^n$.

The *Zimmer program* refers to a number of questions and conjectures posed by Robert Zimmer in the 1980s concerning smooth actions of lattices in higher-rank Lie groups. Roughly, these questions and conjectures aim to establish analogues of known rigidity results for linear representations in the "nonlinear" setting of smooth actions on manifolds. In particular, one may expect that all smooth actions of higher-rank lattices Γ are constructed from standard algebraic actions (such as the actions described in the previous paragraph) and thus, in some sense, can be completely classified.

A first step towards classifying all actions is *Zimmer's conjecture*, which asserts that no nontrivial[1] actions exist

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DOI: http://dx.doi.org/10.1090/noti1643

[1] *An action $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ is* **trivial** *if the image $\alpha(\gamma) \subset \mathrm{Diff}(M)$ is a finite group; otherwise, it is said to be* **nontrivial**.

on closed manifolds whose dimension is less than the dimension of any manifold appearing in an algebraic action. For instance, when $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, the action on the sphere S^{n-1} is smallest dimensional algebraic action and the action on the torus $\mathbb{T}^n$ is smallest dimensional volume-preserving algebraic action. In particular, for lattices in $\mathrm{SL}(n, \mathbb{R})$, Zimmer's conjecture asserts the following.

Zimmer's Conjecture. *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a lattice and let M be a closed manifold.*

(1) *If $\dim(M) < n - 1$, then any homomorphism*
$$\alpha \colon \Gamma \to \mathrm{Diff}^r(M)$$
has finite image.

(2) *If $\dim(M) < n$ and α preserves a smooth volume form, then any homomorphism*
$$\alpha \colon \Gamma \to \mathrm{Diff}^r_{\mathrm{vol}}(M)$$
has finite image.

In particular, Zimmer's conjecture provides an analogue in the context of smooth actions of the finiteness of all linear representations given by the corollary to Margulis's superrigidity theorem above. Prior work towards Zimmer's conjecture primarily focused on actions on the circle and on surfaces, including work by Witte, Burger and Monod, Ghys, Polterovich, and Franks and Handel.

Recent Progress in the Zimmer Program

My talk will focus on recent work on Zimmer's conjecture and the larger Zimmer program due to David Fisher, Sebastian Hurtado, Federico Rodriguez Hertz, Zhiren Wang, and me.

The first main result is the verification of Zimmer's conjecture for actions of certain lattices in $\mathrm{SL}(n, \mathbb{R})$. Namely, when $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ is either

- a cocompact lattice in $\mathrm{SL}(n, \mathbb{R})$ or
- a finite-index subgroup of $\mathrm{SL}(n, \mathbb{Z})$,

Fisher, Hurtado, and I verified that Zimmer's conjecture holds. We also give the conjectured dimension bounds (below which all actions are trivial) for actions of cocompact lattices in certain other matrix groups, including $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}(n, n)$, and $\mathrm{SO}(n, n+1)$. We also give partial results for actions of cocompact lattices in all higher-rank simple Lie groups which show triviality of all actions on manifolds whose dimension is below a certain critical dimension which grows linearly in the rank but which may be lower than the conjectured critical dimension.

At (and above) the critical dimensions $(n - 1)$ and n appearing in Zimmer's conjecture, there are examples of nontrivial algebraic actions of $\mathrm{SL}(n, \mathbb{Z})$. The aim of the Zimmer program is to classify all such smooth actions in terms of algebraic actions. Most results towards such a classification concern volume-preserving Anosov actions on tori (and nilmanifolds). These include a number of results by Katok, Lewis, Margulis, Qian, and Zimmer showing local and global rigidity of volume-preserving Anosov actions of higher-rank lattices on tori.

In recent work with Rodriguez Hertz and Wang, for $n \geq 3$ we study actions of $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ (and more general higher-rank lattices) on tori $\mathbb{T}^d$ (and nilmanifolds). Under certain dynamical hypotheses on the induced action on homology and a mild lifting condition, we show that such an action is, up to a continuous surjective (but possibly noninvertible) change of coordinates, given by an algebraic action. For Anosov actions, the change of coordinates is then shown to be a smooth diffeomorphism. Unlike earlier approaches, our method does not require that the action preserve a volume (or any probability measure).

In another recent and ongoing work with Rodriguez Hertz and Wang, for $n \geq 3$ we study actions of $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ on $(n - 1)$-dimensional manifolds. For such actions, we showed that all nontrivial actions are measurably equivalent to (a finite measurable cover of) the standard action on the sphere; in ongoing work, we aim to show all such actions are equivalent to the natural action on the sphere (or projective space) up to a smooth change of coordinates.

The proofs of the above results combine tools and results from many areas of mathematics, including smooth ergodic theory, homogeneous dynamics, Lie theory, representation theory, and operator algebras. In my talk I will present a number of motivations for the Zimmer program, outline the new results discussed above, and indicate how tools and ideas from various areas of mathematics are combined to establish these results.

ABOUT THE AUTHOR

Aaron Brown's research is in smooth dynamics, ergodic theory, and group actions on manifolds.

Aaron Brown

Tullia Dymarz

Bi-Lipschitz Equivalence of Groups

ABSTRACT. We consider the difference between bi-Lipschitz equivalence and quasi-isometric equivalence of infinite finitely generated groups.

The foundational premise of geometric group theory is that a finitely generated group can be endowed with a canonical metric, unique up to equivalence. This equivalence class should contain all word metrics (those metrics given by fixing a finite generating set and counting the number of generators it takes to transform one element to the other), but it may contain many other kinds of metrics. Gromov started the field of geometric group theory by suggesting that the right equivalence to study is *quasi-isometry*. A quasi-isometric equivalence between metric spaces $(X, d_X), (Y, d_Y)$ requires the existence of a map $f : X \to Y$ and two constants K, C such that for all $x, x' \in X$,

$$-C + \frac{1}{K} d_X(x, x') \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + C$$

and such that the C neighborhood of the image of f is all of Y. This last condition implies that the property is symmetric.

It is not hard to check that all word metrics on a fixed group are quasi-isometric. In addition, many finitely generated groups that arise in geometry, such as fundamental groups of compact Riemannian manifolds, are quasi-isometric to well-studied metric spaces on which they act by isometries, such as the universal cover of the manifold. This phenomenon is captured by Milnor-Švarc's fundamental lemma of geometric group theory: *If a group acts* properly discontinuously *and* cocompactly *on a proper geodesic metric space then it is quasi-isometric to the metric space it acts on.*

The question now is, which groups are quasi-isometric to each other? A simple exercise shows that finite-index subgroups are always quasi-isometric, but there are many examples of groups that are quasi-isometric that do not share any finite-index subgroups. In addition, certain properties that are defined algebraically actually turn out to be geometric, in that they are preserved by quasi-isometry. The prime example of this theme is Gromov's result from the 1980s that shows that any group quasi-isometric to a nilpotent group (nilpotence is a condition defined algebraically) must itself contain a finite-index nilpotent subgroup.

My primary research focus has been on the quasi-isometric classification of polycyclic groups. Algebraically, polycyclic groups lie between nilpotent groups (a geometric class of groups) and solvable groups (not a geometric class of groups). In 2007 Eskin-Fisher-Whyte showed that all groups quasi-isometric to the smallest nonnilpotent polycyclic group are also, up to finite-index, polycyclic and conjectured that being polycyclic is indeed a geometric property. This work is technical and will not be the subject of my talk.

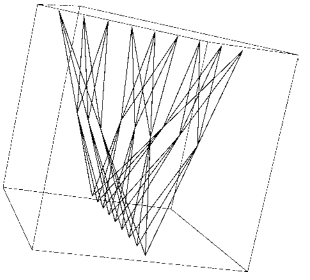

Figure 1. A portion of the Cayley graph for the lamplighter group $(\mathbb{Z}/3\mathbb{Z}) \wr \mathbb{Z}$**, which is quasi-isometric but not bi-Lipschitz equivalent to its index-two subgroup** $(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \wr \mathbb{Z}$**.**

Instead I will talk about another natural equivalence on finitely generated groups which is defined similarly to quasi-isometry except that the additive constant C is required to be zero. This restriction gives us *bi-Lipschitz* equivalence. Under this equivalence the fundamental lemma no longer holds, but all word metrics are bi-Lipschitz equivalent, and surprisingly for certain classes of groups, such as *nonamenable* groups, as proved by Whyte, both types of maps generate the same equivalence. This equivalence was also considered by Gromov, who asked whether it was possible to find groups that are quasi-isometric but not bi-Lipschitz equivalent. In 2010 I found the first examples of such groups. These groups come from the family of so-called *lamplighter groups*, a portion of whose Cayley graphs can be seen in Figure 1.

Incidentally, these examples can be chosen to be a group and one of its finite-index subgroups. Unfortunately there have been very few examples since then, and it is unclear where else to look for them. In my talk I will explore bi-Lipschitz equivalence not only of groups but also of other discrete metric spaces.

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DOI: http://dx.doi.org/10.1090/noti1644

Tullia Dymarz

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Tullia Dymarz works in geometric group theory with a focus on the large scale geometry of solvable groups.

June Huh

The Correlation Constant of a Field

Let G be a finite connected graph, let i, j be distinct edges, and let T be a random spanning tree of G. The probability that i is in T can only decrease by assuming that j is in T:

$$\Pr(i \in T) \geq \Pr(i \in T \mid j \in T).$$

In other words, the number b_{-} of spanning trees containing given edges satisfies

$$\frac{b_i}{b} \geq \frac{b_{ij}}{b_j}.$$

Now let E be a finite spanning subset of a vector space V, let i, j be distinct nonzero vectors in E, and write b_{-} for the number of bases in E containing given vectors. Do we still have

$$\frac{b_i}{b} \geq \frac{b_{ij}}{b_j}?$$

In 1974 Paul Seymour and Dominic Welsh found the first example of a vector configuration over a field of characteristic 2 with $\frac{b\,b_{ij}}{b_i b_j} = \frac{36}{35}$ for some i and j. How large can the ratio be?

Definition. The *correlation constant* of a field k is the supremum of $\frac{b\,b_{ij}}{b_i b_j}$ over all pairs of distinct nonzero vectors i and j in finite vector configurations in vector spaces over k.

This may be an interesting invariant of a field. While studying Hodge-Riemann relations for the intersection cohomology of certain projective varieties, Botong Wang and I noticed that

$$\frac{b\,b_{ij}}{b_i b_j} < 2$$

for any vector configuration. Thus the correlation constant of any field is at most 2. What is the correlation constant of, say, $\mathbb{Z}/2\mathbb{Z}$? Does the correlation constant really depend on the field? We discuss these and other questions.

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DOI: http://dx.doi.org/10.1090/noti1645

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June Huh works at the interface between combinatorics and algebraic geometry.

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Dymarz Figure 1 by Tullia Dymarz.

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