Open Problems Concerning Michell Trusses

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The Dona Ana Bridge in Sena, Mozambique.

We give a brief introduction to the problem of *Michell trusses*, a beautiful and challenging optimization problem related to the construction of bridges that was formulated by Michell [3] in 1904. Activity on the Michell Problem only began gaining momentum about fifty years later, and many interesting questions about it are still open. This problem first appeared in the engineering literature in a formulation that is accessible to any college student. It leads to deep and fascinating mathematical problems, and even the original relaxed problem is not yet fully resolved. It is a good illustration of the obstacles one must overcome when dealing with certain variational problems and of how duality can be a key to characterizing optima.

Our data consist of finitely many force vectors

$$\mathbf{F}_1, \dots, \mathbf{F}_k \in \mathbb{R}^d,$$

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where $d = 2, 3$, and their respective points of application

$$M_1, \dots, M_k \in \mathbb{R}^d.$$

We refer to

$$\mathbf{F} = \sum_{j=1}^{k} \mathbf{F}_j \delta_{M_j}$$

as a force, which is assumed to be of null average and to have zero torque:

$$\sum_{j=1}^{k} \mathbf{F}_j = 0, \qquad \sum_{j=1}^{k} \mathbf{F}_j \wedge M_j = 0. \tag{1}$$

Such a force must have at least two points of application ($k \geq 2$), and the simplest example is

$$\text{beam}(A, B) = (\delta_B - \delta_A)\frac{B - A}{|B - A|},$$

depicted in Figure 1, which represents a beam in tension. Alternatively, we have $-\text{beam}(A, B)$, which represents a beam in compression.

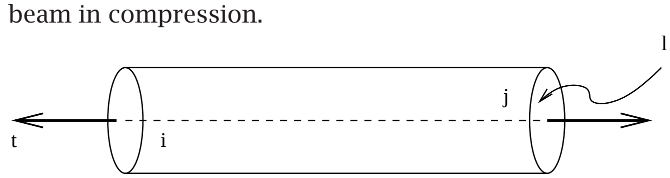

Figure 1. A beam in tension is represented by a force beam (A, B).

An elementary fact in the mechanical engineering literature is that any force $\mathbf{F}$ that satisfies (1) can be decomposed into a finite linear combination of beam(A_i, A_j): there exist $n \in \mathbb{N}$, large enough, $\{A_1, \dots, A_n\} \subset \mathbb{R}^d$, and $\{\lambda_{ij}\}_{i,j=1}^n \subset \mathbb{R}$ such that

$$\mathbf{F} = \sum_{i,j=1}^{n} \lambda_{ij} \text{beam}(A_i, A_j). \tag{2}$$

When this equation holds we say that the frame $[\lambda, \mathcal{A}] = [\{\lambda_{ij}\}, \{A_i\}]$ *withstands* $\mathbf{F}$.

In general the decomposition in (2) is far from unique, and so one seeks the most optimal decomposition. Michell himself proposed $[\lambda, \mathcal{A}]$ to be optimal if it minimizes the cost function

$$\text{Cost}[\lambda, \mathcal{A}] := \sum_{i,j=1}^{n} |\lambda_{ij}|\, |A_i - A_j|, \tag{3}$$

which represents the total volume of a frame, where $|A_j - A_i|$ is the length of the beam in the frame extending from A_i to A_j and having surface area $|\lambda_{ij}|$. By symmetrizing the problem one can (with no loss of generality) assume that the problem's matrix $(\lambda_{ij})_{ij}$ is symmetric. The set $\mathcal{U}$ of *displacements*, $u : \mathbb{R}^d \to \mathbb{R}^d$, is defined as

$$\mathcal{U} = \{u : \ |(u(x) - u(y)) \cdot (x - y)| \leq |x - y|^2\}. \tag{4}$$

The functional $J : \mathcal{U} \to \mathbb{R}$, which represents the total work done by the force $\mathbf{F}$ when the material undergoes displacement u, is defined by

$$J[u] := \int_{\mathbb{R}^d} \langle u; \mathbf{F}(dx) \rangle, \tag{5}$$

plays a crucial role. Indeed if $u \in \mathcal{U}$ and the frame $[\lambda, \mathcal{A}]$ withstands $\mathbf{F}$, then the work may be computed to be

$$J[u] = \sum_{i,j=1}^{n} \lambda_{ij}(u(A_j) - u(A_i)) \cdot \left(\frac{A_j - A_i}{|A_j - A_i|} \right).$$

So immediately we see that

$$J[u] \leq \sum_{i,j=1}^{n} |\lambda_{ij}| \frac{|A_j - A_i|^2}{|A_j - A_i|} = \text{Cost}[\lambda, \mathcal{A}]. \tag{6}$$

Thus, the work is bounded above by the cost,

$$\sup_{u \in \mathcal{U}} J[u] \leq \inf_{(\lambda, \mathcal{A}, n)} \left\{ \text{Cost}[\lambda, \mathcal{A}] \right\}, \tag{7}$$

where the infimum on the right is taken over frames $[\lambda, \mathcal{A}]$ that withstand $\mathbf{F}$ as in (2).

Michell [3] proposed that one should have the following duality principle:

$$\sup_{u \in \mathcal{U}} J[u] = \inf_{(\lambda, \mathcal{A}, n)} \left\{ \text{Cost}[\lambda, \mathcal{A}] \right\}. \tag{8}$$

Let us consider $d = 2$ and the force

$$\mathbf{F} = (\delta_{e_1} - 2\delta_0 + \delta_{-e_1}),$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let us consider a sequence of frames, $[\lambda^n, \mathcal{A}^n]$, as in Figure 2.

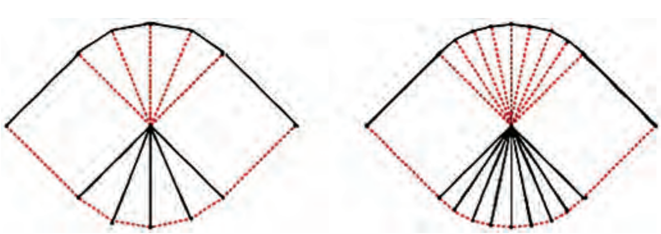

Figure 2. The frames $[\lambda^n, \mathcal{A}^n]$ when $n = 13$ and $n = 21$, where dotted red lines are in tension and thick black lines are in compression.

Bouchitté, Seppecher, and the author [1] identified a specific function $u_* \in \mathcal{U}$ and proved that

$$J[u_*] \geq \text{Cost}[\lambda^n, \mathcal{A}^n] - 0\left(\tfrac{1}{n}\right). \tag{9}$$

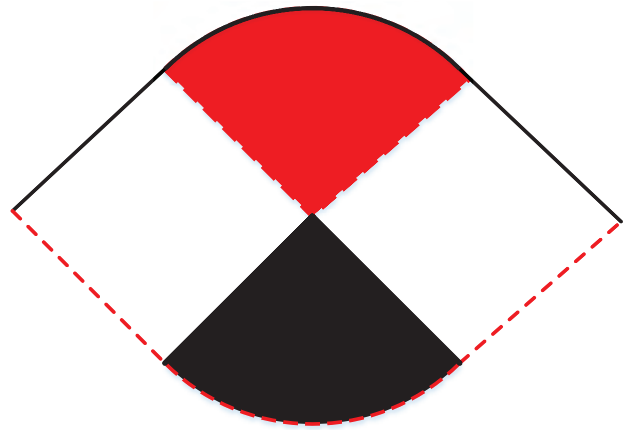

Figure 3. The limit measure σ^*.

By (7) and (9), not only is $([\lambda^n, \mathcal{A}^n])_n$ a minimizing sequence but (8) holds as well.

Therefore, if u_* is a maximizer in (8), then $[\{\lambda_{ij}\}, \{A_i\}]$ is a minimizer in (8) if and only if

$$\lambda_{ij}(u(A_j) - u(A_i)) \cdot (A_j - A_i) = |\lambda_{ij}||A_j - A_i|^2$$

for all $i, j \in \{1, \ldots, n\}$.

Observe that as n goes to ∞, the sequence $([\lambda^n, \mathcal{A}^n])_n$ intuitively "converges" to the measure σ^*, depicted in Figure 3, which clearly fails to belong to the set of frames made from finitely many beams. To understand this convergence more rigorously, we view each term in the sequence of frames as a measure:

$$\sigma := \sum_{i,j=1}^{n} \frac{\lambda_{ij}}{|A_j - A_i|^2} (A_j - A_i) \otimes (A_j - A_i) \mathcal{H}^1_{|[A_i, A-j]},$$

where $\mathcal{H}^1_{|[A_i, A_j]}$ denotes the Hausdorff measure restricted to the segment $[A_i, A_j]$. We can then take the weak limit of the measures to obtain σ^*. It is readily checked that the equilibrium equation (2) can be written in the class of measures as

$$-\operatorname{div}(\sigma) = \mathbf{F} \tag{10}$$

in the sense of distributions.

Thus in order to take into account all possible structures that may appear in the limit, we are forced to search for minimizers in the bigger set, Σ, of *strain tensors*:

$$\Sigma = \{\sigma : \ -\operatorname{div}(\sigma) = \mathbf{F}\}.$$

A strain tensor, σ, is a symmetric matrix whose entries are Radon measures satisfying (10) as distributions. Written in terms of strain tensors,

$$\text{Cost}[\lambda, \mathcal{A}] = C[\sigma] := \int_{\mathbb{R}^d} \rho^0(\sigma),$$

where ρ^0 is the one-homogeneous function that associates to a square symmetric matrix the sum of the absolute values of its eigenvalues. Since the infimum in (8) is taken over the set of all natural numbers $n \in \mathbb{N}$, it can be shown to have its infimum achieved and in fact

$$\sup_{u \in \mathcal{U}} J[u] = \inf_{\sigma \in \Sigma} C[\sigma]. \tag{11}$$

This right-hand side is what is referred to as a *relaxation* of the minimization problem at the right-hand side of (8).

In order to keep track of the stream lines of principal actions in the strain tensors, our proposal in [1] was to look for σ that can be represented by signed Radon measures

$$\gamma = \gamma^+ - \gamma^-$$

defined on the set of curves. The positive part γ^+ of the signed measure corresponds to lines in tension, while the negative part γ^- corresponds to the lines in compression. We shall use the notation

$$|\gamma| := \gamma^+ + \gamma^-.$$

These stream lines form the *Hencky-Prandtl net* when $d = 2$. See Figure 4.

As described in [1], the Hencky-Prandtl net is *a family of orthogonal curves which represent the limits of the families of bars through the optimization process.*

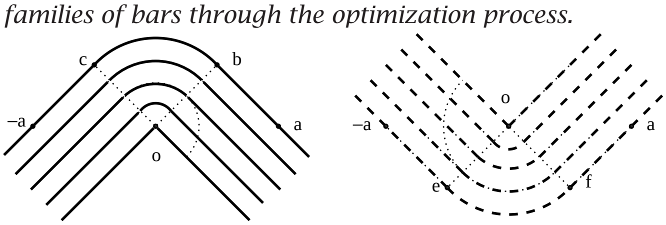

Figure 4. Here we have depicted the Hencky-Prandtl net for $\sigma*$ of Figure 3, with γ^+ depicted on the left and γ^- on the right.

Let $\mathbf{X}$ be the set of $C^{1,1}$ curves of finite length and for $C \in \mathbf{X}$ let t_C denote a unit tangent to C. Any Radon measure γ on $\mathbf{X}$ induces a symmetric matrix $\sigma[\gamma] \in \Sigma$ defined as

$$\sigma[\gamma] = \int_{\mathbf{X}} \left(\int_C \langle \xi(x); t_C \otimes t_C \rangle d\mathcal{H}^1 \right) \gamma(dC).$$

We [1] show that the minimization problem in (11) is equivalent to finding the infimum

$$\text{(12)} \qquad \inf \left\{ \int_{\mathbf{X}} \mathcal{H}^1(C) |\gamma|(dC) \; : \; -\text{div}(\sigma[\gamma]) = \mathbf{F} \right\}.$$

When (12) has a minimizer, γ_*, then the strain tensor, $\sigma[\gamma_*]$, is called a *Michell truss.*

Thanks to Korn's inequality, one sees that (11) admits a maximizer, u_*, which is almost everywhere differentiable. Given C in $\mathbf{X}$, let κ denote the curvature along C and let s be the arc–length parametrization. The tangential component of u_*, denoted as $u_\tau := u_* \cdot t_C$, is Lipschitz along C, whereas the orthogonal component denoted as u_ν is continuous. We show that a necessary and sufficient condition for γ_* to be a minimizer in (12) is

$$\frac{du_\tau}{ds} - \kappa u_\nu = \pm 1 \quad \mathcal{H}^1 \text{ a.e. and } \gamma^\pm \text{ a.e.}$$

Open Problems

I. Does (12) admit a minimizer γ_*? When is $\sigma[\gamma_*]$ uniquely determined?

II. Is there a radius, $r > 0$, such that any curve, C, in the support of γ_* is contained in the ball $B_r(0)$? Are minimizers of $\inf_\Sigma C$ supported by $B_r(0)$?

III. Let u_* be a maximizer of the functional $J[u]$ over the collection of displacements $u \in \mathcal{U}$, as in (4)–(5). Can one identify the set where

$$2E(u_*) := \nabla u_* + \nabla^T u_*$$

is not continuous or not differentiable?

The Dona Ana Bridge in Africa has forty spans and is one of the longest bridges in the world.

We close with a discussion of the first of these open problems in dimension $d = 2$. Note that $u_* \in \mathcal{U}$ if and only if the eigenvalues e_1 and e_2 of the symmetric matrix $E(u_*)$ have their ranges in the set $[-1, 1]$. Let a_1 and a_2 be the eigenvectors of $E(u_*)$, associated to the eigenvalues e_1 and e_2 respectively, so that

$$E(u_*) = e_1 a^1 \otimes a^1 + e_2 a^2 \otimes a^2.$$

Since $J[u]$ is a linear function of u, formally at least its maximizer u_* is an extreme point of the convex set $\mathcal{U}$. One is tempted to assume that

$$|e_1| = |e_2| \equiv 1;$$

however this remains an open question in general.

As a symmetric matrix, σ_* is also diagonalizable, and its eigenvalues λ_1 and λ_2 are signed Radon measures. The duality identity (8) not only forces σ_* to have the same eigenvectors as $E(u_*)$ but also implies that

$$e_1 \lambda_1 \geq 0 \quad \text{and} \quad e_2 \lambda_2 \geq 0.$$

Therefore, (λ_1, λ_2) solves the system

$$\text{(13)} \qquad -\text{div}\left(\lambda_1 a^1 \otimes a^1 + \lambda_2 a^2 \otimes a^2\right) = \mathbf{F}.$$

This is a system of hyperbolic equations in (λ_1, λ_2) whose characteristics would be two families of orthogonal curves if we could prove these characteristics exist. These curves are the missing pieces to build a measure γ_* such that $\sigma_* = \sigma[\gamma_*]$.

It is suspected that if $e_1 e_2 \leq 0$ everywhere, then there is at most one pair (λ_1, λ_2) that satisfies (13) on $\mathbb{R}^2$. This would solve the question of uniqueness in dimension two.

Those who are interested in learning more about these open problems should see the work of the author with Bouchitté and Seppecher [1], as well as his work with

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Dacorogna [2] and references cited within these two papers.

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Wilfrid Gangbo's work in the calculus of variations and partial differential equations is funded by the National Science Foundation. He is dedicated to the promotion of mathematics in the Third World, especially in Africa, his continent of origin.

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