

WHAT IS...

Hypoellipticity?

The Editors

The two top classical partial differential equations (PDEs) $\mathcal{P}u = f$ are Laplace's equation

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which describes, for example, a steady-state temperature distribution $u(x, y)$, and the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0,$$

which describes the small height u of a vibrating string as a function of x and t . These are both linear, not involving powers or, worse, nonlinear functions of u or its derivatives. In other words, the operators are of the form

$$\mathcal{P} = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha.$$

Here ∂^α denotes a general mixed partial derivative depending on the vector α : for example,

$$\partial^{(1,2)} u(x, y) = \frac{\partial^3 u}{\partial x \partial y^2}.$$

For Laplace's equation the degree m (highest order derivative) is 2, the relevant α are $(2, 0)$ and $(0, 2)$, and the associated coefficients a_α are the constants 1 and 1. For the wave equation, again $m = 2$, the relevant α are $(2, 0)$ and $(0, 2)$, and the associated coefficients are 1 and -1 .

Although the equations look similar, their solutions have vastly different smoothness properties. Solutions to Laplace's equation, called harmonic functions, are smooth (infinitely differentiable), while general solutions to the wave equation, including $u(x, t) = g(x - t)$, an arbitrary wave $g(x)$ traveling to the right, need not be smooth (or even continuous once you define what it means for such a function to be a solution). The reason is that the coefficients for the second derivatives are positive

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in Laplace's equation and of mixed sign for the wave equation. For a second-order partial differential operator with highest order terms

$$A \frac{\partial^2}{\partial x^2} + 2B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial y^2},$$

for smoothness it is enough to assume that the matrix

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

is positive definite, which in the case when $B = 0$ reduces to A and C both positive. Since this is the same as the condition for

$$Ax^2 + 2Bxy + Cy^2 = 1$$

to be an ellipse, such PDEs are called elliptic. Ellipticity is sufficient but not necessary for smoothness. For example, solutions to the heat equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

are smooth, even though the equation is not elliptic, because the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is not positive definite. There is no known nice condition that is both necessary and sufficient for smoothness. If all solutions are smooth, the PDE is called *hypoelliptic*.

Definition. A partial differential operator \mathcal{P} is *hypoelliptic* if on every open subset: if $\mathcal{P}u$ is smooth, then u is smooth. (For nonsmooth u , $\mathcal{P}u$ is understood in the distributional sense.)

Example 1. On \mathbb{R} , $\mathcal{P} = d/dx$ is hypoelliptic: if the derivative is smooth, the primitive is smooth.

Example 2. On \mathbb{R}^2 , $\mathcal{P} = \partial/\partial x$ is not hypoelliptic. Indeed, $\mathcal{P}u = 0$ for any function of y alone.

THE GRADUATE STUDENT SECTION

Example 3 (Hörmander). On \mathbb{R}^2 ,

$$\mathcal{P} = \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2}$$

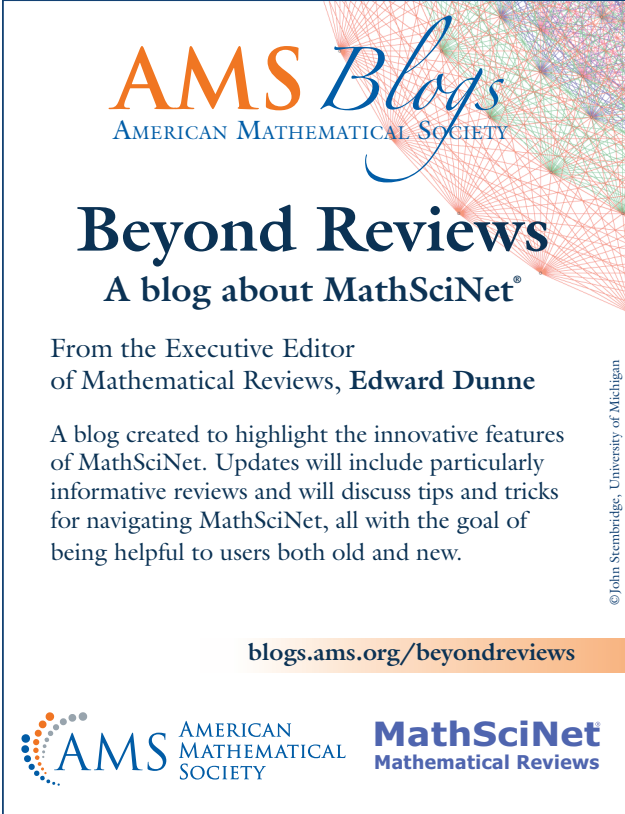
turns out to be hypoelliptic even though it is not elliptic at 0, because the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is not positive definite.

Successively weaker versions of ellipticity that imply hypoellipticity have been discovered under such names as subellipticity and maximal hypoellipticity. The search for a more precise understanding of what is necessary as well as what is sufficient for hypoellipticity goes on.

For more see “WHAT ELSE about...Hypoellipticity?” on p. 424.



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