WHAT IS…

a Hyperbolic 3-Manifold?

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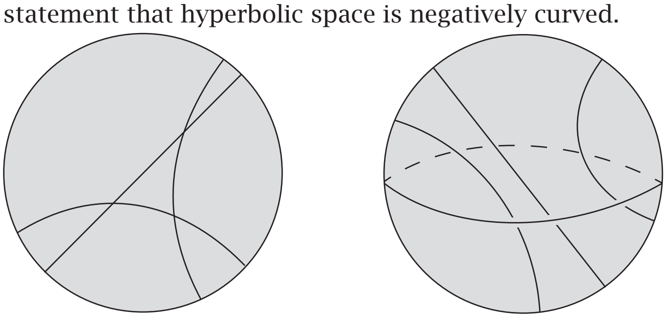
Communicated by Cesar E. Silva

The simplest example of a hyperbolic manifold is hyperbolic geometry itself, which we describe using the Poincaré disk model. In Figure 1 we see the 2-dimensional version $\mathbb{H}^2$ and the 3-dimensional version $\mathbb{H}^3$, each the interior of a unit 2-disk or 3-disk. In both cases, geodesics are diameters or segments of circles perpendicular to the missing boundary. Notice that for any triangle with geodesic edges, the sum of the angles adds up to less than 180 degrees. This choice of geodesics can be used to determine a corresponding metric, which turns out to have constant sectional curvature -1, justifying the statement that hyperbolic space is negatively curved.



(A) (B)

Figure 1. The Poincaré disk models of hyperbolic 2-space $\mathbb{H}^2$ and hyperbolic 3-space $\mathbb{H}^3$ with geodesics that are diameters and segments of circles perpendicular to the missing boundary.

We say that a surface (a 2-manifold) is hyperbolic if it also has a metric of constant sectional curvature -1. We can use this as a definition of a hyperbolic surface, but there are two other helpful ways to think about a hyperbolic surface.

When a surface S has such a metric, we can show that the universal cover of the surface is $\mathbb{H}^2$ and there is a discrete group of fixed-point free isometries Γ of $\mathbb{H}^2$ that act as the covering transformations such that the quotient of $\mathbb{H}^2$ by the action of Γ is the surface.

By choosing a fundamental domain for the group of isometries Γ, we can also think of S as being obtained from a polygon in $\mathbb{H}^2$ with its edges appropriately glued together in pairs by isometries as in Figure 2. In particular, at each point in S, there is a neighborhood isometric to a neighborhood in $\mathbb{H}^2$. So locally, our surface appears the same as $\mathbb{H}^2$.

Among all topological surfaces, how prevalent are the hyperbolic surfaces? Considering compact orientable surfaces without boundary, only the sphere and the torus are not hyperbolic. All other orientable surfaces are hyperbolic as in Figure 3. If we throw in nonorientable surfaces, only the projective plane and the Klein bottle are not hyperbolic. And if we allow punctures, the only additional surfaces that are not hyperbolic are the once-

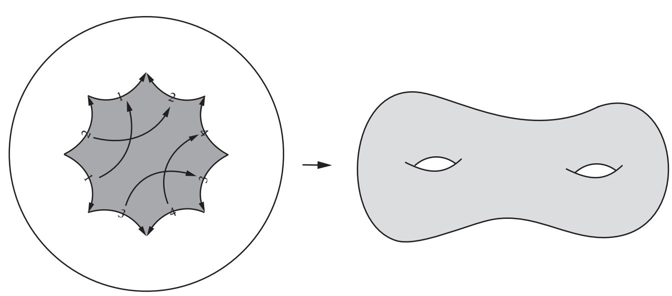


Figure 2. Gluing together pairs of edges of a hyperbolic fundamental domain yields the genus two surface.

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DOI: http://dx.doi.org/10.1090/noti1677

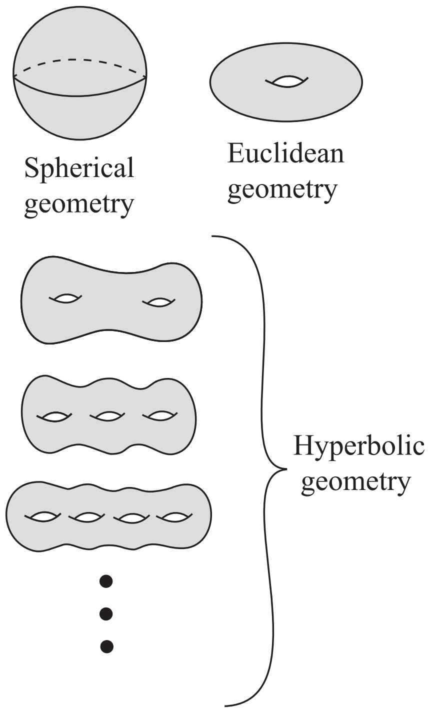

Figure 3. A list of closed orientable surfaces and their respective geometries.

and twice-punctured sphere and the once-punctured projective plane. So among the infinitude of closed surfaces and closed surfaces with arbitrarily many punctures, all but seven are hyperbolic. So if we want to understand the geometries of surfaces, it's all about the hyperbolic case.

A 3-manifold is a topological space M that is locally 3-dimensional. That is to say, every point has a neighborhood in the space that is homeomorphic to a 3-dimensional ball. For instance, the 3-dimensional spatial universe in which we all live is such a 3-manifold. Another example would be to take 3-space (or the 3-dimensional sphere if we want to begin with a compact space) and remove a knot. Then it is still true that this is a 3-manifold, as every point still has a ball about it that is 3-dimensional. We just have to pick the ball small enough to avoid the missing knot.

In the 1970s and 1980s, work of William Thurston (1946–2012) and others led to the realization that many 3-manifolds are hyperbolic. Here again, to be hyperbolic just means that there is a metric of constant sectional curvature -1 or, equivalently, that there is a discrete group of fixed-point free isometries Γ acting on $\mathbb{H}^3$ such that the quotient of the action is M.

A famous example is the figure-eight knot complement. Here, the fundamental domain for the action of the discrete group of isometries is a pair of ideal regular hyperbolic tetrahedra (all angles between faces are $\pi/3$), as in Figure 4.

An ideal hyperbolic tetrahedron is one with geodesic edges and faces such that it is missing its vertices as they sit on the missing boundary of $\mathbb{H}^3$. The sum of the volumes of this pair of ideal regular tetrahedra is $2.0298\ldots$, a number of interest to number theorists as well as topologists, since it is also related to the value of the Dedekind zeta function at 2. (See for instance Zagier's *Inventiones* article from 1986.) This volume was proved to be the smallest hyperbolic volume of any knot by Cao and Meyerhoff in an *Inventiones* article from 2001.

Why is it useful for a 3-manifold to be hyperbolic? One extraordinary advantage is the Mostow–Prasad Rigidty Theorem, which says that if you have a finite volume hyperbolic 3-manifold, its hyperbolic structure is completely rigid. All such structures on a given 3-manifold are isometric. In particular, every such 3-manifold has a unique volume associated with it. We have turned floppy topology into rigid geometry.

Compare that to the Euclidean case. We could take a cube and glue opposite faces straight across. This yields the 3-dimensional torus. But we can make it out of a small cube or a big cube, so there is no unique volume associated to it. We could even trade in the cube for a parallelepiped, and we would still have a valid Euclidean structure on the 3-torus.

On the other hand, the figure-eight knot has a hyperbolic complement with volume $2.0298\ldots$. So we now have an incredibly effective invariant for distinguishing between 3-manifolds. This was an essential tool used in the classification of the 1,701,936 prime knots through 16 crossings by Hoste, Thistlethwaite, and Weeks in 1998.

In the case of knots, volume is not enough to completely distinguish them for two reasons. First, there are nonhyperbolic knots. Thurston showed that knots fall into three categories: they can be torus knots, satellite knots, or hyperbolic knots. A torus knot is a knot that

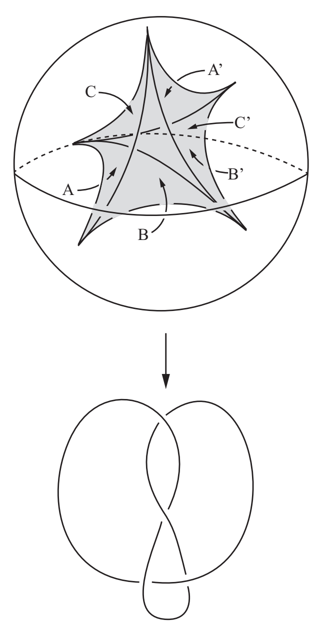

Figure 4. A fundamental domain for the figure-eight knot complement constructed from two ideal regular tetrahedra in $\mathbb{H}^3$.

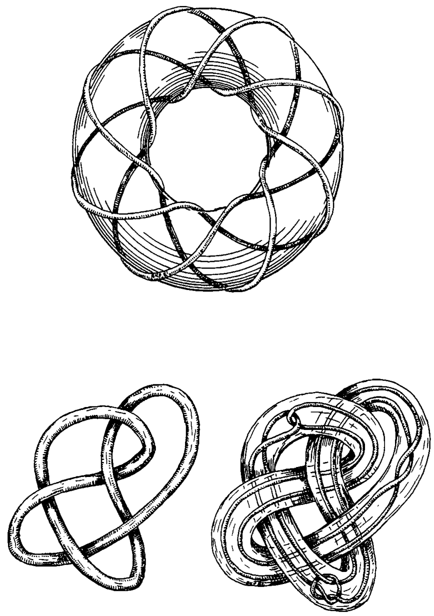

Figure 5. A torus knot at top and a satellite knot, bottom right. Every other knot must be hyperbolic.

lives on the surface of an unknotted torus, as in Figure 5, and is determined by how many times it wraps the long and short way around the torus.

A satellite knot is what you might guess, a knot K that orbits another knot K' in the sense that it exists in a neighborhood of K', which is to say a solid torus with K' as core curve. It is a truly marvelous fact that after excluding just these two categories of knots, all other knots are hyperbolic.

Second, although rare for low crossing number, there can be two different knots with the same volume. For instance, the second hyperbolic knot 5_2 has the same volume as the 12-crossing $(-2, 3, 7)$-pretzel knot as in Figure 6.

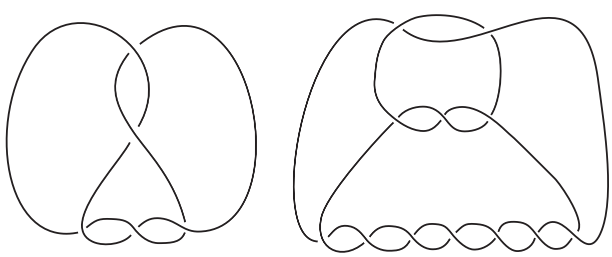

Figure 6. There do exist examples of hyperbolic knots with the same volume, such as the pair pictured here.

What we would like is a complete classification of all closed 3-manifolds. This means we would like a way to "list" them all and to decide, given any two, whether or not they are homeomorphic.

In 1982 William Thurston proposed the Geometrization Conjecture. It says that every closed 3-manifold can be cut open along an essential set of tori and spheres into pieces, and after capping off the spheres with balls, each of the components would be 3-manifolds with one of eight specified geometries, one of which is $\mathbb{H}^3$. In 2003 Grigory Perelman revolutionized low-dimensional topology by proving the Geometrization Conjecture. (He also proved the Poincaré Conjecture in the process, which was a necessary piece in the proof of the larger Geometrization Conjecture.)

So we would like to determine the manifolds with each of the eight geometries. In fact, the manifolds associated to the seven other geometries have been classified and are well understood. There only remains the manifolds that are hyperbolic. Why have we not succeeded in classifying those? The situation is analogous to what happened with surfaces. This is the richest of the geometries, with the preponderance of the manifolds. It is the mother lode.

Thurston also proposed the Virtual Haken Conjecture, implicit in the work of Waldhausen, that every closed 3-manifold satisfying mild conditions (having infinite fundamental group and no essential spheres) either contains an embedded essential surface or possesses a finite cover that does so, thereby allowing the decomposition along the surface into simpler pieces. The proof of the Geometrization Conjecture allowed for a proof of the Virtual Haken Conjecture for all 3-manifolds except hyperbolic 3-manifolds, which is not such a surprise, since again, this is where the action is. It was this case that Ian Agol completed in 2012, thereby settling this fundamental conjecture. Agol received the three million dollar Breakout Prize in mathematics for this and related work.

Research continues forward as we attempt to understand hyperbolic 3-manifolds, their volumes, and other related invariants. This geometric approach to low-dimensional topology has become fundamental to our understanding of 3-manifolds and will continue to play a critical role for years to come.

Additional Reading

Volumes of Hyperbolic Link Complements, Ian Agol, https://www.ias.edu/ideas/2016/agol-hyperbolic-link-complements.

Image Credits

Figures 1–4 and Figure 6 courtesy of Colin Adams.

Figure 5 from *A Topological Picturebook*, courtesy of George Francis.

Photo of Colin Adams courtesy of Alexa Adams.

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ABOUT THE AUTHOR

Colin Adams researches knots, hyperbolic 3-manifolds, and the connections between the two. When not doing math, he likes to write about math, including his comic book, *Why Knot?*, and his novel, *Zombies & Calculus*.