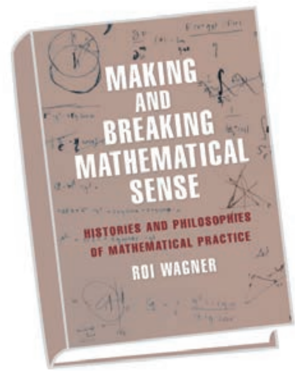
BOOK REVIEW

A Welcome Addition to the Philosophy of Mathematical Practice



by Brendan Larvor

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Making and Breaking Mathematical Sense: Histories and Philosophies of Mathematical Practice
By Roi Wagner
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In his celebrated *Proofs and Refutations*, Lakatos has one of his characters observe that "heuristic is concerned with language-dynamics, while logic is concerned with language-statics." The thought that as mathematics develops, its terms have their meanings stretched, distorted, and replaced, often unbeknownst to the mathematicians using them, is a main theme of Lakatos's essay. In spite of this, few philosophers in the field have picked up and elaborated Lakatos's suggestion that using mathematical terms to prove or disprove theorems changes the meanings of those terms. Most have been content with language-statics—at least until now.

Wagner's book is, among other things, a study in language-dynamics. In fact, Lakatos is not one of Wagner's principal sources, and he is mentioned only twice. Wagner's philosophical hinterland is nineteenth-century German idealism and the post-structural continental philosophy of Deleuze and Derrida.

Mention of post-structuralism may sound alarms in some readers, but one of the delightful features of this book is that it uses post-structuralist ideas without reproducing the jargon or prose-style associated with those traditions. In the first chapter, Wagner tells four different histories of the philosophy of mathematics in the twentieth century. In the second chapter he explains how terms in abbaco mathematical texts had a kind of fluidity about them and could stand for all sorts of things apart from their most obvious referents. Throughout the book, he uses accessible examples to show the dynamism and ambiguity of mathematical terms. The leading post-structuralist and post-modernist Derrida (see Figure 1) appears once, in a pair of quotations about how every meaningful remark can be quoted and can thereby lose its original context. Wagner makes this Derridean moment easy for us in two ways. First, he offers us a path to Derrida through Peirce, and second, he presents a simple example from matrix algebra to show how the same symbol can stand for either an object or an operation and on some occasions seems to stand for both. The tactical re-reading of a matrix

uses post-structuralist ideas without reproducing the jargon or prose-style

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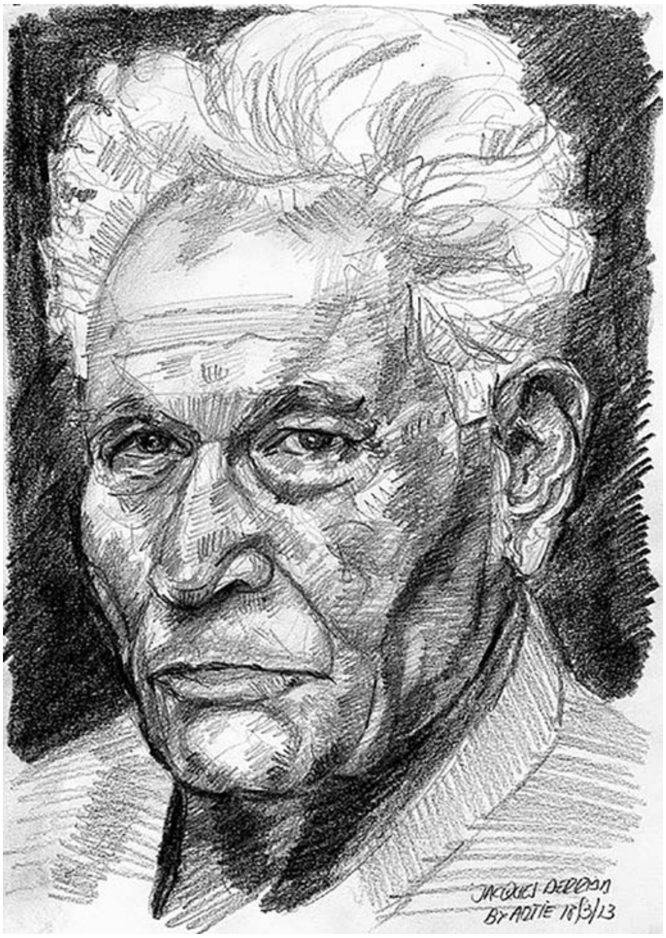

Figure 1. The leading post-structuralist and post-modernist Derrida claimed that every meaningful remark can be quoted and can thereby lose its original context.

as standing for an operation rather than for an object does not seem to be quite what either Peirce or Derrida had in mind, but this hardly matters. The story that Wagner tells about matrix algebra makes sense in its own terms, and it is interesting to see it juxtaposed with a little of the reading through which Wagner learned to think this way.

Similarly, the leading post-modernist Deleuze does not appear in person until a short section in Chapter 5, where he supplies some language and ideas for talking about the use of diagrams in mathematics from the point of view of the theory of embodied cognition. Here as elsewhere, Wagner offers readers the option of skipping the continental bits. In any case, the use of Deleuze is rather indirect—Wagner takes up a book that Deleuze wrote about the painter Francis Bacon called *The Logic of Sensation*. He picks out some of Deleuze's remarks about Bacon's working methods, translates them into a mathematical idiom, and then illustrates them with examples from Euclid. The core idea is quite simple: one of the important features of mathematical representations is that they are open to rule-governed manipulation and revision: algebraic terms can be gathered or multiplied out, lines can be added to Euclidean diagrams, axes can be changed, and so on. When these manual operations have become habitual, the body anticipates them without having to carry them out.

Overall, Derrida has four entries in the index, and Deleuze has five. Kant has twice as many, but the most often mentioned philosopher in this book is Wittgenstein. This is the Wittgenstein of the *Remarks on the Foundations of Mathematics*, who suggests that inconsistencies in mathematics need not be such a big deal so long as there is some practical way of making sure that no one uses an inconsistency to prove anything false. This is the Wittgenstein who notices that the same arithmetical statement might sometimes seem to function as an empirical statement and at other times seem more like a rule of grammar. Wagner attempts to make these two thoughts more plausible than they seem in Wittgenstein by attending more closely to real mathematical practice than Wittgenstein ever did. He cleverly includes quotations from Wittgenstein's discussions with Turing, which remind us that Turing took Wittgenstein seriously even when he was saying apparently crazy things. Turing's sceptical contributions also remind us that Wittgenstein's work was unfinished, that his observations are waiting to be explained by a larger, more perspicacious view of mathematical practice.

Semiosis (the process by which a symbol is connected to its meaning) is one large part of Wagner's argument. The other central idea is his proposal for a "constraints-based" philosophy of mathematics. Mathematics operates under various tensions: between empirical application and free creation, between unification and diversification, between isolation from and integration with other disciplines. The various options and dilemmas in mainstream philosophy of mathematics (Platonism, formalism, nominalism, structuralism, logicism, etc.) arise from trying to resolve these dilemmas once and for all by establishing a common foundation for all mathematics. This cannot work, according to Wagner, because mathematics is too diverse and above all because mathematical practice is too fluid, too semiotically slippery for any such foundational effort to do it justice. The point of a foundation is to fix things still, but living mathematics is mobile. The task of the philosopher is not to resolve the dilemmas but rather to understand the constraints that make mathematics what it is.

proposal for a "constraints-based" philosophy of mathematics

This makes it sound as if Wagner deliberately leaves all philosophical questions about mathematics unresolved. In fact, he does something surprising for an avowed post-structuralist. He says something so definite about the nature of mathematics that he comes close to positing an essence for it.

One of the characteristic features of mathematics is that mathematicians mostly agree on its results. There are disputes, but they almost always get resolved much more quickly than in other sciences, and once resolved are almost never re-opened. Why is this? Here, Wagner says something rather conventional:

> While the vast majority of mathematical everyday disputes are resolved by some sort of semi-formal shorthand,… this alone is not what allows mathematics to be much more consensual than other sciences. The consensus among mathematicians about the validity of proofs has a lot to do with formalization. By formalization I do not mean the translation of an entire proof into a strictly formal language… By formalization I mean a gradual process of piecemeal approximation of formality that is conducted only as far as required to resolve a given dispute. (p. 67)

This works, he explains, because mathematical arguments can be split up into sub-arguments, the disputed parts of which can be formalised. The formalised inferences are of a sort that can be checked by following rules that reliably give the same results regardless of which competent mathematician applies them. Partial formalisation of this sort is, according to Wagner, the final arbitrator of mathematical disputes. It is (as he observes) almost never carried out, but is nevertheless Wagner's explanation for the high level of consensus in mathematics. Indeed, it is one of two criteria that he offers for some intellectual practice to count as mathematics nowadays:

> Allowing myself to oversimplify, I might say that *the contemporary necessary and sufficient conditions of being mathematical is precisely the combination of dismotivation with respect to empirical application and potential formalization serving as highest arbitrator in disputes over the validity of arguments.* (p. 71)

When he identifies dismotivation as a defining characteristic of mathematics, what Wagner means is, roughly, that mathematicians are (as mathematicians) not terribly interested in the potential applications of their work, even if they are doing applied mathematics. More precisely, dismotivation is the process whereby a piece of mathematics floats free of the empirical enquiry that originally formed it. A partial differential equation developed to describe the flow of heat might serve to model something in economics, but the mathematician will not be interested unless this new connection brings with it new mathematics. Once the mathematics used to model some part of reality has developed to the point where it is mathematically interesting, mathematicians will work on it simply as mathematics and feel progressively less vulnerable to embarrassment if the model fails empirically. Whole numbers are useful for counting the sorts of things that can be counted, but number theorists are supremely uninterested in specifying those sorts of things.

Identifying the possibility of formalisation as his explanation of the highly consensual nature of mathematics puts Wagner in a surprisingly conservative position in the philosophy of mathematical practice. Recent work by others has started from the argument that since even partial formalisation almost never happens, something else must account for the consensus. To point to formalisation as the cause of consensus in those cases (the great majority) where it does not happen seems like magical thinking. One way Wagner might respond to this challenge is to broaden his notion of formalisation to include any procedure that can be broken down into rule-governed steps that give the same answer regardless of who carries them out. Before every phone had a calculator on it, people would settle calculation disputes by working slowly through the algorithms they learned in primary school. The rules in Euclid's plane geometry are sufficiently precisely specified that disputes could be settled in the same way, by breaking the argument into elementary steps and carefully checking that the process of proof was properly carried out. In these and similar cases, skilled practitioners can anticipate how the analysis into elementary steps would go (as Wagner notes). Perhaps what explains consensus in mathematics is not formalisation in the contemporary sense, but rather that there should be some procedures available to practitioners of mathematics that break down into elementary steps, that give consistent and reliable results and that are recognised as authoritative. Perhaps the formalised inferences of contemporary research mathematics have these features pre-eminently but not uniquely.

Be that as it may, this is the picture of mathematics that Wagner sets out in the first four chapters: mathematics as a collection of semiotically fluid practices that respond to external and internal constraints (commercial and military needs, capacities and limits of the human brain, inscription technologies, ideological and theoretical demands arising from mathematics itself and nearby disciplines, etc.) marked by dismotivation and high levels of consensus. Change and growth are explained by language-dynamics, as theorised by Derrida, while consensus and stability are explained by language-statics, made possible by formalisation.

The fifth and sixth chapters broach a new topic: the cognitive basis of mathematics. The chief business of Chapter 5 is to review the state of knowledge regarding the neural basis of elementary mathematics and to criticise the well-known theory of Lakoff and Núñez. Wagner pursues this theory in some detail, but the gist of his critique has three points. Lakoff and Núñez claim that mathematics arises in human thinking thanks to some basic metaphors. Wagner argues that this single model of a cognitive metaphor results in a very impoverished picture of the relations between various domains of mathematics. It explains how inferences can be mapped from one domain to another, but not how any other sort of transfer might take place. Second, because they have just one kind of relation (namely, cognitive metaphor), they end up constructing a rigidly hierarchical picture of mathematics that is structurally similar to the formal models of mathematics developed by foundationalist programmes in the philosophy of mathematics. No one who has absorbed the lessons of Wagner's first four chapters could find that satisfactory. Finally, Wagner suggests that the cognitive metaphor, as defined by Lakoff and Núñez, looks rather like a partial isomorphism as understood in modern, abstract mathematics. The explanation they offer for the human capacity for mathematical thinking therefore seems to run the risk of circularity.

The final chapter is interesting for philosophers curious about the history of their discipline and the genesis of Wagner's ideas. He leads us through a discussion of German idealism as found in Fichte, Schelling, and Hermann Cohen and into a brief meditation on the applicability of mathematics. The guiding thread here is the thought that the (allegedly) unreasonable applicability of mathematics arises from the fact that our world is increasingly formatted mathematically. Wagner follows this line as far as he can, but confesses in the end that it does not lead all the way to conviction.

This book is a welcome addition to the philosophy of mathematical practice, partly for the richness of its historical and contemporary mathematical examples, which this review has underplayed in favour of the more philosophical content. It is also valuable because it deploys rich stocks of philosophical theory in a sophisticated manner. This is needed in the philosophy of mathematical practice, much of which currently tends towards a kind of Baconian (the philosopher, not the painter) hope that case studies and borrowings from related empirical disciplines will spontaneously assemble themselves into a coherent account of mathematics. In Wagner's book, we find an impressive exhibition of the way in which philosophical theory can mediate between impossibly general questions (What is mathematics? How do proofs prove?) and detailed studies (How is algebra logically related to geometry in the third book by Rafael Bombelli?). It is not, of course, the last word on any of these subjects. No work of philosophy ever is.

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Photo of Brendan Larvor courtesy of University of Hertfordshire.

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ABOUT THE REVIEWER

Brendan Larvor specialises in the history and philosophy of mathematics.