BOOK REVIEW

Reverse Mathematics

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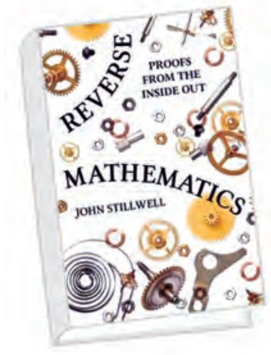

Reverse Mathematics: Proofs from the Inside Out
By John Stillwell
Princeton University Press
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There are several monographs on aspects of reverse mathematics, but none can be described as a "general audience" text. Simpson's *Subsystems of Second Order Arithmetic* [3], rightly regarded as a classic, makes substantial assumptions about the reader's background in mathematical logic. Hirschfeldt's *Slicing the Truth* [2] is more accessible but also makes assumptions beyond an upper-level undergraduate background and focuses more specifically on combinatorics. The field has been due for a general treatment accessible to undergraduates and to mathematicians in other areas looking for an easily comprehensible introduction to the field.

With *Reverse Mathematics: Proofs from the Inside Out* [5], John Stillwell provides exactly that kind of introduction. The book is aimed at upper-level undergraduates and professional mathematicians who are interested in the details of arithmetization and in seeing several examples of the methods used in reverse mathematics but who do not have previous knowledge of mathematical logic or computability theory.

When Is an Axiom *Necessary*?

Sometimes, when we prove a mathematical theorem, every step in the proof seems to be somehow required by the theorem at hand. Other times, we look at a particular step with skepticism. Have we used a sledgehammer to drive a nail, applying a very strong theorem to a problem that could be solved with simpler means? Are the techniques used in the proof genuinely *necessary* to obtain the theorem at hand? This question is a central motivation of the field of reverse mathematics in mathematical logic.

Mathematicians have long investigated the problem of the Parallel Postulate in geometry: which theorems require it, and which can be proved without it? Analogous questions arose about the Axiom of Choice: which theorems genuinely *require* the Axiom of Choice for their proofs?

In each of these cases, it is easy to see the importance of the background theory. After all, what use is it to prove a theorem "without the Axiom of Choice" if the proof uses some other axiom that already implies the Axiom of Choice? To address the question of necessity, we must begin by specifying a precise set of background axioms—our *base theory*. This allows us to answer the question of whether an additional axiom is necessary for a particular proof, as follows.

> *When a theorem is proved from the right axioms, the axioms can be proved from the theorem.*

Suppose we find that a theorem T is provable from our base theory together with an additional axiom A that is not provable in the base theory. To show that A is necessary, we can try to prove a *reversal*: we assume T as if it were an axiom, together with our base theory, and prove A as if it were a theorem. If we can do this, we have shown that A and T are equivalent, relative to the base theory. Moreover, any other axiom A' which allows us to prove T over our base theory would also allow us to prove A. The reversal shows that A is, in a precise sense, the *weakest* axiom that, together with the base theory, allows us to prove T.

Many results of this kind have been obtained. For example, the Parallel Postulate is equivalent to Playfair's axiom in absolute geometry, and in topology the Axiom

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Figure 1. Harvey Friedman proposed the program of reverse mathematics at the 1974 International Congress of Mathematicians.

of Choice is equivalent to Tychonoff's theorem over Zermelo–Fraenkel set theory.

Reverse mathematics studies the strength of everyday mathematical theorems in this way. At the 1974 International Congress of Mathematicians Harvey Friedman [1] (see Figure 1) laid out the founding vision for this program: "When a theorem is proved from the right axioms, the axioms can be proved from the theorem." The key to this analysis is to choose a base system strong enough to formalize the theorems we want to study but not so strong that it proves those theorems outright. Friedman proposed using specific base theories from *second-order arithmetic* instead of geometry or set theory. Equally importantly, he proposed looking at fundamental theorems of mathematics—results such as the Bolzano–Weierstrass theorem in calculus—rather than more esoteric theorems in set theory or topology.

Stephen Simpson [3] (Figure 2) rephrased the main question as: "Which set existence axioms are required to prove the theorems of everyday, non-set-theoretic mathematics?" In many cases, the required axioms turn out to be very modest.

Second-Order Arithmetic

Second-order arithmetic is a family of formal systems for studying the natural numbers, real numbers, and many other basic mathematical objects. It takes as given only two fundamental types of objects: "numbers," which are intended to represent natural numbers, and "sets," which are intended to represent sets of natural numbers. Work of Weyl, Hilbert and Bernays, and Feferman showed that much of elementary real analysis can be studied in second-order arithmetic via *arithmetization*, in which more complicated objects are represented—*coded*—as numbers or sets of numbers.

Which axioms are required to prove the theorems of everyday mathematics?

For example, an integer can be coded as a pair of natural numbers with the correct difference, and a rational number can be coded as a pair of integers with the correct ratio. A real number can be coded as a Cauchy sequence of rational numbers, perhaps with a fixed rate of convergence. By using more complex coding systems, we can also represent $\mathbb{R}^n$ for each n, open subsets of $\mathbb{R}$, and continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. Algebraic structures such as countable groups and fields and countable vector spaces over countable fields can also be coded into sets of natural numbers. In combinatorics, countable graphs and countable partitions of countable sets can be formalized directly into second-order arithmetic.

Not everything can be coded this way. Second-order arithmetic is not able to talk about arbitrary subsets of $\mathbb{R}$ nor about objects of very high cardinality. In this way, it is more suitable for "ordinary" theorems that talk about countable algebraic objects or complete separable metric spaces.

There are several motivations for using second-order arithmetic. It is a concrete and relatively weak foundational system, and the provability of ordinary mathematical theorems in this setting shows that stronger systems such as set theory or topos theory are not required for

Figure 2. Stephen Simpson has proved many key results in reverse mathematics. His monograph *Subsystems of Second Order Arithmetic* [3] is the standard graduate-level reference text.

these theorems. Another motivation is the close relationship between second-order arithmetic and computability theory. This relationship is one of the keys to the success of reverse mathematics: by working in second-order arithmetic, we can use a powerful toolbox of methods from computability theory to study theorems that, at first glance, seem unrelated to computation.

Subsystems of Second-Order Arithmetic

Subsystems of second-order arithmetic are simply axiom systems for second-order arithmetic which can have varying levels of strength. Following tradition, subsystems are often named with short acronyms, many of which have subscripts or superscripts indicating particular variations. The weakest subsystem usually encountered is known as RCA_0. It has axioms saying that $\mathbb{N}$ is a discrete ordered semiring and a set of relatively weak induction axioms. RCA_0 also has set existence axioms which say, essentially, that if we have sets $B_1, \ldots, B_k$ and a set A is Turing computable from these sets, then the set A must exist. The acronym "RCA" stands for "recursive comprehension axiom," where "recursive" is used as a synonym of "computable."

A model M of RCA_0 consists of a set of numbers $\mathbb{N}^M$, which may or may not be the ordinary natural numbers, and a collection of subsets of $\mathbb{N}^M$. Crucially, we do not require that all subsets of $\mathbb{N}^M$ must be included. Instead, we rely on the set existence axioms to know that particular sets will be included in the model, that is, to know they will "exist." Allowing the model to contain only some subsets of $\mathbb{N}^M$ also avoids the standard proof that all models of Peano's axioms are isomorphic, because that proof requires quantifying over all subsets.

The standard model of second-order arithmetic consists of the ordinary natural numbers and every subset of the natural numbers. Another important model of RCA_0 has the ordinary natural numbers but only includes the Turing computable subsets of the naturals. We say that this model, called REC, "believes that every set is computable." If a theorem is provable in RCA_0, then the theorem is true in REC. Thus, in particular, a theorem provable in RCA_0 cannot imply the existence of uncomputable sets of natural numbers. We often view RCA_0 as a formalization of computable analysis.

The remaining subsystems usually encountered in reverse mathematics (see Figure 3) consist of RCA_0 together with additional axioms. One such subsystem, ACA_0, consists of RCA_0 with one additional axiom: "if $f : \mathbb{N} \to \mathbb{N}$ is a function, the range of f must exist." This seemingly innocuous statement is not provable in RCA_0, as it is not true in REC. In terms of computability, the new axiom is equivalent to saying that for each set $A \subseteq \mathbb{N}$ we may form the Turing jump A'. (The Turing jump of a set is the range of a particular function computable from the set.) Thus, in a model of ACA_0, the collection of sets is closed under Turing jump. We often view ACA_0 as a formalization of Weyl's predicative analysis. The acronym "ACA" stands for "arithmetical comprehension axiom."

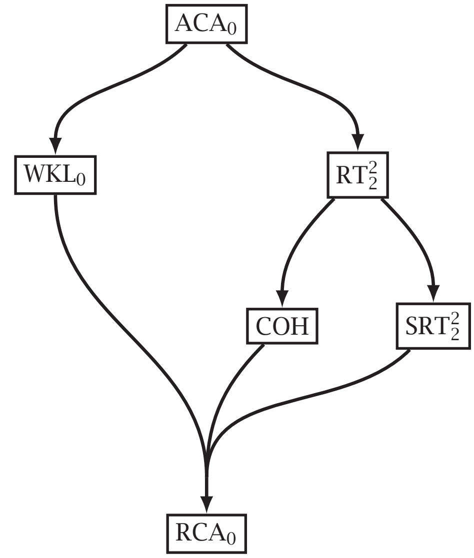

Figure 3. Relationships between six subsystems. WKL_0 and RT^2_2 form an incomparable pair below ACA_0. RT^2_2 itself can be split into two strictly weaker systems, COH and SRT^2_2, which together are equivalent to RT^2_2.

There are many subsystems between RCA_0 and ACA_0. One of these, WKL_0, is named after a weak form of König's lemma and is related to several theorems of analysis and countable algebra. A second, RT^2_2, is a fragment of the infinite version of Ramsey's theorem. These systems are incomparable: neither proves the axioms of the other over RCA_0. Moreover, RT^2_2 can be *split* into a "cohesive set" principle COH and a "stable version" SRT^2_2. Each of these is strictly weaker than RT^2_2, but taken together their axioms are equivalent to the axiom of RT^2_2 over RCA_0. Figure 3 shows the relationship between the subsystems just mentioned.

The measure of strength in reverse mathematics gives a complex, nonlinear hierarchy. However, many ordinary theorems of mathematics turn out to be equivalent, over RCA_0, to the axioms of one of five linearly ordered subsystems (the reason for this phenomenon is not completely clear). Simpson named these subsystems the "Big Five" (see Figure 4).

The weakest three of the Big Five are RCA_0, WKL_0, and ACA_0. These three are sufficient for almost all theorems of a standard undergraduate mathematics curriculum that can be stated in second-order arithmetic. Beyond them are two stronger systems, ATR_0 and Π^1_1-CA_0. The Big Five subsystems are shown in Figure 4 along with a mathematical theorem representative of each.

Stillwell's *Reverse Mathematics*

Stillwell focuses on RCA_0, WKL_0, ACA_0, and on a few carefully chosen mathematical results from introductory real analysis. He carefully limits the amount of logic and computability included in the main portion of the text.

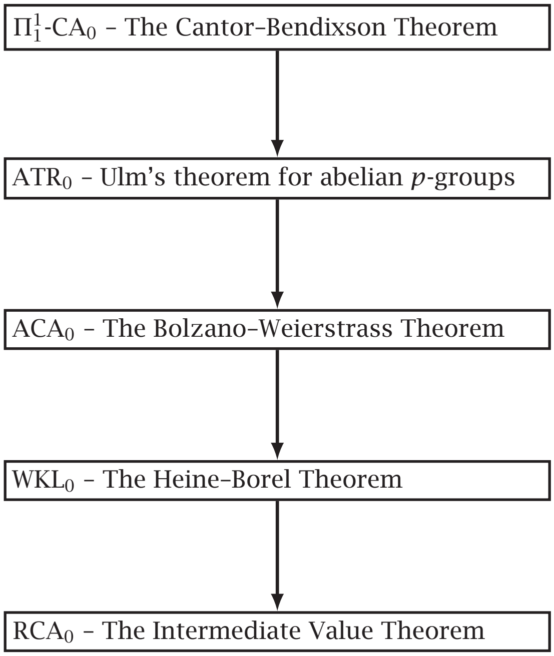

Figure 4. The "Big Five" subsystems of second-order arithmetic and a representative theorem equivalent to each.

The later portion includes some comments on the logical details, but still at a level intended for a general reader. This necessarily means that some of the beautiful interactions between computability, second-order arithmetic, and mathematical theorems are less clearly visible, but it is very much in line with the *mathematical* character of reverse mathematics.

A key aspect of reverse mathematics, which can sometimes be obscured by the logical methods employed, is an underlying love of the mathematics being studied. Many researchers in reverse mathematics gravitate towards well-known, basic theorems of mathematics (the Bolzano–Weierstrass theorem, Ramsey's theorem, Hilbert's basis theorem, etc.) in order to understand these theorems better. The mathematics itself provides a central motivation for the logical analysis being performed.

Accordingly, work in reverse mathematics requires a detailed understanding of both the logical tools and the mathematics being studied. More advanced treatments may make substantial assumptions about the reader's background in both of these areas. Stillwell presents a careful introduction to a portion of elementary real analysis, its arithmetization, and its provability in ACA_0 without assuming the reader is already fluent in the details.

The standard proof of a result can sometimes be formalized directly into a desired subsystem of second-order arithmetic. In other cases, a new proof must be discovered if the previous proof used methods that were stronger than necessary. In either case, the reverse mathematics analysis of a theorem leaves us with more information than the mere fact that the theorem is provable. Stillwell's treatment of several theorems of elementary real analysis demonstrates this in a way unobscured by the underlying logical machinery.

Stillwell shows how ACA_0 is equivalent over RCA_0 to the Bolzano–Weierstrass theorem ("every bounded sequence of reals has a convergent subsequence"). He also proves that WKL_0 is equivalent over RCA_0 to the Heine–Borel theorem in the form "if a sequence of rational intervals covers $[0,1]$, then some finite subsequence is also a cover." Along the way, he clearly demonstrates how these results are related to particular statements about infinite paths through trees, which is what allows computability methods to be applied to the theorems.

In Zermelo–Fraenkel set theory, the Bolzano–Weierstrass theorem and the Heine–Borel theorem are both provable, and so they are trivially equivalent. By looking at them in a weaker base system such as RCA_0, we see that there is, in a precise sense, a difference between open cover compactness and sequential compactness on the real line. This kind of *separation result* illustrates the additional information that can be obtained through reverse mathematics. We understand the two theorems better by seeing an intrinsic way in which they differ.

Because the author avoids most of the logical machinery, the book will not prepare readers to jump directly into research. The introductory treatment also omits subsystems stronger than ACA_0, which are required to prove theorems such as the Cantor–Bendixson theorem. These higher subsystems are closely related to generalized computability theory in much the same way that the weaker systems are related to Turing computability. So, if it does its job, the book will leave a reader asking for more.

What could be the next step after reading this book? The first four chapters of Simpson [3] give a thorough survey of arithmetization and the reverse mathematics of mathematical theorems in RCA_0, WKL_0, and ACA_0. Hirschfeldt [2] provides an introduction to many recent results in the reverse mathematics of combinatorics. Solomon [4] gives an introductory, but more technical, summary of several reverse mathematics results in algebra. Beyond these is a large and growing research literature.

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ABOUT THE AUTHOR

Carl Mummert's research is in mathematical logic (particularly reverse mathematics, second-order arithmetic, and computability theory), topology, and combinatorics. His teaching incorporates techniques from Inquiry Based Learning (IBL).