Linear Hydrodynamic Stability

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Introduction

Hydrodynamic instability and turbulence are ubiquitous in nature, such as the air flow around an airplane of Figure 1. Turbulence was also the subject of paintings by Leonardo da Vinci and Vincent van Gogh (Figure 2). Turbulence is the last unsolved problem of classical physics. Mathematically, turbulence is governed by the Navier–Stokes equations, for which even the basic question of well-posedness is not completely solved. Indeed, the global well-posedness of the 3D Navier–Stokes equations is one of the seven Clay millennium prize problems [1]. Hydrodynamic stability goes back to such luminaries as Helmholtz, Kelvin, Rayleigh, O. Reynolds, A. Sommerfeld, Heisenberg, and G. I. Taylor. Many treatises have been written on the subject [5].

Linear hydrodynamic stability theory applies the simple idea of linear approximation of a function through its tangent to function spaces. Hydrodynamicists have been trying to lay down a rigorous mathematical foundation for linear hydrodynamic stability theory ever since its beginning [5]. Recent results on the solution operators of Euler and Navier–Stokes equations [2] [3] [4] imply that the simple idea of tangent line approximation does not apply because the derivative does not exist in the inviscid case, thus the linear Euler equations fail to provide a linear approximation to inviscid hydrodynamic stability. Even for the linearized Navier–Stokes equations, instabilities are often dominated by faster than exponential growth, especially when the viscosity is small.

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Figure 1. Hydrodynamic instability and turbulence are ubiquitous in nature, such as the turbulent vortex air flow around the tip of an airplane wing.

Linear Approximation in the Simplest Setting

Let us start with a real-valued function of one variable

$$y = f(x).$$

If the function is differentiable, then

$$\Delta y = f'(x)\Delta x + o(|\Delta x|).$$

Thus in a small neighborhood of $\Delta x = 0$, the tangent line

$$dy = f'(x)dx$$

offers a linear approximation. Sometimes there is a nice change of variables

$$\Delta\eta = \Delta y + g(\Delta y),$$

Figure 2. Vincent van Gogh's painting "Starry Night" exhibits a turbulent fluid pattern.

where $g(0) = g'(0) = 0$ such that

$$\Delta\eta = f'(x)\Delta x$$

in a small neighborhood of $\Delta x = 0$. This is called linearization. The same idea can be applied to differential equations. Let us consider a simple first order differential equation,

$$\frac{dx}{dt} = f(x).$$

Denote by $S^t(x_0)$ the solution operator

$$x(t) = S^t(x_0),$$

where $x(0) = x_0$. The solution operator can be viewed as a family of maps parametrized by t. If $S^t(x_0)$ is differentiable in x_0, then

$$\Delta x(t) = \frac{d}{dx_0}S^t(x_0)\Delta x_0 + o(|\Delta x_0|).$$

In a small neighborhood of $\Delta x_0 = 0$, the tangent line

$$dx(t) = \frac{d}{dx_0}S^t(x_0)dx_0$$

offers a linear approximation, and $dx(t)$ satisfies the linear equation

$$\frac{d}{dt}(dx) = f'(x)dx,$$

whereas Δx satisfies

$$\frac{d}{dt}(\Delta x) = f'(x)\Delta x + o(|\Delta x|).$$

Sometimes there is a nice change of variables

$$\Delta\eta = \Delta x + g(\Delta x)$$

where $g(0) = g'(0) = 0$ such that

$$\frac{d}{dt}(\Delta\eta) = f'(x)\Delta\eta$$

in a small neighborhood of $\Delta\eta = 0$. This is called linearization of the differential equation.

Mathematical Foundation of Inviscid Linear Hydrodynamic Stability Theory

We start with the basic incompressible, inviscid (no viscosity) Euler equations for the velocity $u(t,x)$ of a fluid:

$$\partial_t u + u\cdot\nabla u = -\nabla p, \ \nabla\cdot u = 0$$

under various boundary conditions, where the spatial dimension is either 2 or 3. Here p is the pressure. In order to establish linear hydrodynamic stability theory, first of all the (nonlinear) Euler equations must be well posed. The usual well-posedness requires that there is a unique solution which depends on the initial condition continuously. Linear hydrodynamic stability theory requires a stronger well-posedness: the solution needs to depend on the initial condition *differentiably*. The well-posedness of the Euler equations depends critically on the function space in which the Euler equations are posed. First of all, the function space consists of functions that satisfy the boundary condition. Second, a norm is defined on the function space so that every function with a finite norm belongs to the function space. The most natural function spaces for the well-posedness of Euler equations are the Sobolev spaces H^n. A fluid velocity function $u(t,x_1,x_2)$ belonging to H^n must have a finite Sobolev norm which is the square root of the integral over the spatial domain of the sum of squares of n^{th} order partial derivatives of velocity components u_j:

(0.1)

$$\|u\|_n^2 = \int \sum_{\ell+m\leq n,\ j=1,2}\left[\left(\frac{\partial}{\partial x_1}\right)^{\ell}\left(\frac{\partial}{\partial x_2}\right)^{m} u_j\right]^2 dx_1dx_2,$$

and similarly for spatial dimension 3. Under the boundary conditions of either decaying at infinity or spatial periodicity, the Euler equations are locally well posed (for a short time) in the Sobolev spaces H^n when $n > 1 + d/2$, where d is the spatial dimension. In two spatial dimensions, the local well-posedness can be extended to global (all time) well-posedness. There are also well-posedness results for other boundary conditions, and boundary conditions usually do not pose any substantial issue for well-posedness.

Once we know well-posedness, we can define a solution map S^t (a flow, see Figure 3) in the function space:

$$S^t(u(0)) = u(t), \ \text{ for any } u(0)\in H^n,$$

where $u(t)$ is the solution starting at the initial condition $u(0)$. The well-posedness of the Euler equations in H^n implies that $S^t(u(0))$ is continuous in $u(0)$. To study the stability of any solution $u(t)$, one needs to introduce a perturbation in its initial condition

$$u(0) + \Delta u(0),$$

which generates a new solution to the Euler equations

$$\hat{u}(t) = S^t(u(0) + \Delta u(0)).$$

The stability of the solution $u(t)$ is studied via investigating the growth or decay of the difference

$$\begin{aligned}\Delta u(t) &= \hat{u}(t) - u(t)\\ &= S^t(u(0)+\Delta u(0)) - S^t(u(0)).\end{aligned}$$

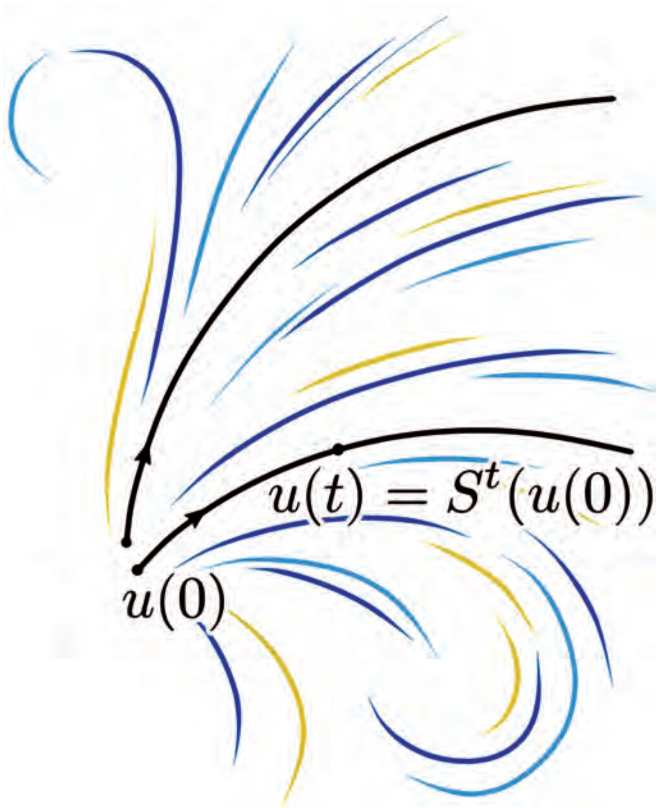


Figure 3. A solution of the Euler equation may be viewed as an orbit in the function space, and the solution map S^t is the flow map.

The linear hydrodynamic stability theory must be based on the linear approximation to $\Delta u(t)$ as a function of $\Delta u(0)$, which requires that $S^t(u(0))$ is differentiable in $u(0)$. If $S^t(u(0))$ were differentiable in $u(0)$, then we would have

$$\Delta u(t) = \left[\nabla_{u(0)} S^t(u(0))\right] \Delta u(0) + o\left(\|\Delta u(0)\|_n\right), \tag{0.2}$$

where $\nabla_{u(0)}$ represents the derivative in $u(0)$ and $\| \ \|_n$ is the H^n norm (0.1). In a small neighborhood of $\Delta u(0) = 0$, we would have the linear approximation

$$du(t) = \left[\nabla_{u(0)} S^t(u(0))\right] du(0). \tag{0.3}$$

Indeed, $du(t)$ would satisfy the linearized Euler equations

$$\partial_t du(t) + du \cdot \nabla u + u \cdot \nabla du = -\nabla dp, \ \nabla \cdot du = 0. \tag{0.4}$$

Unfortunately, H. Inci ([2], 2015) proved that $S^t(u(0))$ is nowhere differentiable in either 2 or 3 spatial dimensions under decaying boundary condition at infinity. That is, at any initial condition in H^n, the solution operator $S^t(u(0))$ is not differentiable in $u(0)$. Recently, Inci and the author proved that differentiability fails for other boundary conditions as well. The derivative (0.2) never exists, the differential (0.3) never exists, and the linearized Euler equations (0.4) always fail to provide a linear approximation for the inviscid hydrodynamic stability. In conclusion, *the inviscid linear hydrodynamic stability theory always fails to have a rigorous mathematical foundation!*

There are different ways for the derivative (0.2) to fail to exist. The most common way is that the norm of the derivative $\nabla_{u(0)} S^t(u(0))$ is infinite. In such a case, the amplification from $\Delta u(0)$ to $\Delta u(t)$ can be expected to be much faster than the exponential growth associated with any unstable eigenvalue of the linearized Euler equations (0.4). This is the phenomenon that we call "rough dependence upon initial data" [4]. Such superfast growth can reach substantial magnitude in very short time during which the exponential growth is very small as in Figure 6 (see p. 1258). Thus the classical inviscid instability due to unstable eigenvalues does not capture the true inviscid instability of superfast growth.

Rayleigh Equation as an Equation for a Directional Differential

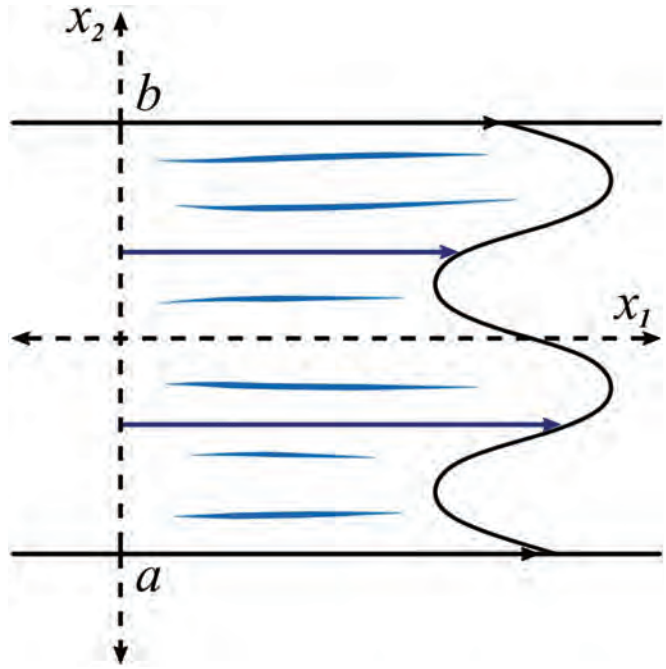


Figure 4. Simple channel flows for which the approximating Rayleigh equation cannot capture the dominant instability of superfast growth.

Now we focus on a 2D channel flow as in Figure 4, where x_1 is the streamwise direction and x_2 is the transverse direction with the boundaries at $x_2 = a, b$. The classical interest is to study the linear stability of the steady state

$$u_1 = U(x_2), \ u_2 = 0,$$

which satisfies the boundary condition

$$u_2 = 0 \ \text{ at } x_2 = a, b.$$

The solution operator is not differentiable at the steady state, so the linearized Euler equations at the steady state cannot provide a linear approximation. Nevertheless, the classical inviscid linear hydrodynamic stability theory assumed that the solution operator was differentiable. Formally starting from the linearized Euler equations, Rayleigh derived the so-called Rayleigh equation as follows. Let

(0.5) $$du = (\partial_{x_2}\psi, -\partial_{x_1}\psi),\ \psi = \phi(x_2)e^{ik(x_1-\sigma t)}.$$

Then $\phi(x_2)$ satisfies the Rayleigh equation

(0.6) $$(U-\sigma)(\phi'' - k^2\phi) - U''\phi = 0,$$

with the boundary condition

$$\phi(a) = \phi(b) = 0.$$

Setting $t = 0$ in (0.5), we have

(0.7) $$du(0) = (\partial_{x_2}\psi(0), -\partial_{x_1}\psi(0)),\ \psi(0) = \phi(x_2)e^{ikx_1}.$$

One can view the $du(0)$ in (0.7) as a single Fourier mode in x_1 out of the whole Fourier series in x_1. Indeed, (0.5) represents a directional differential with the direction of $du(0)$ specified by k and $\phi(x_2)$. Even though the full differential $du(t)$ does not exist, the directional differential (0.5) can still exist once the Rayleigh equation (0.6) produces an eigenfunction. Thus the directional differential (0.5) generated from the Rayleigh equation (0.6) cannot capture the nature of the full differential (0.3), which does not exist. The most common way for the nonexistence of (0.3) is that the norm of the derivative $\nabla_{u(0)}S^t(u(0))$ is infinite. This will result in superfast growth of certain perturbations $\Delta u(0)$, which is much faster than any exponential growth predicted by Rayleigh equation. Such superfast growth can reach substantial magnitude in very short time. The superfast growing perturbations are not the directional perturbation (0.5) of Rayleigh. Instability is usually dominated by the fastest growing perturbation. Thus the Rayleigh equation cannot capture the dominant inviscid instability of superfast growth, which can reach substantial magnitude in very short time—short term unpredictability.

High Reynolds Number Linear Hydrodynamic Stability Theory

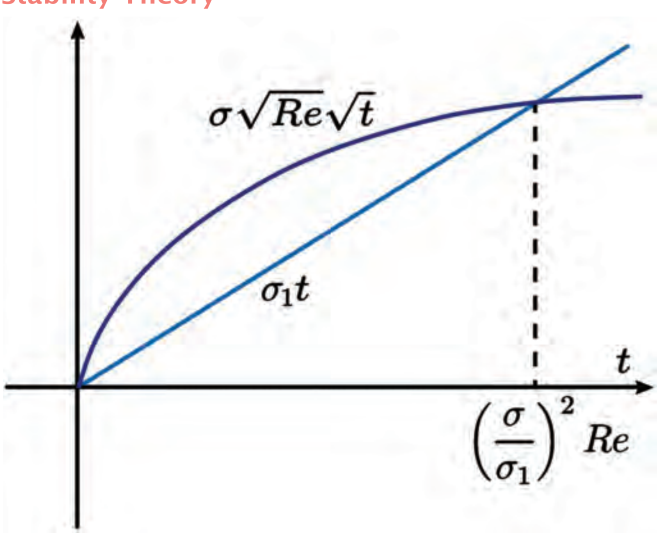
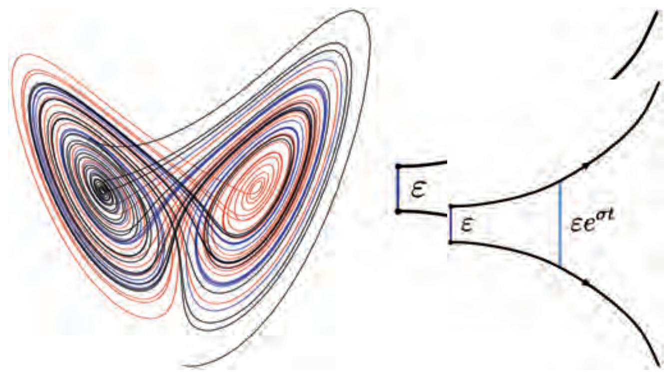


Figure 5. The upper-bound bound (0.11) on the gradient of the flow has two terms. For small t, the one involving the Reynolds number Re is larger.

(a) Ocean Turbulence



(b) Lorenz Attractor

Figure 6. (a). Low viscosity yields rough dependence on initial conditions, as in ocean turbulence. (b). Higher viscosity yields smoother but still sensitive dependence on initial conditions, as in the famous Lorenz-Attractor chaos.

We now consider the incompressible Navier–Stokes equations (with viscosity)

$$\partial_t u + u\cdot\nabla u = -\nabla p + \frac{1}{Re}\nabla^2 u,\ \nabla\cdot u = 0$$

under various boundary conditions, where the spatial dimension d is either 2 or 3. Here Re is the Reynolds number, large when the viscosity is small. Under the boundary conditions of either decaying at infinity or spatial periodicity, like Euler equations Navier–Stokes equations are locally well posed in the Sobolev space H^n when $n > 1 + d/2$. In two spatial dimensions, the local well-posedness can be extended to global well-posedness. As for the Euler equations, given an initial condition $u(0)$ in H^n, we consider the Naiver-Stokes solution $u(t) = S^t(u(0))$. Unlike for the Euler equations, the solution operator $S^t(u(0))$ for the Navier–Stokes equations is not only continuous but also everywhere differentiable in $u(0)$. Then we have

(0.8) $$\Delta u(t) = \left[\nabla_{u(0)}S^t(u(0))\right]\Delta u(0) + o\left(\|\Delta u(0)\|_n\right).$$

In a small neighborhood of $\Delta u(0) = 0$, we have the linear approximation

(0.9) $$du(t) = \left[\nabla_{u(0)}S^t(u(0))\right]du(0).$$

As a result, $du(t)$ satisfies the linearized Navier–Stokes equations

(0.10)
$$\partial_t du(t)+du\cdot\nabla u+u\cdot\nabla du = -\nabla dp+\frac{1}{Re}\nabla^2 du,\ \nabla\cdot du = 0.$$

The norm of the derivative $\nabla_{u(0)}S^t(u(0))$ as a map which maps $du(0)$ to $du(t)$, is defined as

$$\|\nabla_{u(0)}S^t(u(0))\| = \sup_{du(0)}\frac{\|du(t)\|_n}{\|du(0)\|_n}.$$

Under either decaying at infinity or periodic boundary condition in both two and three spatial dimensions, the norm of the derivative $\nabla_{u(0)}S^t(u(0))$ is bounded as follows [3]

$$\|\nabla_{u(0)}S^t(u(0))\| \le e^{\sigma\sqrt{Re}\sqrt{t}+\sigma_1 t},\ t\in[0,T], \tag{0.11}$$

where

$$\sigma_1 = \frac{\sqrt{2e}}{2}\sigma,\ \sigma = \frac{8}{\sqrt{2e}}\sup_{\tau\in[0,T]}\|u(\tau)\|_n,$$

and $[0,T]$ is a time interval on which well-posedness holds. The exponent of the bound has two parts: the first part depends on the square root of the Reynolds number Re and t, while the second part is independent of Re as in Figure 5. As Re approaches infinity, the bound also approaches infinity in agreement with the intuition that the norm of the derivative approaches its inviscid counterpart, which is infinite. The peculiar feature of the first part is the square root in both t and Re. At $t=0$, the time derivative of the first part is infinite. During the time interval $t\in(0,\frac{2}{e}Re)$, the first part remains greater than the second part. When the Reynolds number Re is large (and the viscosity is small), this time interval is very large. Thus, when the Reynolds number Re is large, the first part corresponds to superfast growth. We also call such superfast growth as "rough dependence upon initial data" or "short term unpredictability" as in the turbulent waves of Figure 6(a). The second part corresponds to the exponential growth with unstable eigenvalues, the sensitive dependence of chaos as in Figure 6(b). Examples show that as the Reynolds number approaches infinity, the unstable eigenvalues approach the corresponding inviscid unstable eigenvalues.

Numerical simulations on the linearized Navier–Stokes equations (0.10) under periodic boundary condition have verified the $e^{c\sqrt{t}}$ nature in (0.11) for the amplification of abundant perturbations (0.9) along abundant base solutions. Our conclusion is that high Reynolds number turbulence appears from the outset as superfast amplifications of perturbations.

Conclusion

We discover that the linearized Euler equations with their exponential growth of perturbations due to unstable eigenvalues fail to provide a good approximation, because they miss the actual super-exponential growth. Even though the linearized Navier–Stokes equations can provide a decent approximation for high viscosity, they still miss the initially dominant super-exponential growth of perturbations.

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ABOUT THE AUTHOR

Y. Charles Li's research focuses on chaos in partial differential equations and theoretical aspects of fluid turbulence. He also has interest in nanotechnology, mathematical biology, and complex systems.

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