Progressions in Reasoning in K–12 Mathematics

Dev Sinha

"I think that it is impossible for some of our students to learn to do proofs," explained a colleague of mine. My belated response is that all students can indeed learn to do proofs. College faculty have been asking aspiring math majors to make a huge jump, from not being responsible for providing reasoning in entry-level college courses and most of their K–12 experience to making formal proofs. To ease this transition, our department at the University of Oregon has recently created "lab" courses for first-year students in addition to our "bridge" requirement, to help students by degrees gain experience with proof-based mathematics.

More broadly, we should afford all students, at all levels, practice with gradually more demanding reasoning, something which educators would call engaging in progressions in reasoning [CCSS18]. Teachers, mathematics educators, and mathematicians have been working together to develop and study such learning progressions for reasoning[1] in K–12 mathematics (see for example [SBK09, Knu02, KCJS02]). In this article I share perspective as a research mathematician who has developed and implemented reasoning-focused tasks for K–12 students of all grades, for both aspiring and current teachers, and for many types of undergraduates.

Historically, explicit calls for student reasoning had been all but absent in school curricula until Euclidean geometry. But mathematical argument does not need to wait until the onset of puberty. Indeed, children naturally ask "why questions," and based on research both the National Council of Teachers of Mathematics Principles and Standards for School Mathematics [NCTM00] and Common Core State Standards in Mathematics, in particular its third Practice Standard [CCSS10], call for more reasoning and proof throughout school mathematics.

Such calls are being answered. For example using pictures such as those given in Figure 1, many current curricula provide opportunities to reason about the sum of two odd numbers in the second or third grade.

While such arguments through pictures lack the formal trappings of proof, engaging with such arguments is valuable experience. Moreover, visual learning of arithmetic has a clear basis in the research literature [PB13, Ans16]. One could place this activity in a coherent learning progression across K–12 by revisiting arithmetic of even and odd numbers through other arguments, either based on place value and case analysis or grounded in algebra. Such a learning sequence could advance in high school with activities such as showing that the sum of two squares cannot be one less than a multiple of four.

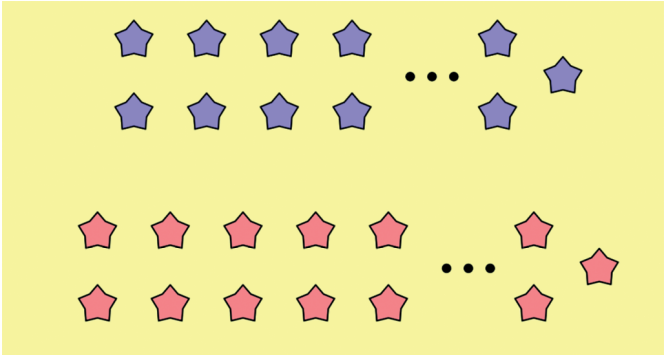

Figure 1. Pictures which can support opportunities for second and third graders to reason about the sum of two odd numbers.

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Communicated by Notices *Associate Editor William McCallum.*

[1] *We use the terms "reasoning" and "proof" interchangeably. There is a continuum of formalism in reasoning and proof, and it is pedagogically useful for students to engage at different levels, even within the same classroom activity. We prefer to emphasize this range rather than making only a binary distinction based on some hard-to-define cutoff for rigor and formalism.*

DOI: http://dx.doi.org/10.1090/noti1871

Reasoning about even and odd numbers has been used fruitfully in a number of settings. Colleagues at the University of Oregon and I use this material as an introduction to different levels of formalism for undergraduates who aspire to be elementary school teachers [BHS]. Patrick Callahan, a mathematician who led the California Math Project, often has schools evaluate student reasoning by asking students at different grade levels to explain why the sum of odd numbers is even. Callahan reports that high schoolers generally fare no better than grade schoolers, and when presented with the argument through variables, they (including advanced students) commonly report that they did not realize it was "allowed" for variables to be used in that way. Deborah Ball and her colleagues at the University of Michigan have used student-driven discussion about the definition of even numbers as a strong component of teacher training [Bas05, BB03].

The Common Core State Standards for Mathematics were designed through progressions [CCSS18]. While the Common Core can be read as policy or as informing pedagogy, to a knowledgeable reader they also suggest proofs for all of K–12 mathematics, short of concepts that require limits, in particular working rigorously with functions over all real numbers. The commutative property of multiplication, for example, should be established through noticing that a rectangular array and its transpose are in bijective correspondence. Strong curricula engage students in such proofs in age-appropriate ways.

The Common Core also asks that the canon of elementary mathematics be taught consistently with how mathematicians practice mathematics. The multiplication table is not just a set of facts but also a rich locale for conjecture and proof. Students who type $1/7$ or $\sqrt{5}$ into a calculator and get an "answer" can be asked what that answer means in terms of inequalities, for practice at using definitions as well as reinforcement of estimation and number sense. And the story of the law of exponents, which goes from having a simple verification for positive whole exponents to being the driver of the definition for all other exponents, is a great example of the art of mathematical definition. (I enjoy the parallel between the law of exponents and the homotopy lifting property, which went from being a property of fiber bundles to being Serre's definition of a fibration.)

Reasoning at the high-school level can reach, for example, the circle of ideas centered on the fact that the sum of the first n odd numbers is n^2. The statement itself is ripe for conjecture through seeing cases, and it is substantial work for students to make their conjectures precise. A graphical argument, as in Figure 2, is readily accessible, and then provides an opportunity to press for details—for example, why the number of blocks added in each step increases by two. Algebraically, there are multiple arguments: an inductive argument, the standard trick for summing arithmetic sequences, and one through calculating successive differences (a case in which "simplifying" has a purpose) and employing reasoning central to the fundamental theorem of calculus. Indeed, such analysis provides a first explanation, in the discrete setting, for why quadratic functions should model total displacement in the presence of constant acceleration.

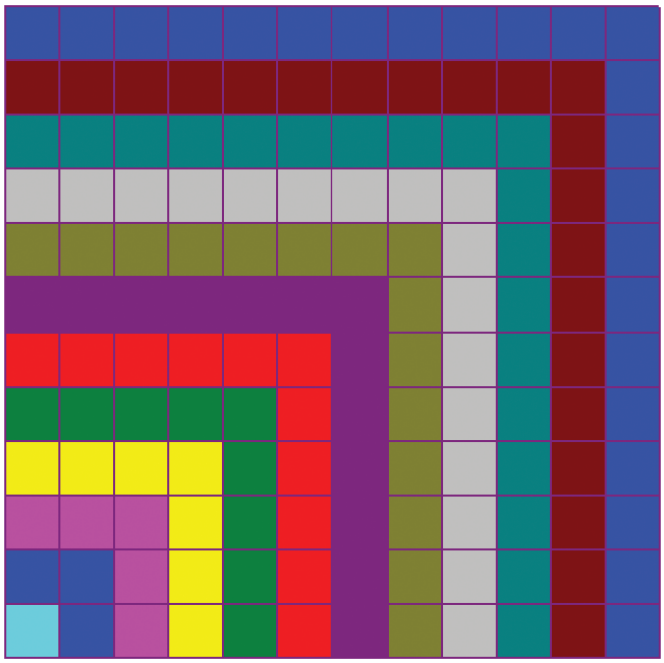

Figure 2. A graphical argument that the sum of the first n odd numbers is n^2.

Asking students to provide and critique reasoning is more time-intensive than only demonstrating reasoning to them or omitting reasoning altogether. But the development of communities of reasoning is sorely needed, especially at this moment. Students benefit immensely from establishing truth through logic that is accessible to all, with contributions from themselves and peers, rather than having mathematics exclusively "handed down" through authority of teacher and textbook. Such communities foster responsibility for acknowledging errors, understanding them as a way to progress to correct understanding.

In addition to presenting mathematics consistently with the practices and values of our community, increased reasoning will pay dividends through deeper retention as well as greater transfer of reasoning ability (see for example Chapter 3 of [NRC00]). Facility with application is also supported, as students who have access to reasoning can more flexibly use mathematics to understand the world. For example, concrete visual models are now regularly used to reason about dividing fractions and in particular "remainders," as needed for applications. If it takes $1/4$ of a ton of steel to make a car, and a factory has $8\ 1/3$ tons of steel, dividing, we see that there could be 33 cars made, and there will be $1/12$ of a ton of steel left over. This is $1/3$

of the amount needed to make another car, as one sees in the quotient 33 ⅓, but in some contexts 1/12 would be the correct "remainder," a quantity which is not accessible if one only learns to "flip and multiply."

Progressions occur not only in the mathematics itself, as definitions and theorems build on previous such, but across many domains. A familiar domain to mathematicians is the amount of abstraction. Related, but less familiar, is the sophistication of representation, including progression from tactile models to pictures to diagrams to variables. On the more cognitive side, there are progressions in student autonomy, including whether tools are called for explicitly or demanded through problem-solving. There are progressions in how much of the process of doing mathematics is engaged in, for example in formulation of conjectures and counterexamples. Demands in language will also progress. Strong curricula attend to all of these types of progression, and address them through research-based pedagogy, in particular active learning [LBHea14, Bea16, MAA18].

Teachers need to be prepared to implement curricula that demand that students supply their reasoning. As the mathematical community plays substantial roles in their preparation—especially that of future high-school teachers—we make some recommendations for teacher training and for other related matters below. For comprehensive recommendations for teacher preparation, see [BLS+12, AMT17].

1. Restructure entry-level college courses so students are asked to autonomously provide reasoning, through proofs or in the context of applications.[2] The rest of the world, including future teachers, takes our choices for these classes as a signal about the nature of our subject. Currently, they deduce that math is only about accurately reproducing procedures!

2. Develop profound understanding of the K–12 mathematics progressions in mathematics courses for future elementary and secondary teachers.

3. Provide math majors interested in teaching with opportunities to connect formal college-level mathematics with school mathematics, as being developed in multiple NSF-funded projects: the Mathematical Association of America's META Math project; the Association of Public and Land-Grant Universities' MODULES project; and the ULTRA project run by Rutgers, Columbia, and Temple Universities. At some point—perhaps masters-level or other professional preparation—such students should learn the arguments underlying the rules for arithmetic and thus algebra, as preparation for teaching algebra as reasoning and being able to appeal to or even fill in background. Knowledge of such arguments is also helpful for entry-level college teaching.

4. Encourage development of opportunities for reasoning in high-school curricula, in particular the use of algebra as a tool of reasoning, for example to establish divisibility rules or in counting problems. Reasoning about authentic applications should also be developed, and play an especially prominent role in high-school math.

5. Support K–12 educators as they move away from harmful practices such as acceleration without understanding and tracking. Students should be challenged through depth of understanding, which is achievable in mixed-ability classrooms. Tracking has negative consequences for all students, especially those from disadvantaged backgrounds. The benefits of such system shifts, as recommended by the National Council of Teachers of Mathematics [NCTM18], have been borne out by data, for example in the work of San Francisco Unified School District [BF14, SFU18].

[2] *For example, in my department we've replaced our Intermediate Algebra offerings with classes which apply algebra to mathematical modeling.*

ACKNOWLEDGMENT. I would like to thank Yvonne Lai and Jenny Ruef for their help in writing this piece.

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Credits

Photo of Dev Sinha and Figure 1 are courtesy of the author. Figure 2 is courtesy of Dennis DeTurck.