Algebraic and Topological Tools in Linear Optimization

Jesús A. De Loera

The Linear Optimization Problem

This is a story about the significance of diverse viewpoints in mathematical research. I will discuss how the analysis of the linear optimization problem connects in elegant ways to algebra and topology. My presentation has two sections, grouped under the guiding light of these areas. These mathematical areas, while often considered to be pure mathematics, in fact connect deeply to many other important computational problems besides linear optimization (see [5] and the many references therein). Clearly, the famous quest for the "most efficient" algorithm possible to solve the linear optimization problem shapes my story, too; this is a challenge that many consider to be one of the top mathematical challenges of the century (see, e.g., [14]). My narrative is informal, thus I will not give all details, but I hope enough intuition will entice others to learn more about these methods. My target reader is a non-expert mathematician, say a graduate student curious about how algebra connects to optimization, but I also hope to give experts quick pointers to remarkable new activity since 2010. For the sake of space I was asked to leave out the majority of references, but the interested reader can obtain a longer version, with all missing references, by contacting me.

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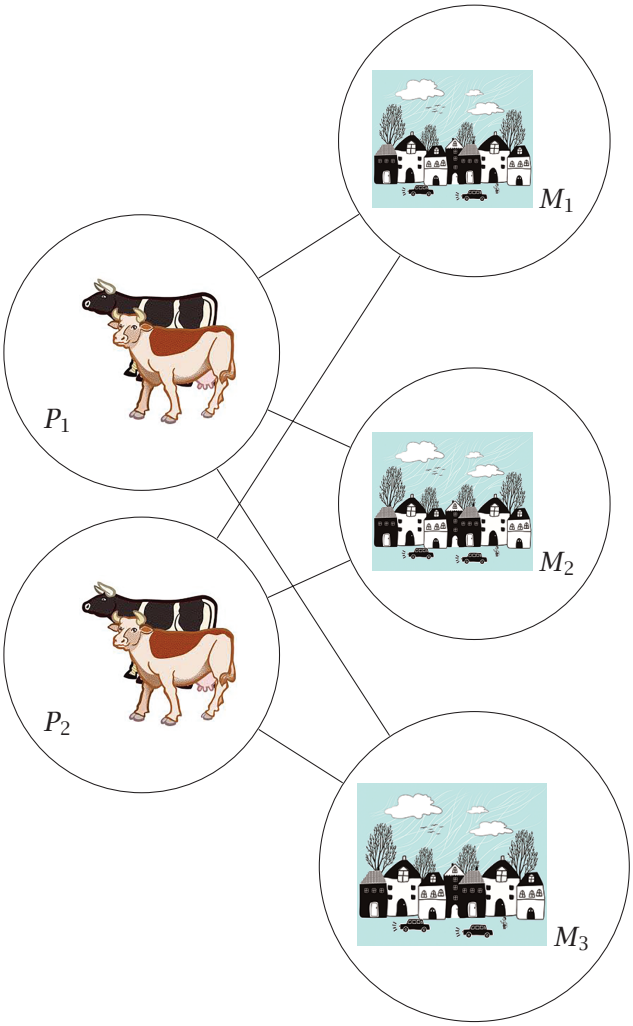

Figure 1. Two milk production plants supply three cities.

The hero of my story is the *linear optimization problem*. This is the basic, but fundamental, computational problem of maximizing or minimizing a linear function subject to the condition that the solution must satisfy a given set of linear inequalities and equations. One wishes to find a vector $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ such that we maximize $c_1x_1 + c_2x_2 + \dots c_dx_d$ subject to

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,d}x_d &\le b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,d}x_d &\le b_2, \\ &\vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,d}x_d &\le b_n. \end{aligned}$$

Here $a_{i,j}, b_i, c_j$ are assumed to be integers. The same problem presented in matrix-vector notation is summarized as

$$\max\{\, \mathbf{c}^{\mathcal{T}}\mathbf{x} \;:\; A\mathbf{x} \le \mathbf{b},\ \ \mathbf{x} \in \mathbb{R}^n \,\}.$$

As a very concrete example, imagine there are two milk production plants, P_1 and P_2, that supply three cities, M_1, M_2, and M_3, with fresh milk. Facility P_i bottles s_i gallons of milk, and city M_j consumes r_j gallons. There is a cost C_{ij} for transporting one gallon of milk from plant P_i to city M_j. See Figure 1 for an illustration of the transportation problem. The problem is to meet the market requirements, i.e., satisfy demand with the available supply, doing so at a minimum transportation cost. If the amount of milk to be shipped from P_i to M_j is given by x_{ij}, then we can write a *linear program* modeling the optimization challenge of minimizing $\sum_{i=1}^{2}\sum_{j=1}^{3} C_{ij}x_{ij}$ subject to

$$\begin{aligned} \textstyle\sum_{i=1}^{2} x_{ij} &\ge r_j && \text{for each city,} \\ \textstyle\sum_{j=1}^{3} x_{ij} &\le s_i && \text{for each milk plant,} \\ x_{ij} &\ge 0. \end{aligned}$$

Research on linear optimization can be traced back at least to Fourier, but the subject really starts developing at full speed around the time of World War II. In fact, it is around 1939–1941 that Kantorovich and Koopmans investigated the simple type of transportation problem we saw earlier. They later received the Nobel Prize in Economics for it. Dantzig (see Figure 2), von Neumann, Gale, Kuhn, and Tucker were crucial in the first developments of the subject in the late forties. The name *linear program* is old, and the word *programming* was not about computer programming, but was used as a synonym for planning. Following that old tradition I will call *linear programs*, or LPs for short, the instances of the linear optimization problem.

If distributing milk efficiently is not your favorite way to engage with mathematics, consider the following way to think about linear programs: With a simple reformulation adding auxiliary variables, the inequality system can always be rewritten as the solutions of a system of linear *equations* over the *non-negative* real numbers, $\max\{\, \mathbf{c}^{\mathcal{T}}\mathbf{x} \;:\; A\mathbf{x} = \mathbf{b},\ \ \mathbf{x} \ge 0,\ \mathbf{x} \in \mathbb{R}^n \,\}$. Thus linear optimization can be thought of as *linear algebra with non-negative variables*. In fact, Fourier's initial algorithm was a process of variable elimination quite similar to Gaussian elimination. Today we know it as the Fourier–Motzkin elimination algorithm (it was rediscovered by T. S. Motzkin).

Why do LPs matter? Well, LPs are simple yet very expressive models, which include as special cases several important problems: for instance, the challenges of finding the shortest paths on a graph, the maximum flow on a network, the minimum weight spanning tree or matching on a graph, and all two-player zero-sum games can be formulated directly as linear programs. More importantly, all other parts of optimization rely heavily on linear optimization as a pillar for computation and theory. For example, optimization problems with discrete variables are most often reduced via branching to the repeated use of linear programming. Linear programs are also used in various approximation schemes for combinatorial and non-linear optimization. But the impact of linear optimization goes well beyond optimization itself and reaches other areas of

mathematical research: e.g., in combinatorics and graph theory and in discrete geometry, for the solution of Kepler's conjecture. Exciting new applications continue to appear, and the impact of linear optimization is also palpable in practical ventures (e.g., airlines, oil industry, etc.) because there are very fast software packages that can solve concrete problems with millions of variables in reasonable time. For those wishing to learn more, there are several excellent books ([10, 13]) and surveys (such as [8, 11, 15]) covering the theory of linear optimization.

Before we start, a word about the geometry behind linear programs. The *feasibility region* of a linear program, i.e., the set of possible solutions, is always a *convex polyhedron*. Polyhedra are beautiful jewel-like objects that have attracted mathematicians for centuries. Here a polyhedron $P \subset \mathbb{R}^d$ is the set of solutions of a system of linear inequalities of the form $A\mathbf{x} \leq \mathbf{b}$, where A is an integer matrix with d-dimensional row vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and $\mathbf{b}$ is an integer vector. In this way, the *input size* is given by d variables, n constraints, and the maximum binary size L of any coefficient of the data. We assume that row vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ span $\mathbb{R}^d$. Thus $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq b\}$. When P is bounded, we call it a *polytope*. Two important types of polytopes are *simple* and *simplicial* polytopes. Simple polytopes are those where at every vertex, d inequalities meet with equality, for example a cube. *Simplices* are d-dimensional polytopes with exactly $d+1$ inequalities. A *simplicial* polytope is one whose faces are all simplices, e.g., an octahedron or an icosahedron. Not all polytopes fit in these two types (e.g., the polytope in Figure 3), but most arguments in linear optimization go through them. For example, simple polytopes correspond to non-degenerate linear programs, which are typically run in computation. The beautiful geometry of polytopes is clearly presented in [17]. We now begin the story.

A Topological Point of View

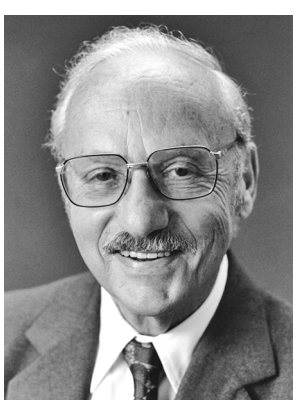

Figure 2. George Dantzig (1914–2005).

Dantzig's simplex method from 1947 [4] is one of the most common algorithms for solving linear programs. It can be viewed as a family of combinatorial local search algorithms on the graph of a convex polyhedron. More precisely, the search is done over a finite graph, the one-skeleton of the polyhedron or *graph of the polyhedron*, which is composed of the zero- and one-dimensional faces of the feasible region (called *vertices* and *edges*). The search moves from a vertex of the one-skeleton to a better neighboring vertex according to some *pivoting rule* that selects an improving neighbor. The operation of moving from one vertex to the next is called a *pivot step* or simply a *pivot*. Geometrically the simplex method traces a finite path on the graph of the polytope. See Figure 3.

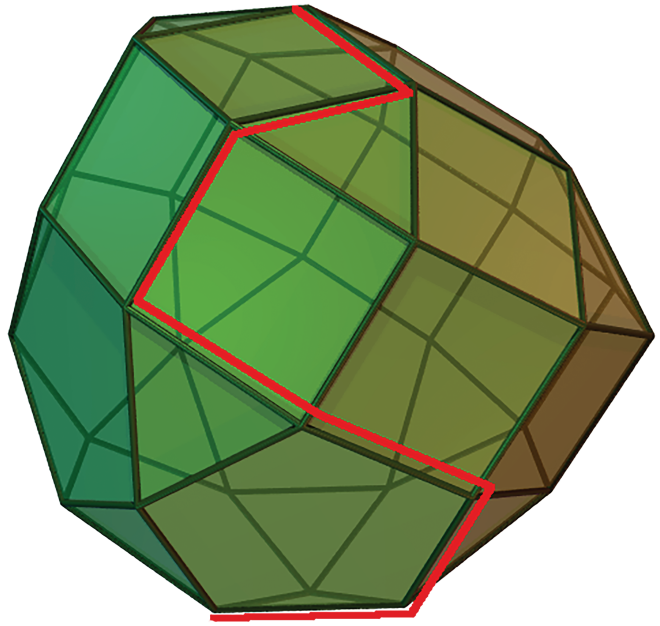

Figure 3. The simplex algorithmic steps trace a path on the graph of a polyhedron.

Today, after sixty years of use, and despite competition from other algorithms, the simplex method is still widely popular and useful. The simplex method was even named as one of the most influential algorithms in the twentieth century, but we still do not completely understand its theoretical performance sufficiently well to explain its practical performance. *Is there a polynomial-time version of the simplex method?* Such an algorithm would allow the solution of a linear program with a number of pivot steps that is a polynomial in d, n, and L. This is a very famous problem that has received a lot of attention. Suggested by this complexity question, there is a related geometric puzzle about the *diameter* of the graph of a polytope used to make the simplex walk. The diameter is the length of the longest shortest path among all possible pairs of vertices; e.g., for a three-dimensional cube the diameter is three. It remains a well-known open problem whether there is always a polynomial bound on the diameter. If a counterexample exists, then a polynomial simplex method would be impossible. So what are current bounds for the diameter?

The best upper bounds for the diameter of polytopes, valid for all polytopes, originated in a groundbreaking paper by Kalai and Kleitman. If we denote by n the number of facets of a polytope, and d its dimension, then their result asserts a bound of $n^{\log(d)+1}$. This was improved by Todd and then, most recently, Sukegawa with the current

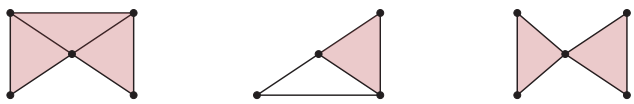

Figure 4. Some simplicial complexes (from left to right): a pure strongly connected complex, a non-pure complex, a non-strongly connected complex.

record of $(n-d)^{\log O(d/\log(d))}$. In a different influential approach Larman proved that the diameter is no more than $2^{d-3}n$. This bound was improved by Barnette to $\frac{2^{d-3}}{3}(n-d+5/2)$. Remarkably this bound shows that, in fixed dimension d, the diameter must be *linear* in the number of facets. Could the diameter of all polytopes be in fact linear in terms of n, d? We will come back to this later!

A key point I would like to stress is that the proofs of these general upper bounds use only very limited (metric) geometric properties of polytopes; the coordinates and equations defining an LP do not play a big role. The bounds hold for more abstract combinatorial objects called *simplicial complexes.*

A simplicial complex K is a finite collection of simplices that are glued to each other in a structured manner. If $\sigma \in K$, then all its faces (which are smaller simplices too!) are also in K. Thus a simplicial complex must be closed under containment. The intersection of any two elements of K is another element of K. The *dimension* of a simplex is equal to the number of vertices minus one (e.g., a triangle is two-dimensional). A simplex of K that is not a face of another simplex is called *maximal*. A simplicial complex K is called *pure* if all of its maximal faces are of the same dimension. Maximal faces of a pure simplicial complex are called *facets*. For a pure simplicial complex K, one may form its *dual graph*, which is a graph whose vertex set is given by the facets of K and whose edge set is given by pairs of facets in K that differ in a single element. A simplicial complex is *strongly connected* if its dual graph is a connected graph. See Figure 4.

Pure simplicial complexes are excellent topological abstractions of polytopes because the boundary complex of a simplicial $d+1$-polytope is always a pure simplicial complex of dimension d, and there is a direct way to go from non-degenerate linear optimization problems (known to have the largest diameters anyway) to pure simplicial complexes by using the *polarity* operation. This is illustrated in Figure 5, where facets turn into vertices and vertices into facets under polarity to obtain an octahedron that is simplicial (all faces are now triangles). Note that the n facets of a simple polytope turn into n vertices of a simplicial complex.

A path on the edge of the cube becomes a path on the (triangles) simplices of an octahedron. In higher dimensions the distance between two facets of the simplicial complex, F_1, F_2, is the length s of the shortest simplicial path $F_1 = f_0, f_1, \ldots, f_s = F_2$. Consecutive d-dimensional simplices in the path must share a common $(d-1)$-dimensional face. The diameter of a simplicial complex is the maximum over all distances between all pairs of facets. Note that the diameter of K equals the graph theoretic diameter of its dual graph.

The *topological* approach to finding the diameter of simple polytopes instead explores the diameter of their corresponding (polar) simplicial complexes. We remind the reader that now n refers both to the number of facets of the simple polytope and to the number of vertices of its (polar) simplicial complex. This idea has a long history, starting with the introduction of *abstract polytopes* by Adler and Dantzig [1]. Mani and Walkup, Kalai, Billera and Provan, and Klee and Kleinschmidt were some of the pioneers. The message is that special simplicial complexes have nicely bounded diameter. Coordinates and coefficients do not matter; distances and angles do not matter. We present here a "taste" of topological results about diameter simplicial complexes.

First of all, topological abstraction is justified given our current bounds. Eisenbrand, Hähnle, Razborov, and Rothvoss worked with a class of simplicial complexes called *normal complexes* which again include all those coming from linear programs. They proved that both the Larman-style bounds and the Kalai–Kleitman-style bound hold in fact for normal simplicial complexes. Their work led to the following tantalizing conjecture:

Conjecture 1 (Hähnle, 2014). *The diameter of every normal $(d-1)$-complex with n vertices, including simplicial d-polytopes with n vertices, is at most $(n-1)d$.*

Overall, nice topological or combinatorial conditions give rise to good diameter bounds. Consider for example the following property: Peeling off a simplicial complex, piece by piece, is an important tool in combinatorial topology; e.g., a *shelling* of a pure simplicial complex is an ordering of its facets in which, at each step, the ith facet intersects nicely with the union of the other previous facets. Shellings do not exist for all pure complexes, but remarkable results have been shown for shellable complexes. For our purposes *vertex decomposability* is very important.

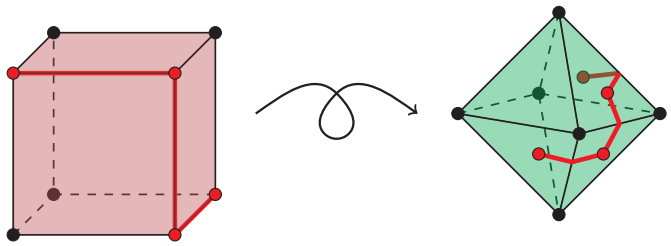

Figure 5. An example of the polarity operation: The polar of a cube is an octahedron and vice versa.

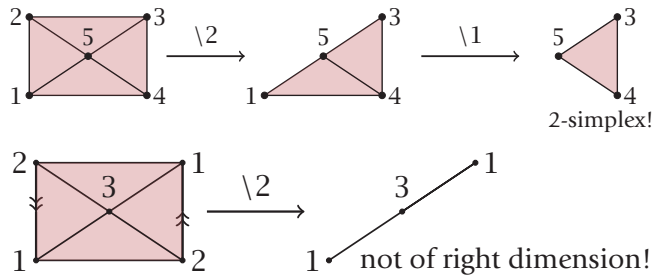

Figure 6. An example of a weakly 0-decomposable complex (on top) and a Möbius band, which is not weakly 0-decomposable.

Billera and Provan conceived a natural way to prove a linear bound on the diameter that relies on vertex decomposition properties of complexes. They introduced the notion of a *weakly k-decomposable complex*. A d-dimensional simplicial complex Δ is weakly k-decomposable if it is pure of the same dimension, and either Δ is a single d-simplex, or there exists a face τ of Δ (called a *shedding face*) such that $\dim(\tau) \leq k$ and $\Delta \backslash \tau$ is d-dimensional and weakly k-decomposable. So one recursively "peels off" the simplicial complex using a sequence of faces so that finally we arrive at a (full-dimensional) simplex. In Figure 6 we show an example of a weakly 0-decomposable complex through a possible shedding order of vertices (in three steps), but when the same simplicial complex is made into a Möbius band identifying one pair of opposite sides, the complex is not anymore weakly 0-decomposable.

The reason why weak-decomposability is so interesting for bounding diameters is the following theorem:

Theorem 2 (Billera, Provan, 1980). *If Δ is a weakly k-decomposable simplicial complex, $0 \leq k \leq d$, then*

$$\operatorname{diam}(\Delta) \leq 2f_k(\Delta),$$

where $f_k(\Delta)$ is the number of k-faces of Δ. In the case of weakly 0-decomposable, we have the following linear *bound ($n = f_0(\Delta)$):*

$$\operatorname{diam}(\Delta) \leq 2f_0(\Delta) = 2n.$$

Note that if all simplicial polytopes were weakly 0-decomposable, then the diameter would be linear, being no more than twice the number of facets of the polar simple polytope. All simplicial d-dimensional polytopes are weakly d-decomposable because they are *shellable* (see, e.g., [17] for an introduction). The question is then, which simplicial polytopes are weakly 0-decomposable? More strongly, is there a fixed constant $k < d$ for which all simplicial polytopes are weakly k-decomposable? If this were true for $k = 0$ (weakly 0-decomposable), then the desired linear bound would be achieved! These challenges have been settled. De Loera and Klee constructed examples of simplicial polytopes that are not weakly 0-decomposable disproving this method as an approach for a linear bound of the diameter for polytopes. Interestingly, the counterexamples are explicit from the transportation problems, like those in the introduction, with two milk factories and at least five cities to service.

As a word of warning, we note that going into too much topological generality will certainly not give a useful bound for the diameter, at least not one sufficiently good for linear optimization. This is evident from the pioneering work by Santos, and more recently Criado and Santos; they showed

Theorem 3 (Criado, Santos 2017). *If $H_s(n,d)$ denotes the maximum diameter of a pure strongly connected $(d-1)$-complex with n vertices, then*

$$\frac{n^{d-1}}{(d+2)^{d-1}} - 3 \leq H_s(n,d) \leq \frac{1}{d-1}\binom{n}{d-1} \simeq \frac{n^{d-1}}{d!}.$$

Thus the diameter of many complexes grows as $c_d n^{d-1}$ for a constant c_d depending only on d.

They also showed that similar exponential behavior appears for simplicial *pseudo-manifolds* (to be a pseudo-manifold every $(d-2)$-dimensional simplex of the complex is contained in exactly two maximal d-dimensional ones). On the other hand, linear programs are associated to simplicial polytopes, which are simplicial spheres, a much more restricted type of simplicial complex.

Some constructions of spheres and balls with "large" diameter were presented by Mani and Walkup with exciting new improvements by Criado and Santos. Nevertheless, today all known simplicial spheres and balls have diameter bounded by only $1.25n$. Regarding construction of polytopes with "large" diameter, Warren Hirsch conjectured in 1957 that the diameter of the graph of a polyhedron defined by n inequalities in d dimensions is at most $n - d$. Dantzig popularized the conjecture in his classic book, and it became known as the *Hirsch conjecture*. Counterexamples in the *unbounded* case were found quickly by Klee and Walkup, but it took fifty-three years of hard work to build a counterexample to the *Hirsch conjecture* for polytopes. In his historic paper [12] Francisco Santos showed

Theorem 4 (Santos, 2010).

- *There is a 43-dimensional polytope with 86 facets and of diameter at least 44 (this result has now been improved).*
- *There is an infinite family of non-Hirsch polytopes with n facets and diameter $\sim (1+\epsilon)n$ for some $\epsilon > 0$. This holds true even in fixed dimension.*

A key observation of Santos's construction was an extension of a well-known result of Klee and Walkup. They showed that the Hirsch conjecture could be proved true from just the case when $n = 2d$. In that case the problem

is to prove that given two vertices u and v that have no facet in common, one can pivot from one to the other in d steps so that at each pivot we abandon a facet containing u and enter a facet containing v. This was named the *d-step conjecture* (see [8]).

The construction of Santos's counterexample uses a variation of this result for a family of polytopes called *spindles*. Spindles are polytopes with two distinguished vertices u,v such that every facet contains either u or v but not both. Examples of a spindle include the cross polytopes, the cube, and the polytope in Figure 7. Spindles can be seen as the overlap of two pointed cones (as shown in Figure 7). The *length of a spindle* is the distance between this special pair of vertices.

Santos's strong d-step theorem for spindles says that from a spindle P of dimension d with $n > 2d$ facets and length λ one can construct another spindle P' of dimension $d+1$ with $n+1$ facets and length $\lambda+1$. Since one can repeat this construction again and again, each time increasing the dimension, length, and number of facets of the new spindle by one unit, we can repeat this process until we have $n = 2d$ (number of facets is twice the dimension). In particular, if a spindle P of dimension d with n facets has length greater than d, then there is another spindle P' of dimension $n-d$ with $2n-2d$ facets and length greater than $n-d$ that violates the Hirsch conjecture. Santos constructed such a five-dimensional spindle. As of today the work on constructing lower bounds still leaves open the possibility of a linear diameter for polyhedra.

An Algebraic Point of View

In 1984 Narendra Karmarkar presented an algorithm to solve LPs that used a different principle from the simplex method. At each iteration of the algorithm a point in the interior of the polytope will be generated. His paper [7] started the revolution of *interior-point methods* and gave an alternative proof of polynomiality of linear programming. Originally Karmarkar presented his work in terms of projective transformations, but later it was shown his algorithm was equivalent to an earlier idea. One replaces the linear objective function with something more complicated. A *barrier function* is added to the original linear objective function. A barrier function has a singularity at the boundary of the polyhedron and thus prevents the points at each iteration from leaving the feasible region. Barrier functions originated in non-linear programming during the 1960s when Fiacco and McCormick showed they defined a smooth curve, the barrier trajectory or the *central path*, that converges to the solution of the constrained problem from the strict interior of the feasible region, completely avoiding the boundary. Interior-point methods have had a profound impact in modern optimization. Interior-point methods are used not just for linear programming, but for non-linear optimization and other forms of convex optimization, e.g., semidefinite programming (see, e.g., [16] and the references therein).

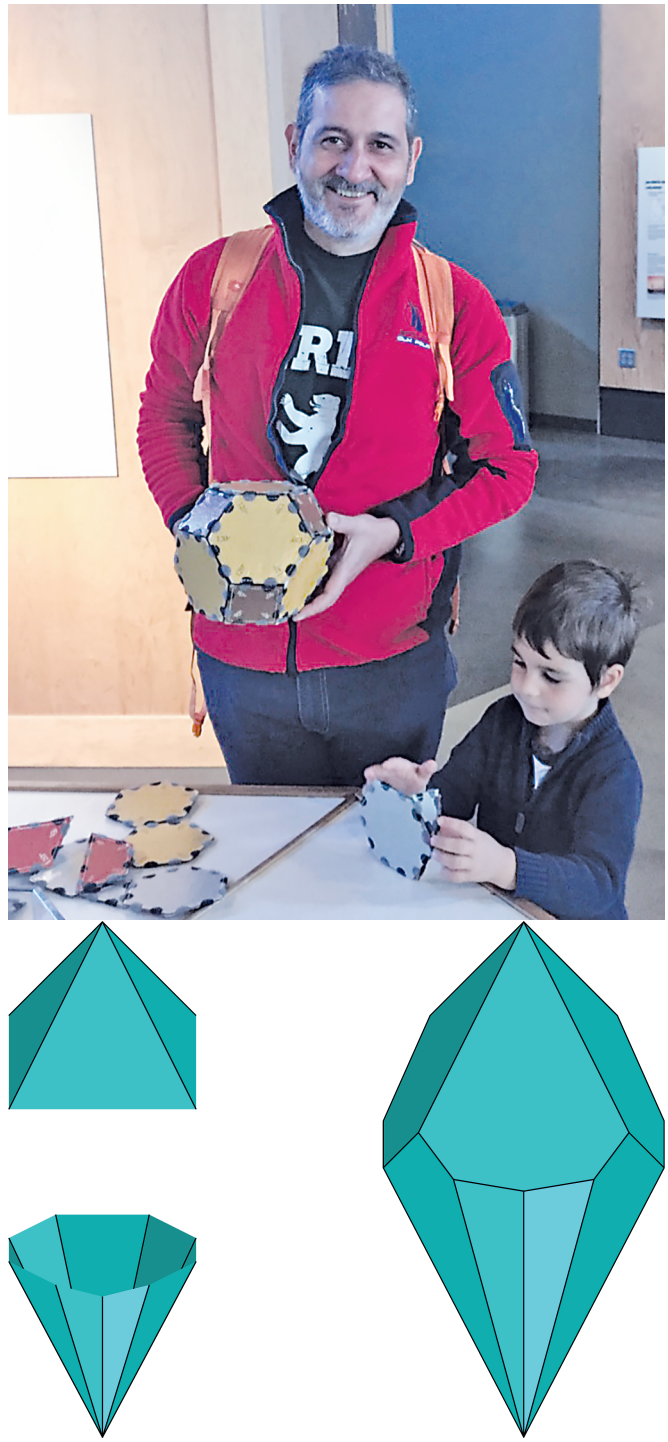

Figure 7. Francisco Santos and a younger geometer (top) and a three-dimensional spindle (bottom).

A logarithm function is a classical choice to use as the barrier function. For a concrete example, consider the problem to maximize $\mathbf{c}^T\mathbf{x}$ subject to the conditions

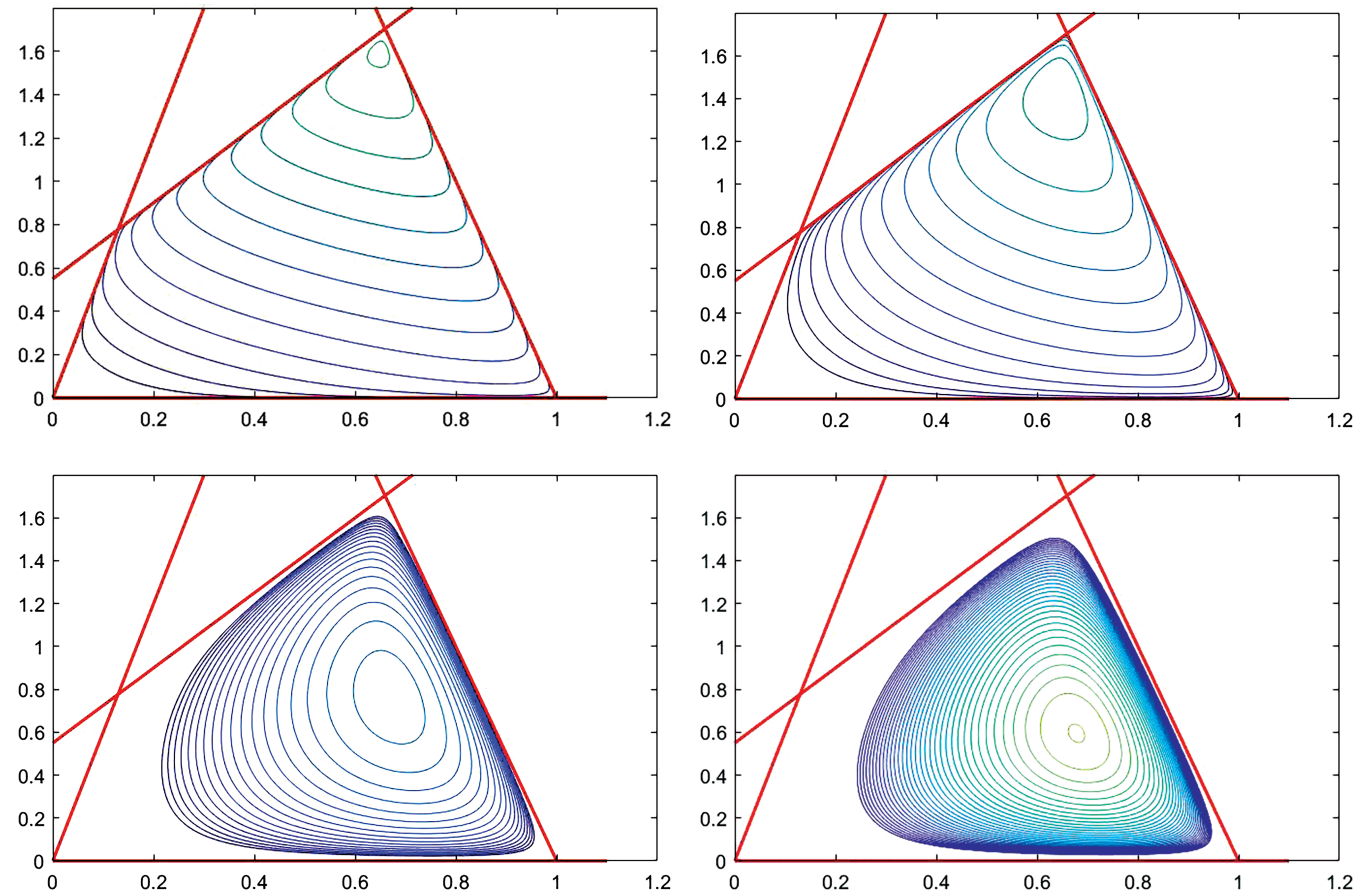

Figure 8. The level sets of a logarithmic barrier function f_λ at four different values of λ. Top left: λ is close to zero, thus the optimum of f_λ is near the LP optimum. Bottom right: λ is a large positive number, thus the optimum of f_λ is near the analytic center of the polyhedron.

$A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$; the *logarithmic barrier function* for this problem will be $f_\lambda(\mathbf{x}) := \mathbf{c}^T\mathbf{x} + \lambda \sum_{i=1}^n \log x_i$. It is natural to use Newton's method to find unconstrained maximizers of the logarithmic barrier function for a sequence of positive values of λ converging to zero. This numerical approach yields, for finitely many values of λ, a piece-wise approximation to the central path. See Figure 8.

Let me show you now an amazing mathematical connection to algebra! This discussion will be motivated by another longstanding open problem. With the existence of polynomial-time algorithms for linear programming, we can be more ambitious and ask, *is there a strongly polynomial-time algorithm that decides the feasibility of the linear system of inequalities $Ax \leq b$?* Strongly polynomial-time algorithms take a number of steps bounded only by a polynomial function of the number of variables and constraints. In particular, the size of the coefficients would not matter. Some instances of linear optimization in which we know we can do that include network type LPs, LPs with at most two variables per inequality, and combinatorial LPs (those with bounded maximum sub-determinant).

It is natural to ask whether interior-point methods can be adjusted or adapted to work as strongly polynomial-time algorithms for all LPs. We will see here a partial negative answer of this question for interior-point methods. Some fascinating properties on the differential geometry of the central paths will become relevant, and to answer them the methods of tropical algebraic geometry will be key. While typically interior-point methods are seen as a family of numerical-approximation algorithms, the progress came from looking at the intrinsic algebraic and symbolic nature of interior-point methods and new combinatorial tools, such as tropical geometry, to analyze systems of polynomial equations.

To set up the story, recall the fundamental fact that all linear optimization problems come in pairs, the pair of linear programming problems in *primal* and *dual* formulations (with slack variables):

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0; \quad (1)$$

$$\text{Minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} - \mathbf{s} = \mathbf{c} \text{ and } \mathbf{s} \geq 0. \quad (2)$$

As before, here A is an $n \times d$ matrix. The fundamental

results of duality theory, available in all good books on the subject (e.g., [13]), allow us to set up the basic algebraic formulation of the *primal-dual* interior-point methods:

$$A\mathbf{x} = \mathbf{b}\,,\ A^T\mathbf{y} - \mathbf{s} = \mathbf{c},\ \text{ and } x_i s_i = \lambda \quad \text{for } i = 1, 2, \ldots, d. \tag{3}$$

Perhaps the key observation for this section is that *this is a system of polynomial equations (quadratic and linear), and thus its solutions can be studied as an algebraic variety*. This algebraic variety is the *central curve*. Algebraic thinking about the central curve goes back to pioneering work by Bayer and Lagarias in the early days of interior-point methods. Today we are taking this further.

It is known that for each value of $\lambda > 0$, the system of polynomial equations (3) (with interior feasible points) has a unique real solution $(\mathbf{x}^*(\lambda), \mathbf{y}^*(\lambda), \mathbf{s}^*(\lambda))$ with the properties $\mathbf{x}^*(\lambda) > 0$ and $\mathbf{s}^*(\lambda) > 0$. The point $\mathbf{x}^*(\lambda)$ is precisely the optimal solution of the logarithmic barrier function $f_\lambda(\mathbf{x})$ we saw earlier for problem (1).

The parametrized set of solutions, given by the changing parameter λ, is the *central path*. These solutions for $\lambda \to 0$ have a limit point $(\mathbf{x}^*(0), \mathbf{y}^*(0), \mathbf{s}^*(0))$ which satisfies equation (3) when $\lambda = 0$, which defines optimal solutions, and thus in the limit we reach an optimum point of the linear program. Traditionally the central path is only followed approximately: discrete incremental steps that follow the path are generated by applying a Newton method to the equations. Similarly, tradition dictates the central path only connects the optimal solution of the linear programs in question with its *analytic center* within *one* single cell, with $s_i \geq 0$. Our plan now is to break with tradition: we look at the algebraic curve with all *exact* solutions defined by the system of equations (3).

In Figure 9 we see a depiction of one difference between the numeric point of view and the algebraic point of view of the central path. The central path is just a small portion of the entire explicit central curve that in reality extends beyond a single feasibility region (given by different sign constraint choices on variables). The central (algebraic) curve passes through all the vertices of a hyperplane arrangement defined by the LP constraints.

One way to estimate the number of Newton steps needed to reach the optimal solution is to bound the *curvature* of the central path. The intuition is that curves with small curvature are easier to approximate with fewer line segments. This idea has been investigated by various authors, and it has yielded interesting results. For example, S. Vavasis and Y. Ye found that the central path contains no more than n^2 crossover events (turns of a special type). This finding led to an interior-point algorithm whose running time depends only on the constraint matrix A. The notion of curvature we need is the *total curvature* of the central curve. It is defined as the degree of the map to the unit sphere assigning to each point of the curve the unit tangent vector at that point. This assignment is called the *Gauss map*. For a curve in the plane this is the sum of arc lengths on the unit circle. In Figure 10 we illustrate the tracing of an arc on the unit circle for one curve segment in the sequence of tangent vectors.

Dedieu, Malajovich, and Shub showed that the *average* total curvature of the primal, the dual, and the primal-dual central paths of the strictly feasible polytopes defined by (A, b) is at most $2\pi(d-1)$ (primal), at most $2\pi d$ (dual), and at most $2\pi d$ (primal-dual). In particular, it is independent of the number n of constraints. Later De Loera, Sturmfels, and Vinzant obtained bounds for the total curvature in terms of the degree of the Gauss maps of the curve. They also computed the degree, arithmetic genus, and defining prime ideal of the central curve and their primal dual projections. Their techniques used classical formulas from algebraic geometry. Unfortunately, for practical applications, the more relevant quantity is not the total curvature of the entire algebraic curve but rather the total curvature in and around the usual portion of the central path within a specific polytope region, going from the analytic center to an optimum. A. Deza, T. Terlaky, and Y. Zinchenko investigated the total curvature in a series of interesting papers. They conjectured that the largest possible total curvature of the associated central path with respect to all cost vectors is no more than $2\pi n$, where n is the number of facets of the polytope.

In an exciting 2018 paper [2], X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig disproved the Deza–Terlaky–Zinchenko conjecture, and they also showed that (certain) logarithmic-barrier interior-point methods can never be strongly polynomial for linear programming!

Theorem 5 (Allamigeon et al. 2018 [2]).

- *There is a parametric family of linear programs in $2d$ variables with $3d+1$ constraints, depending on a parameter $t > 1$, such that the number of iterations of any primal-dual path-following interior-point algorithm with a log-barrier function that iterates in the wide neighborhood of the central path is exponential*

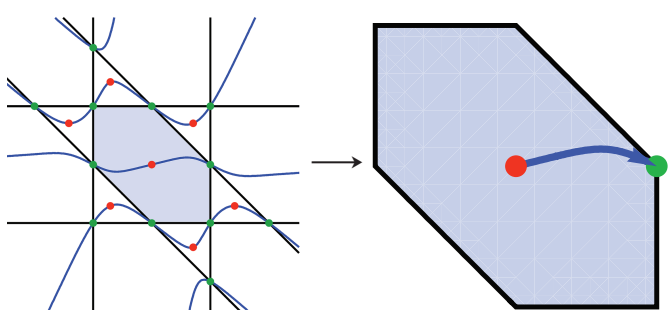

Figure 9. One difference from the central path (left) to the central curve (right) is more points of solution exist.

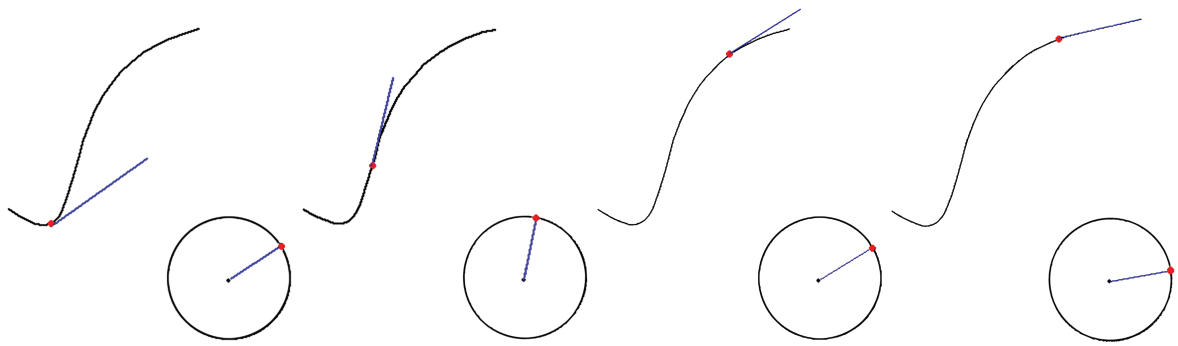

Figure 10. Illustration of the total curvature as the total traversed arc length of the unit tangent vectors of a curve.

in d for all t sufficiently large.

- *There exists an explicit family of linear programs with $3d+1$ inequalities in $2d$ variables where the central path has a total curvature that is exponential in d.*

This is exciting news, but even more surprising, these remarkable counterexamples were produced using new tools from *tropical geometry*. Tropical methods have been used before in optimization, game theory, and control theory. Tropical geometry is today a very active field of study, and I can only scratch its surface here. For more details see [9].

Tropical geometry is algebraic geometry over the *max-plus* semiring $(\mathbb{R}_{\max}; \oplus, \odot)$ where the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ is endowed with the operations $\mathbf{a} \oplus \mathbf{b} = \max(\mathbf{a}, \mathbf{b})$ and $\mathbf{a} \odot \mathbf{b} = \mathbf{a} + \mathbf{b}$. The max-plus semirings are also called *tropical semirings*. In this way we have some funny arithmetic, e.g., $1 \oplus 3 = 3$ and $5 \odot 0 = 5$. Tropical arithmetic is associative, distributive, the additive identity is $-\infty$, and the multiplicative identity is 0. Tropical arithmetic extends of course to matrices and polynomials, and so one can define tropical varieties, tropical polyhedra, and much more. For example, a *tropical halfspace* is the set of vectors x satisfying a "linear" inequality in the tropical semiring; this translates to those vectors that satisfy $\max(\alpha_1^+ + x_1, \dots, \alpha_n^+ + x_n, \beta^+) \geq \max(\alpha_1^- + x_1, \dots, \alpha_n^- + x_n, \beta^-)$. Just like convex polyhedra are the feasible sets of linear optimization, now tropical polyhedra will play a role for feasible solution sets. A *tropical polyhedron* is simply a finite intersection of tropical halfspaces. Figure 11 shows an example with five halfspaces (marked by colors); a tropical pentagon is indicated in gray.

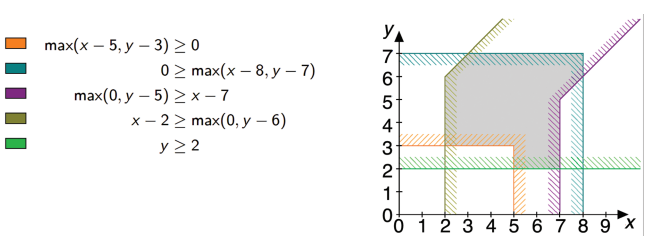

Figure 11. An example of a tropical polyhedron.

It turns out that for every algebraic variety, there is a way to obtain a tropical variety. The *tropicalization* of an algebraic variety V is a process that yields a polyhedral complex $T(V)$ in $\mathbb{R}^d$. The combinatorial complex $T(V)$ has many of the key properties of the original algebraic variety. For instance, if V is a planar algebraic curve over an algebraically closed field, then $T(V)$ is a planar graph. Several key features of the algebraic variety V are easier to see in its tropicalization $T(V)$. For example, if V is irreducible, then $T(V)$ is connected. Figure 12 shows one example of a genus three Riemann surface (it is given by a smooth degree four homogeneous polynomial equation on three complex variables). Its tropicalization preserves the homological information (number of holes).

For our purposes the key idea is that any tropical polyhedron is actually the tropicalization of a linear program with coefficients over a field of absolutely convergent real-valued Puiseux series K (these are power series that allow for negative and fractional exponents). As K is an ordered real-closed field, the basic results of linear programming (Farkas' lemma, strong duality, etc.) still hold true, and the central path of such a linear program is well-defined. The elements of K are real-valued functions. The key point is: a linear program over the field K encodes an entire parametric family of traditional linear programs over the reals $\mathbb{R}$, and the central path on K describes all central paths of this parametric family. The tropicalization of the K-linear programs allows us to see the behavior of the central path, now shown as a piece-wise linear path inside a tropical polyhedron. This piece-wise linear path is the *tropical central path*, defined as the image under the tropicalization. Consider the following example. The Puiseux polyhedron $\mathcal{P} \subset K^2$ is defined by five inequalities:

$$\begin{aligned} \mathbf{x}_1 + \mathbf{x}_2 &\leq 2 \\ t\mathbf{x}_1 &\leq 1 + t^2\mathbf{x}_2 \\ t\mathbf{x}_2 &\leq 1 + t^3\mathbf{x}_1 \\ \mathbf{x}_1 &\leq t^2\mathbf{x}_2 \\ \mathbf{x}_1, \mathbf{x}_2 &\geq 0\,. \end{aligned} \tag{4}$$

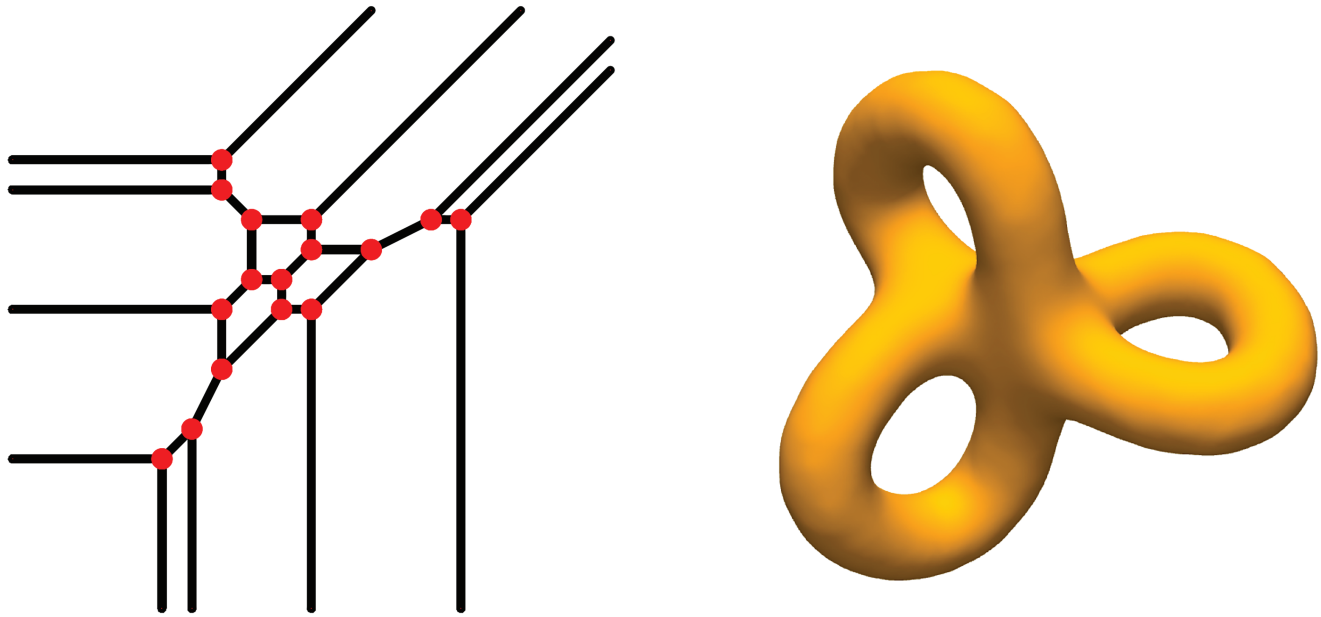

Figure 12. The picture on the left shows a smooth plane tropical curve of degree 4 and genus 3. This arises as the tropicalization of a smooth complex algebraic curve of the same degree and the same genus. As a manifold that curve is a smooth surface of genus 3, and this is the picture on the right. The pictures illustrate that the graph-theoretic genus corresponds to the topological genus.

The tropicalization of $\mathcal{P}$ is described by five *tropical* linear inequalities:

$$\begin{aligned} \max(x_1, x_2) &\leq 0 \\ 1 + x_1 &\leq \max(0, 2 + x_2) \\ 1 + x_2 &\leq \max(0, 3 + x_1) \\ x_1 &\leq 2 + x_2 \,. \end{aligned} \tag{5}$$

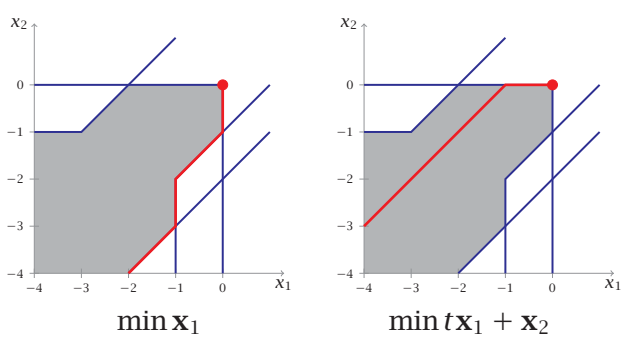

Figure 13. Two tropical central paths corresponding to two different objective functions.

Figure 13 shows the tropicalization of the Puiseux polyhedron from equation (4) (the shaded region), but it also shows two different tropical central paths for two different objective functions. One can see that the tropical central path on the left degenerated to a vertex-edge path, akin to the simplex method moving through the boundary of the polyhedron. The tropicalization allows for simpler calculations. The total curvature of an algebraic curve becomes, under the tropicalization, a sum of piece-wise linear angular turns. The problem of computing total curvature becomes a problem of adding polygonal angles.

Conclusions

I hope the reader has seen the power of algebraic and topological tools in the analysis and creation of linear optimization algorithms. While here I focused on the theory of linear optimization, algebraic-geometric-topological techniques have had impact in other parts of optimization, too. For example, real algebraic geometry has been decisive for global optimization problems with polynomial constraints. Through the theory of sums of squares and semidefinite programming one can compute convex optimization approximations to difficult highly non-convex problems [3]. Algebraic methods were used in integer programming through Graver bases tools [6].

If you can see the benefits of having a diversity of mathematical perspectives, can you imagine the result of having a larger, more diverse group of mathematicians working on solving problems and on finding new applications of mathematics?

References

[1] Adler I, Dantzig GB. Maximum diameter of abstract polytopes: Pivoting and extensions, *Math. Programming Stud.* (1):20–40, 1974, DOI 10.1007/bfb0121238. MR0496673

[2] Allamigeon X, Benchimol P, Gaubert S, Joswig M. Log-barrier interior point methods are not strongly polynomial, *SIAM Journal on Applied Algebra and Geometry*,

no. 1 (2):140–178, 2018. https://doi.org/10.1137/17M1142132

[3] Blekherman G, Parrilo PA, Thomas RR (eds.) *Semidefinite optimization and convex algebraic geometry*, MOS-SIAM Series on Optimization, vol. 13, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2013. MR3075433

[4] Dantzig GB. Maximization of a linear function of variables subject to linear inequalities, *Activity Analysis of Production and Allocation*; 1951:339–347. MR0056260

[5] De Loera J, Goaoc X, Meunier F, Mustafa N. The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg, To appear in *Amer. Math. Bull.*:arXiv:1706.05975, 2017.

[6] De Loera J, Hemmecke R, Köppe M. *Algebraic and geometric ideas in the theory of discrete optimization*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2012. https://epubs.siam.org/doi/abs/10.1137/1.9781611972443

[7] Karmarkar N. A new polynomial-time algorithm for linear programming, *Combinatorica*, no. 4 (4):373–395, 1984, DOI 10.1007/BF02579150. MR779900

[8] Kim ED, Santos F. An update on the Hirsch conjecture, *Jahresber. Dtsch. Math.-Ver.*, no. 2 (112):73–98, 2010, DOI 10.1365/s13291-010-0001-8. MR2681516

[9] Maclagan D, Sturmfels B. *Introduction to tropical geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015. MR3287221

[10] Matoušek J, Gärtner B. *Understanding and using linear programming*, Springer-Verlag, New York, 2007.

[11] Megiddo N. On the complexity of linear programming, *Advances in economic theory* (Cambridge, MA, 1985); 1989:225–268. MR1117042

[12] Santos F. A counterexample to the Hirsch conjecture, *Ann. of Math. (2)*, no. 1 (176):383–412, 2012, DOI 10.4007/annals.2012.176.1.7. MR2925387

[13] Schrijver A. *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, Ltd., Chichester, 1986. A Wiley-Interscience Publication. MR874114

[14] Smale S. Mathematical problems for the next century, *Mathematics: frontiers and perspectives*; 2000:271–294. MR1754783

[15] Todd MJ. The many facets of linear programming, *Math. Program.*, no. 3, Ser. B (91):417–436, 2002, DOI 10.1007/s101070100261. ISMP 2000, Part 1 (Atlanta, GA). MR1888985

[16] Wright MH. The interior-point revolution in optimization: history, recent developments, and lasting consequences, *Bull. Amer. Math. Soc. (N.S.)*, no. 1 (42):39–56, 2005, DOI 10.1090/S0273-0979-04-01040-7. MR2115066

[17] Ziegler GM. *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR1311028

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