

Invariants of Rings via Equivariant Homotopy

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What is algebraic K -theory? Why is it interesting? What tools can we use to understand it? These are some of the basic questions my talk at the AMS Fall Central Sectional Meeting will aim to address.

Algebraic K -theory is an invariant of rings which illustrates a fascinating interplay between algebra and topology. In the 1950s and 1960s, definitions of lower algebraic K -groups first emerged out of work of Grothendieck, Whitehead, Bass, Milnor, and others. For a ring R , one can define these lower K -groups— $K_0(R)$, $K_1(R)$, and $K_2(R)$ —using constructions from algebra. For example, the group $K_1(R)$ is $GL(R)/E(R)$, the quotient of the infinite general linear group by a subgroup generated by certain elementary matrices. From their inception, these lower K -groups had strong connections to topology. Indeed, quotients of $K_1(\mathbb{Z}[G])$, the first algebraic K -theory of group rings, play an essential role in the classification of high-dimensional manifolds via the s -cobordism theorem.

The early 1970s saw a major advance in the field when Quillen defined higher algebraic K -theory, i.e., groups $K_n(R)$ for $n > 2$. Interestingly, while lower K -groups could be defined algebraically, Quillen's definition of higher algebraic K -theory relied on tools from topology. Indeed, he defined these algebraic K -groups to be the homotopy groups of a particular space associated to the ring R . Quillen later won the Fields Medal for his work in this area.

There was immediate interest in computing these higher algebraic K -groups, yet such computations have proven to be very difficult. Even for basic rings like \mathbb{Z} and $\mathbb{Z}/p^2\mathbb{Z}$, the algebraic K -groups still aren't completely known today. Interest in K -theory calculations remains strong, however, due to the important role that K -theory plays across mathematical fields. The modern field of algebraic K -theory lies at the crossroads of topology, algebraic geometry, and number theory, with applications to motivic homotopy theory, classification of manifolds, class field theory, etc. Explicit computations of K -theory groups have important applications to these areas.

One might try to understand algebraic K -theory by

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approximating it with more computable ring invariants. One classical invariant of rings is Hochschild homology, which is indeed related to algebraic K -theory via a “trace” map. The Hochschild homology of a ring R is defined using homological algebra as the homology of a chain complex formed from tensor powers of R . This definition is purely algebraic, and the Hochschild homology groups of a ring are computationally much more accessible than algebraic K -theory. However, one loses a lot of the information in this approximation. This is not so surprising. We would not expect all of the information in algebraic K -theory to be captured by a purely algebraic invariant.

One of the foundational ideas of modern algebraic topology is that one can often take constructions from algebra and form topological analogues of these constructions by replacing the ground ring \mathbb{Z} with a topological object called the sphere spectrum. It is possible to form a topological analogue of Hochschild homology in this way. Topological Hochschild homology (THH), defined originally by Bökstedt, receives a trace map from algebraic K -theory and serves as a better approximation to K -theory than classical Hochschild homology.

Further, topological Hochschild homology has a circle action, which gives it rich additional structure. Understanding this equivariant structure on THH leads to an even better approximation to algebraic K -theory, called topological cyclic homology (TC). Topological cyclic homology was originally defined by Bökstedt, Hsiang, and Madsen, who also constructed a trace map from K -theory to TC, lifting the trace map to THH:

$$K_q(R) \rightarrow TC_q(R) \rightarrow THH_q(R).$$

In nice situations, the cyclotomic trace map from K -theory to TC is close to an equivalence, and topological cyclic homology captures a great deal of information about algebraic K -theory. So the question becomes, how computable is topological cyclic homology? Note that to compute TC one needs to understand THH not just as a topological object but as a topological object with a circle action.

Equivariant homotopy theory is a branch of algebraic topology which studies topological objects with a group action. Algebraic K -theory itself is not an equivariant object, yet it is often the tools of equivariant homotopy theory that most easily facilitate its computation. This is a remarkable fact about equivariant homotopy theory: it can be used with great success to study some questions which on the surface are not equivariant. Tools from equivariant homotopy theory have proven extremely powerful in studying algebraic K -theory. Indeed, the approach outlined above has resulted in many important algebraic K -theory calculations, and in the last few years there have been significant advances in this field. In this talk I will give an introduction to algebraic K -theory and its applications and talk about modern methods to compute

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algebraic K -theory groups. I will explain the role of equivariant homotopy theory in this story and describe my recent work in these areas.



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