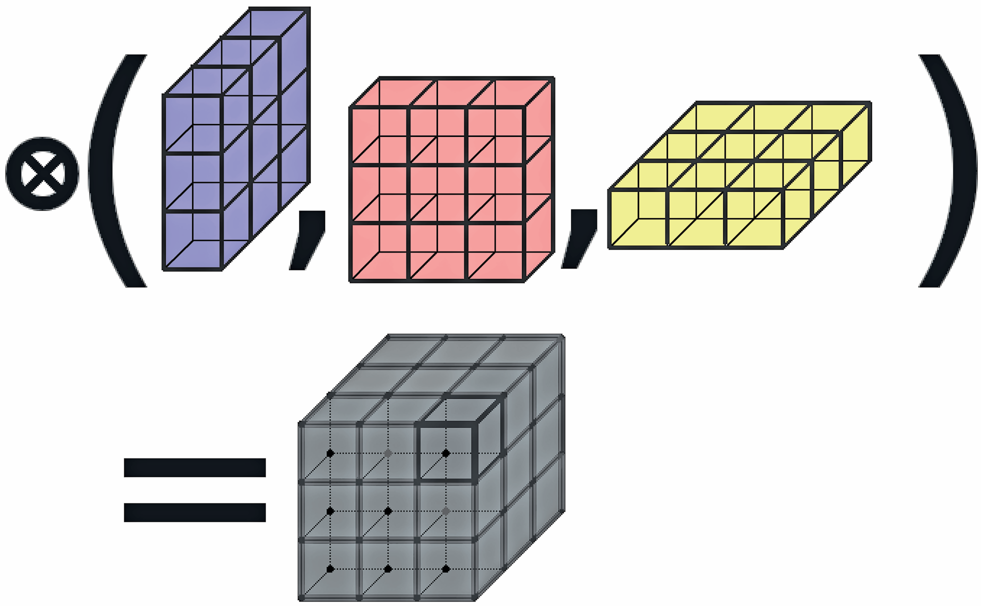
A Hypermatrix Analog of the General Linear Group



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Matrices are so ubiquitous and so deeply ingrained into our mathematical lexicon that one naturally asks: "Are there higher-dimensional analogs of matrices; more importantly why bother with them at all?" In short, hypermatrices are higher-dimensional matrices. Hypermatrices are important because they broaden the scope of matrix concepts such as spectra and group actions. Hypermatrix algebras also illuminate subtle aspects of matrix algebra. In this article we describe an instance where the transition from matrices to hypermatrices shines a light on subtle details of invariant theory and leads to new insights.

1. What are Hypermatrices?

Hypermatrices are multidimensional matrices. More formally a hypermatrix $\mathbf{H}$ is a function which maps a Cartesian product of finite subsets of consecutive integers to members of a field or a skew field $\mathbb{K}$:

$$\mathbf{H} : \{0, \dots, n_0 - 1\} \times \cdots \times \{0, \dots, n_{m-1} - 1\} \to \mathbb{K}.$$

Such a hypermatrix is of order m and of size $n_0 \times \cdots \times n_{m-1}$. Investigations on hypermatrices were initiated during the 19th century in Cayley's work on hyperdeterminants [Cay45]. To some extent, the history of hypermatrix investigations resembles the history of matrix investigations. The study of matrix and hypermatrix invariants preceded independent investigations of their algebras. The present note focuses on an invariant theory perspective on the transition from matrices to hypermatrices. There are numerous other approaches to generalize matrix results to hypermatrices. For instance, other approaches focus instead on discriminants, determinants, resultants, and spectra. [GKZ94, GF17, Qi05, DLDMV00, Tuc66, Lim05, Lim21]

2. Are Tensors Hypermatrices?

The distinction between tensors and hypermatrices closely mirrors the distinction between abstract linear transformations and their matrix representations. Recall that an abstract linear transformation specified over finite-dimensional $\mathbb{K}$-vector spaces has an orbit of possible matrix representations. For instance if $\mathbf{M} \in \mathbb{K}^{m \times n}$ represents

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an abstract linear transformation $T : V \to W$ with respect to bases $\mathcal{B}_V = \{\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}\}$ and $\mathcal{B}_W = \{\mathbf{w}_0, \ldots, \mathbf{w}_{m-1}\}$, then $\mathbf{XMY}^{-1} \in \mathbb{K}^{m\times n}$ represents the same linear transformation T with respect to new bases $\mathcal{B}'_V = \{\mathbf{Yv}_0, \ldots, \mathbf{Yv}_{n-1}\}$ and $\mathcal{B}'_W = \{\mathbf{Xw}_0, \ldots, \mathbf{Xw}_{m-1}\}$ for arbitrary invertible matrices $\mathbf{X}$ and $\mathbf{Y}$. Recall that the matrix product of $\mathbf{A} \in \mathbb{K}^{m\times n}$ with $\mathbf{B} \in \mathbb{K}^{n\times p}$ satisfies the equality

$$\left(\mathbf{XAY}^{-1}\right)\left(\mathbf{YBZ}^{-1}\right) = \mathbf{X}\left(\mathbf{AB}\right)\mathbf{Z}^{-1},$$

meaning that the matrix product is equivariant under the action of the general linear group. In other words, the matrix product is a well-defined operation on tensor products of pairs of vector spaces $S \in U \otimes V^*$ and $T \in V \otimes W^*$ such that $ST \in U \otimes W^*$, where V^* and W^* denote dual spaces to V and W, respectively. The First Fundamental Theorem of Invariant Theory [Pro07, LY13] asserts that it is impossible to specify a binary product whose operands are taken from tensor products of triplets of vector spaces $T_1 \in U_1 \otimes V_1 \otimes W_1$ and $T_2 \in U_2 \otimes V_2 \otimes W_2$ and outputs a member of a tensor product of a triplet of vector spaces. Later we describe a ternary product whose operands are taken from tensor products of triplets of vector spaces.

The *tensorial orbit* of an abstract linear transformation represented by $\mathbf{M} \in \mathbb{K}^{m\times n}$ (relative to arbitrarily chosen bases) is the subset of all matrices of the same size which account for all possible changes of bases and described by

$$\left\{\mathbf{AMB} \; : \; \begin{array}{c}\mathbf{A}, \mathbf{B} \text{ are respectively} \\ m\times m \text{ and } n\times n \text{ invertible matrices over } \mathbb{K}\end{array}\right\}.$$

For instance, the tensorial orbit of $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ whose entries are taken from the finite field with two elements denoted $\mathbb{F}_2$ is

$$\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \right.$$
$$\left.\begin{pmatrix}0&1\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}.$$

In particular, the tensorial orbit of a zero matrix is always a singleton, whose unique member is the zero matrix itself. Zero matrices are the only matrices whose tensorial orbits are singletons. Members of the tensorial orbit of the $n\times n$ identity matrix are all the elements of the general linear group over $\mathbb{K}$ denoted $\mathrm{GL}_n(\mathbb{K})$. It is comprised of all $n\times n$ invertible matrices whose entries are taken from the field or skew field $\mathbb{K}$. When investigating abstract linear transformations via their matrix representations, it is crucial to distinguish matrix properties which depend on the choice of coordinate system from properties which are independent of this choice.

A property common to all members of a tensorial orbit is called a *tensorial invariant*. Properties common to some specified suborbit of the tensorial orbit are also called tensorial invariants relative to the specified suborbit. For instance, the size of the tensorial orbit (i.e., the cardinality) of an abstract linear transformation prescribed over a finite field is a tensorial invariant. It is well known that there are $n+1$ distinct tensorial orbits over $\mathbb{C}^{n\times n}$. Each orbit collects $n\times n$ matrices having the same rank. In particular over $\left(\mathbb{F}_m\right)^{n\times n}$ (where $\mathbb{F}_m$ denotes the finite field with m elements), the largest orbit is $\mathrm{GL}_n(\mathbb{F}_m)$ and its size is $\prod_{0\le k<n}\left(m^n - m^k\right)$. Two other well-known matrix tensorial invariants include the rank and the nullity.

3. Hypermatrix Tensorial Invariant

Classical invariant theory is the study of properties of polynomials which are unaffected by a linear change of variables. Such properties are not tied to any specific basis for the dual space generating their coordinate ring [Olv99]. Classically, third-order hypermatrices admit tensorial orbits very similar to their matrix counterparts. Tensorial orbits of third-order hypermatrices are also devised via a change of basis. In other words, a member of the tensorial orbit of $\mathbf{A} \in \mathbb{K}^{m\times n\times p}$ is obtained by applying three separate invertible linear transformations to columns, rows, and depth vectors of $\mathbf{A}$, respectively. Each column, row, and depth vector of $\mathbf{A}$ lies in the three $\mathbb{K}$-vector spaces $\mathbb{K}^{m\times 1\times 1}$, $\mathbb{K}^{1\times n\times 1}$, and $\mathbb{K}^{1\times 1\times p}$, respectively. For instance, there are eight distinct tensorial orbits over $\left(\mathbb{F}_2\right)^{2\times 2\times 2}$ of sizes listed in increasing order:

$$1,\ 12,\ 18,\ 18,\ 18,\ 27,\ 54,\ 108.$$

Incidentally, hypermatrix tensorial invariants like the tensor rank are defined by analogy to their matrix counterparts. Technically, a tensor does not refer to any particular hypermatrix along its tensorial orbit but instead refers to the orbit as a whole. Unfortunately, in the majority of references which discuss applications of tensor decompositions, the word "tensor" is widely used to mean hypermatrix.

Around the early 1990s, D. M. Mesner and P. Bhattacharya [MB90, MB94] generalized association schemes to higher-order combinatorial structures. In the process, they pioneered a ternary hypermatrix product operation which generalizes the matrix product. The Bhattacharya–Mesner product opened up a path to investigations of new kinds of tensorial orbits defined over skew fields, which we call *non-classical tensorial orbits*. The main motivation for investigating non-classical tensorial orbits was to characterize invariants of linear actions prescribed over skew fields [GER11, GF17, GF20]. By contrast, the main motivation for investigating classical tensorial orbits is to determine invariants of linear actions prescribed over fields. Recall that an ordered matrix pair is conformable if the number of columns of the first matrix in the pair matches the number of rows in the second matrix of the pair. In other words, conformable pairs are matrix pairs for which the matrix product is meaningful in the specified order. The Bhattacharya–Mesner product naturally generalizes the matrix product to a ternary product operation defined for a conformable triplet of third-order hypermatrices

$$\mathbf{A} \in \mathbb{K}^{m\times\ell\times p},\ \mathbf{B} \in \mathbb{K}^{m\times n\times\ell},\ \text{and}\ \mathbf{C} \in \mathbb{K}^{\ell\times n\times p},$$

denoted $\text{Prod}(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathbb{K}^{m\times n\times p}$ with entries given by

$$\text{Prod}(\mathbf{A}, \mathbf{B}, \mathbf{C})[i, j, k] = \sum_{0\le t<\ell} \mathbf{A}[i, t, k]\,\mathbf{B}[i, j, t]\,\mathbf{C}[t, j, k]$$

$$\text{for all} \begin{cases} 0 \le i < m, \\ 0 \le j < n, \\ 0 \le k < p. \end{cases}$$

Accordingly, the transpose of $\mathbf{A} \in \mathbb{K}^{m\times n\times p}$ is denoted $\mathbf{A}^\top \in \mathbb{K}^{n\times p\times m}$, and permutes the entries of $\mathbf{A}$ by affecting a cyclic permutation of indices,

$$\mathbf{A}^\top[i, j, k] = \mathbf{A}[k, i, j]\ \text{for all} \begin{cases} 0 \le i < m, \\ 0 \le j < n, \\ 0 \le k < p. \end{cases}$$

For notational convenience we adopt the notation convention

$$\mathbf{A}^{\top^2} := \left(\mathbf{A}^\top\right)^\top,\ \mathbf{A}^{\top^3} := \left(\mathbf{A}^{\top^2}\right)^\top = \mathbf{A}.$$

When $\mathbb{K}$ is a field, the following transpose identity reminiscent of the matrix counterpart holds:

$$\text{Prod}(\mathbf{A},\ \mathbf{B},\ \mathbf{C})^\top = \text{Prod}(\mathbf{B}^\top,\ \mathbf{C}^\top,\ \mathbf{A}^\top).$$

When $m = n = p = 1$, $\text{Prod}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ corresponds to a third-order analog of an inner product. Its operands form a conformable triplet made of a row vector, a depth vector, and a column vector. In the case where $p = 1$, $\text{Prod}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ expresses the natural action of $\mathbf{B} \in \mathbb{K}^{m\times n\times\ell}$ on a pair of matrix depth slices (also called frontal slices) $(\mathbf{A}, \mathbf{C}) \in \mathbb{K}^{m\times\ell\times 1} \times \mathbb{K}^{\ell\times n\times 1}$. More precisely, in this case $\text{Prod}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ yields a matrix whose (i, j) entry is the 3-way inner product of the ith row of $\mathbf{A}$ (as a matrix), jth column of $\mathbf{C}$ (as a matrix), and the (i, j) line of $\mathbf{B}$ (a vector). Finally, when $\ell = 1$, $\text{Prod}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ corresponds to a third-order analog of an outer product. Its operands form a conformable triplet made of a column slice (i.e., an element of $\mathbb{K}^{m\times 1\times p}$), a depth slice (i.e., an element of $\mathbb{K}^{m\times n\times 1}$), and a row slice (i.e., an element of $\mathbb{K}^{1\times n\times p}$). The Bhattacharya–Mesner algebra is therefore a non-associative algebra with a ternary composition rule. Note that the Bhattacharya–Mesner outer product corresponds to the trilinear form associated with the well-known matrix multiplication tensor whose fundamental importance was discovered by Strassen [Str69]. More precisely, recall from Strassen's investigations of the arithmetic complexity of matrix multiplication over $\mathbb{F}_2$ [Str69] the trilinear form

$$\sum_{i,j,k} \mathbf{A}[i, k]\,\mathbf{B}[k, j]\,\mathbf{C}[i, j]\,.$$

We see that the hypermatrix which underlies the said trilinear form expresses an instance of the Bhattacharya–Mesner outer product. Naturally, the Bhattacharya–Mesner rank of $\mathbf{A} \in \mathbb{K}^{m\times n\times p}$ is the smallest number of Bhattacharya–Mesner outer-product summands which add up to $\mathbf{A}$. Recall that every outer product is a Bhattacharya–Mesner product instance where $\ell = 1$. More importantly, the Bhattacharya–Mesner rank of a third-order hypermatrix gives a lower bound on the number of ordinary matrix products required to simultaneously express its entries. In short, the Bhattacharya–Mesner rank gives a lower bound on the matrix multiplication complexity, a third-order hypermatrix. Hereafter unless otherwise specified, hypermatrix outer products will refer to the Bhattacharya–Mesner outer product (i.e., a product instance where $\ell = 1$).

In hindsight, the Bhattacharya–Mesner product compactly describes general systems of linear equations over skew fields such as the field of quaternions. For instance, if we wanted to express a general linear system of p equations with ℓ unknowns, for $\ell = 2$ and $p = 3$ we would have

$$\begin{cases} a_{000}\,x_{000}\,b_{000} & + & a_{010}\,x_{001}\,b_{100} & = & c_{000}, \\ a_{001}\,x_{000}\,b_{001} & + & a_{011}\,x_{001}\,b_{101} & = & c_{001}, \\ a_{002}\,x_{000}\,b_{002} & + & a_{012}\,x_{001}\,b_{102} & = & c_{002}. \end{cases} \tag{1}$$

Hence we could do so with hypermatrices

$$\mathbf{A} \in \mathbb{K}^{1\times 2\times 3},\ \mathbf{B} \in \mathbb{K}^{2\times 1\times 3},\ \text{and}\ \mathbf{c} \in \mathbb{K}^{1\times 1\times 3},$$

via a single constraint in the entries of the $1\times 1\times 2$ unknown $\mathbf{x}$ as

$$\text{Prod}(\mathbf{A},\ \mathbf{x},\ \mathbf{B}) = \mathbf{c}. \tag{2}$$

When $\mathbb{K}$ is a skew field the system described in (1) is not expressible as a system of equations of either form

$$\mathbf{Mx} = \mathbf{r}\ \text{ or } \mathbf{xM} = \mathbf{r},$$

where the matrix $\mathbf{M}$ collects all coefficients occurring in the system of equations. There are variants of the Gauss–Jordan elimination procedure which enables us to solve such general systems over skew fields [GF20, GR97, GR91, GR92].

We explain here how non-classical invariants depart from their classical counterparts. For this purpose, we draw attention to the simplest non-trivial illustration. Namely the determination of fundamental invariants of quadratic binary forms. Classically, a quadratic binary form is a homogeneous polynomial of degree 2 in commuting variables x_0 and x_1 of the form

$$p(x_0,\ x_1) = a_0 x_0^2 + a_1 x_0 x_1 + a_2 x_1^2,$$

where $a_0, a_1, a_2 \in \mathbb{C}$. Consider the natural action of elements of the special linear group

$$\mathrm{SL}_2(\mathbb{C}) := \left\{ \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} : q_{00}q_{11} - q_{10}q_{01} = 1 \right\}$$

on the polynomial $p(x_0, x_1)$ prescribed by the linear map

$$\begin{pmatrix} x_0 \\ \\ x_1 \end{pmatrix} \mapsto \begin{pmatrix} q_{00}\, x_0 + q_{01}\, x_1 \\ \\ q_{10}\, x_0 + q_{11}\, x_1 \end{pmatrix}.$$

This action maps $p(x_0, x_1)$ to the new polynomial

$$b_0 x_0^2 + b_1 x_0 x_1 + b_2 x_1^2,$$

where

$$\begin{cases} b_0 = a_0 q_{00}^2 + a_1 q_{00} q_{10} + a_2 q_{10}^2, \\ \\ b_1 = 2\, a_0 q_{00} q_{01} + a_1 q_{01} q_{10} + a_1 q_{00} q_{11} + 2\, a_2 q_{10} q_{11}, \\ \\ b_2 = a_0 q_{01}^2 + a_1 q_{01} q_{11} + a_2 q_{11}^2. \end{cases}$$

It is well known and easy to see that

$$a_1^2 - 4a_0 a_2 = b_1^2 - 4b_0 b_2$$

expresses an invariant of $p(x_0, x_1)$ to the action of $\mathrm{SL}_2(\mathbb{C})$. Perhaps less obvious is the fact that this invariant is the unique fundamental invariant of a quadratic binary form [Olv99]. Meaning that it generates all other invariants of $p(x_0, x_1)$ under the action of $\mathrm{SL}_2(\mathbb{C})$, much in the same way that the set of elementary symmetric polynomials in n variables generates the ring of symmetric polynomials in n variables.

In contrast with the classical setting, in non-classical settings the simplest quadratic binary forms are polynomials in non-commuting variables x_0, x_1 of the form

$$\begin{aligned} &P(x_0, x_1) \\ &= a_0 x_0 b_0 x_0 c_0 + a_1 x_0 b_1 x_1 c_1 + a_2 x_1 b_2 x_0 c_2 + a_3 x_1 b_3 x_1 c_3, \end{aligned}$$

where a_i, b_i, c_i are members of a skew field. In the non-classical setting, we seek invariants with respect to the general linear action prescribed by the map

$$\begin{pmatrix} x_0 \\ \\ x_1 \end{pmatrix} \mapsto \begin{pmatrix} u_{000}\, x_0\, v_{000} + u_{010}\, x_1\, v_{100} \\ \\ u_{001}\, x_0\, v_{001} + u_{011}\, x_1\, v_{101} \end{pmatrix}.$$

One recognizes at once from such a map the Bhattacharya–Mesner product of a conformable triplet made up of hypermatrices of sizes $1 \times 2 \times 2$, $1 \times 1 \times 2$, and $2 \times 1 \times 2$. The map is compactly described as

$$\mathbf{x} \mapsto \mathrm{Prod}(\mathbf{U},\ \mathbf{x},\ \mathbf{V}). \tag{3}$$

Unfortunately, very little is known about invariants of polynomials in non-commutative variables. As an initial step in the study of non-classical invariants, we investigate non-commutative invariants of linear forms. Such investigations suggest distinctive third-order analogues of general linear groups. But first we briefly mention an alternative formulation of the eigenvalue/eigenvector problem over skew fields suggested by (3). In this context, the eigenvalue/eigenvector problem is specified via a pair of matrix slices

$$\mathbf{A} \in \mathbb{K}^{1 \times \ell \times p} \text{ and } \mathbf{B} \in \mathbb{K}^{\ell \times 1 \times p},$$

by an equation of the form

$$\mathrm{Prod}(\mathbf{A},\ \mathbf{v},\ \mathbf{B}) = \lambda_{\mathrm{L}}\, \mathbf{v}\, \lambda_{\mathrm{R}}.$$

The vector $\mathbf{v}$ of size $1 \times 1 \times \ell$ is the eigenvector, while scalars from the skew fields λ_{L} and λ_{R} denote the left and right eigenvalues, respectively.

4. Third-Order Analog of the General Linear Group

We now describe a third-order analog of invertible matrices. Recall that the general linear group over an arbitrary field $\mathbb{K}$ (possibly a skew field) is defined as follows:

$$\mathrm{GL}_m(\mathbb{K}) := \left\{ \begin{array}{c} \mathbf{A} \in \mathbb{K}^{m \times m} : \exists\, \mathbf{B} \in \mathbb{K}^{m \times m} \\ \text{subject to} \\ \mathbf{B}\,\mathbf{A}\,\mathbf{X} = \mathbf{X} \text{ for all } \mathbf{X} \in \mathbb{K}^{m \times n} \end{array} \right\}. \tag{4}$$

We preface our description of a third-order analog of $\mathrm{GL}_m(\mathbb{K})$ by emphasizing that when $\mathbb{K}$ is a skew field, general linear maps such as the one described in (3) are specified via a pair of matrix slices

$$\mathbf{A} \in \mathbb{K}^{1 \times \ell \times p} \text{ and } \mathbf{B} \in \mathbb{K}^{\ell \times 1 \times p}$$

by a map of the form

$$\mathbf{x} \mapsto \mathrm{Prod}(\mathbf{A},\ \mathbf{x},\ \mathbf{B}).$$

Since entries of $\mathbf{A}$ exclusively left-multiply entries of $\mathbf{x}$, while entries of $\mathbf{B}$ exclusively right-multiply entries of $\mathbf{x}$, one anticipates that the inverse map is also specified via a similar pair of matrix slices. Bearing in mind this subtle detail and taking into account the ternary nature of the Bhattacharya–Mesner product, we define a third-order analog of the general linear group by analogy to (4) as follows:

$$\mathrm{GL}_{m \times n \times p}(m \times p \times p,\ p \times n \times p,\ \mathbb{K}) :=$$
$$\left\{ \begin{array}{c} (\mathbf{A}, \mathbf{B}) \in \mathbb{K}^{m \times p \times p} \times \mathbb{K}^{p \times n \times p} : \exists\, (\mathbf{C}, \mathbf{D}) \\ \in \mathbb{K}^{m \times p \times p} \times \mathbb{K}^{p \times n \times p} \text{ subject to} \\ \mathrm{Prod}(\mathbf{C}, \mathrm{Prod}(\mathbf{A}, \mathbf{X}, \mathbf{B}), \mathbf{D}) = \mathbf{X} \text{ for all } \mathbf{X} \in \mathbb{K}^{m \times n \times p} \end{array} \right\}. \tag{5}$$

In short, third-order analogues of $\mathrm{GL}_n(\mathbb{K})$ are defined in terms of invertible hypermatrix pairs. Two pairs of hypermatrices are said to be *hypermatrix inverse pairs* $(\mathbf{A}, \mathbf{B}) \in \mathbb{K}^{m \times p \times p} \times \mathbb{K}^{p \times n \times p}$ and $(\mathbf{C}, \mathbf{D}) \in \mathbb{K}^{m \times p \times p} \times \mathbb{K}^{p \times n \times p}$ if

$$\mathrm{Prod}(\mathbf{C}, \mathrm{Prod}(\mathbf{A}, \mathbf{X}, \mathbf{B}), \mathbf{D}) = \mathbf{X} \text{ for all } \mathbf{X} \in \mathbb{K}^{m \times n \times p}. \tag{6}$$

A computer search reveals that there are 6666 distinct such inverse pairs $(\mathbf{A}, \mathbf{B})$, $(\mathbf{C}, \mathbf{D}) \in (\mathbb{F}_2)^{2 \times 2 \times 2} \times (\mathbb{F}_2)^{2 \times 2 \times 2}$ subject to (6). Whereas, conformability constraints in the matrix setting restrict invertibility to square matrices, conformability constraints for third-order hypermatrices afford us more flexibility in the sizes of invertible pairs of hypermatrices as seen in (5). Note that over any field or skew field, there are subsets of invertible pairs of hypermatrices which form a group defined in terms of the Bhattacharya–Mesner product. In other words, for members of such subsets, we have

$$(\mathbf{C}, \mathbf{D}), (\mathbf{A}, \mathbf{B}) \in \mathbb{K}^{m\times p\times p} \times \mathbb{K}^{p\times n\times p} \quad \text{and } (\mathbf{U}, \mathbf{V}) \in \mathbb{K}^{m\times p\times p} \times \mathbb{K}^{p\times n\times p}.$$

We define a group composition law as follows:

$$\begin{gathered}(\mathbf{C}, \mathbf{D}) \circ (\mathbf{A}, \mathbf{B}) = (\mathbf{U}, \mathbf{V}) \Leftrightarrow \\ \mathrm{Prod}(\mathbf{C}, \mathrm{Prod}(\mathbf{A}, \mathbf{X}, \mathbf{B}), \mathbf{D}) = \mathrm{Prod}(\mathbf{U}, \mathbf{X}, \mathbf{V}) \\ \text{for all } \mathbf{X} \in \mathbb{K}^{m\times n\times p}.\end{gathered}$$

The pair $(\mathbf{U}, \mathbf{V})$ is always uniquely determined by the pairs $(\mathbf{A}, \mathbf{B})$ and $(\mathbf{C}, \mathbf{D})$. The simplest illustration of such a group is given by a third-order analog of the group of invertible diagonal matrices, called *scaling hypermatrices.* A pair $(\mathbf{A}, \mathbf{B}) \in \mathbb{K}^{m\times p\times p} \times \mathbb{K}^{p\times n\times p}$ is an invertible pair of scaling hypermatrices if their entries are such that

$$\mathbf{A}[i, t, k] = \begin{cases} \alpha_{it} \in \mathbb{K}\backslash\{0\} & \text{if } 0 \le t = k < p, \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } \begin{cases} 0 \le i < m, \\ 0 \le t < p, \\ 0 \le k < p, \end{cases}$$

$$\mathbf{B}[t, j, k] = \begin{cases} \beta_{tj} \in \mathbb{K}\backslash\{0\} & \text{if } 0 \le t = k < p, \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } \begin{cases} 0 \le t < p, \\ 0 \le j < n, \\ 0 \le k < p. \end{cases}$$

If $(\mathbf{A}, \mathbf{B}) \in \mathbb{K}^{m\times p\times p} \times \mathbb{K}^{p\times n\times p}$ is a scaling hypermatrix pair, then

$$\forall \mathbf{X} \in \mathbb{K}^{m\times n\times p}, \ \mathrm{Prod}(\mathbf{A}, \mathbf{X}, \mathbf{B})[i, j, k] = \alpha_{ik}\, \mathbf{X}[i, j, k]\, \beta_{kj} \quad \text{for all } \begin{cases} 0 \le i < m, \\ 0 \le j < n, \\ 0 \le k < p. \end{cases}$$

The entry constraints for the pair $(\mathbf{A}, \mathbf{B}) \in \mathbb{K}^{m\times p\times p}\times\mathbb{K}^{p\times n\times p}$ described above are similar to requiring in the matrix case that all diagonal entries be the only non-vanishing entries. Let $(\mathbf{C}, \mathbf{D}) \in \mathbb{K}^{m\times p\times p} \times \mathbb{K}^{p\times n\times p}$ denote the inverse pair of some given pair of scaling hypermatrices $(\mathbf{A}, \mathbf{B}) \in \mathbb{K}^{m\times p\times p}\times\mathbb{K}^{p\times n\times p}$. Then the entries of $(\mathbf{C}, \mathbf{D})$ are obtained by simply inverting all non-zero entries of $(\mathbf{A}, \mathbf{B})$ respectively as follows:

$$\mathbf{C}[i, t, k] = \begin{cases} \alpha_{it}^{-1} & \text{if } 0 \le t = k < p, \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } \begin{cases} 0 \le i < m, \\ 0 \le t < p, \\ 0 \le k < p, \end{cases}$$

$$\mathbf{D}[t, j, k] = \begin{cases} \beta_{tj}^{-1} & \text{if } 0 \le t = k < p, \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } \begin{cases} 0 \le t < p, \\ 0 \le j < n, \\ 0 \le k < p. \end{cases}$$

More generally,

$$\forall\ (\mathbf{U}, \mathbf{V}) \in \mathrm{GL}_{m\times n\times p}(m \times p \times p,\ p \times n \times p,\ \mathbb{K})$$

$\mathrm{Prod}(\mathbf{U}, \mathbf{H}, \mathbf{V})$ expresses the hypermatrix action of $(\mathbf{U}, \mathbf{V})$ onto $\mathbf{H}$, and the inverse of $(\mathbf{U}, \mathbf{V})$ simply undoes this action. Moreover, such actions are akin to a change of basis of sort. Since the Bhattacharya–Mesner product is non-associative (in reference here to ternary composition trees of products), there are six distinct non-classical tensorial orbits for a given $\mathbf{H} \in \mathbb{K}^{m\times n\times p}$. One of which is given by

$$\left\{ \begin{array}{c} \mathrm{Prod}(\mathbf{Q}^{\top^2}, \mathbf{P}^{\top^2}, \mathrm{Prod}(\mathrm{Prod}(\mathbf{U}, \mathbf{H}, \mathbf{V}), \mathbf{F}^{\top}, \mathbf{E}^{\top})) \\ \text{such that} \\ (\mathbf{U}, \mathbf{V}), (\mathbf{E}, \mathbf{F}), (\mathbf{P}, \mathbf{Q}) \\ \in \mathrm{GL}_{m\times n\times p}(m \times p \times p,\ p \times n \times p,\ \mathbb{K}) \end{array} \right\}.$$

A hypermatrix property shared by all members of a non-classical tensorial orbit is called a *non-classical tensorial invariant.* A simple illustration of a non-classical tensorial invariant is the orbit size over a finite field. For instance, over $(\mathbb{F}_2)^{2\times2\times2}$, every non-zero hypermatrix lies within the same non-classical tensorial orbit of size $(2^8 - 1)$.

5. Third-Order Analog of the Fundamental Theorem of Linear Algebra

The conventional tensor rank (classically defined as the smallest number of vector outer-product summands required to add up to a given hypermatrix) is a well-known classical invariant. Unfortunately, relative to this notion of rank, the Fundamental Theorem of Linear Algebra (FTLA) (also known as the rank-nullity theorem) is known not to hold for hypermatrices of order greater than 2 [Lan17]. The Bhattacharya–Mesner rank [GF20] is a non-classical invariant to the action of invertible hypermatrix pairs. In many respects, the Bhattacharya–Mesner rank is similar to the matrix rank. For instance, for all $\mathbf{A} \in \mathbb{K}^{m\times n\times p}$

$$(\text{Bhattacharya–Mesner rank of } \mathbf{A}) \le \min(m, n, p).$$

The Bhattacharya–Mesner rank also yields a generalization of FTLA. We recall from [GF20] that the nullity of

$\mathbf{A} \in \mathbb{C}^{m\times n\times p}$ is defined to be the maximum number of all zero frontal slices (i.e., a frontal slice consisting solely of zero entries) appearing in $\mathrm{Prod}(\mathbf{U},\mathbf{A},\mathbf{V})$, where

$$(\mathbf{U},\mathbf{V}) \in \mathrm{GL}_{m\times n\times p}(m\times p\times p,\ p\times n\times n,\ \mathbb{K}).$$

This definition is a third-order analog of the matrix nullity of $\mathbf{M} \in \mathbb{K}^{n\times n}$ defined as the maximum number of zero columns appearing in $\mathbf{MQ}$, where $\mathbf{Q} \in \mathrm{GL}_n(\mathbb{K})$. Incidentally, the third-order analog of the FTLA [GF20] relates the nullity to the Bhattacharya–Mesner rank.

Theorem. *Given any $\mathbf{A} \in \mathbb{C}^{m\times n\times p}$, where $p = min(m,n,p)$, then $\mathbf{A}$ has nullity $(p-r)$ if and only if the Bhattacharya–Mesner rank of $\mathbf{A}$ is $r \le p$.*

The Bhattacharya–Mesner rank also suggests a generalization of the notion of linear independence. Fortunately, this generalization is easily described using ordinary matrices. A matrix subset $\{\mathbf{M}_t : 0 \le t < p\}$ is *left-right diagonally dependent* if we can solve for diagonal matrix sets

$$\{\mathbf{D}_t : 0 \le t \ne \tau < p\} \subset \mathbb{K}^{m\times m} \quad \text{and} \quad \{\mathbf{D}'_t : 0 \le t \ne \tau < p\} \subset \mathbb{K}^{n\times n}$$

such that

$$\mathbf{M}_\tau = \sum_{0\le t\ne\tau<p} \mathbf{D}_t\,\mathbf{M}_t\,\mathbf{D}'_t. \tag{7}$$

Over any field, if each diagonal matrix in the sets $\{\mathbf{D}_t : 0 \le t < p\}$ and $\{\mathbf{D}'_t : 0 \le t < p\}$ is of the form

$$\mathbf{D}_t = \alpha_t \mathbf{I}_m \ \text{ and } \ \mathbf{D}'_t = \beta_t \mathbf{I}_n, \text{ where } \alpha_t, \beta_t \in \mathbb{K} \quad \forall\, t \in \{0,1,\dots,p-1\},$$

then the condition described by (7) is equivalent to the usual notion of linear dependence. The condition described by (7) reduces to the usual notion of linear dependence when each matrix in the set $\{\mathbf{M}_t : 0 \le t < p\}$ has at most one non-zero entry. In the latter setting the sharp upper bound on the maximum number of left-right diagonally independent matrices is $m \cdot n$. Alternatively, when every column of each matrix in the set $\{\mathbf{M}_t : 0 \le t < p\}$ is non-zero, the trivial upper bound on the maximum number of left-right diagonally independent matrices reduces to $\min(m,n)$ by dimension arguments applied to the column spaces of the matrices. More generally, for an arbitrary matrix subset $\{\mathbf{M}_t : 0 \le t < p\}$, the maximum number of diagonally independent matrices when $n = m = 2$ is equal to 2. Surprisingly, when $m = n > 2$, the maximum number of left-right diagonally independent matrices is equal to $(n-1)$ [GF20] which is less than the $\min(m,n)$ upper bound. Moreover, a third-order hypermatrix $\mathbf{A} \in \mathbb{C}^{m\times n\times p}$ (where $r \le p = \min(m,n,p)$) is full rank if and only if its matrix frontal slices are left-right diagonally independent [GF20]. Over skew fields, determining the maximum number of left-right diagonally independent matrices is an important open problem because it is a crucial first step towards the characterization of invariants of polynomials of degree 1 in non-commutative variables.

6. Concluding Remarks

We have tried to make the case that hypermatrices offer a natural extension to the general linear group. The Bhattacharya–Mesner ternary algebra is close enough to the algebra of matrices to offer a magnifying glass for investigating subtle details of invariant theory. In particular, the Bhattacharya–Mesner algebra suggests a natural and fruitful generalization of the Fundamental Theorem of Linear Algebra [GF20] as well as a generalization of the Spectral Theorem [GF17, GER11]. Subsequent insights stemming from a deeper understanding of hypermatrix algebras are likely to suggest new techniques in invariant theory and closely related fields, such as harmonic analysis *and* combinatorics.

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