

C. S. Seshadri (1932–2020)

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and David Mumford*

An Overview of the Life and Work of C. S. Seshadri

Vikraman Balaji

1. Introduction

Conjeevaram Srirangachari Seshadri was born on February 29, 1932, in Kanchipuram, a small temple town west of Chennai. Among the prominent temples in Kanchipuram are the Varadharaja Perumal Temple for Vishnu as well as the Ekambaranatha Temple which is the *prithivi kshetra* or *earth abode* of Shiva. Seshadri was the eldest among twelve children of his parents, Sri C. Srirangachari (a well-known advocate in Chengleput, a town 60 kms south of Chennai) and Srimati Chudamani. Seshadri's entire schooling was in Chengleput. He joined Loyola College, Chennai in 1948. M. S. Narasimhan, his long-term friend and collaborator, also joined Loyola the very same year. Seshadri graduated from there in 1953 with a BA (Hons) degree in mathematics. Seshadri married Sundari in 1962. They had two sons, Narasimhan (who lives in Zürich with his wife, daughter, and son) and Giridhar (who lives in Chennai with his wife and two sons).

During his years at college, S. Narayanan and Father C. Racine played a decisive role in Seshadri's taking up mathematics as a profession. Fr. Racine was a Jesuit priest who had worked under Élie Cartan in Paris. Seshadri

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Communicated by Notices Associate Editor Steven Sam.

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DOI: <https://doi.org/10.1090/noti2383>

joined the Tata Institute of Fundamental Research, Mumbai as a student in 1953. He received his PhD from the Bombay University in 1958 for his thesis entitled "Generalised multiplicative meromorphic functions on a complex manifold." His thesis adviser was Professor K. Chandrasekaran who shaped the mathematical career of Seshadri as he did for many others. Seshadri spent the years 1957–60 in Paris, where he came under the influence of many great mathematicians of the French school, like Chevalley, Cartan, Schwartz, Grothendieck, and Serre.

He was a member of the faculty of the School of Mathematics at the Tata Institute from 1960 until 1984, where he was responsible for establishing an active school of algebraic geometry. He moved to the Institute of Mathematical Sciences, Chennai in 1984. In 1989, Seshadri became the director of the Chennai Mathematical Institute, which was then called the SPIC Mathematical Institute, founded by A. C. Muthiah.

Seshadri was a recipient of numerous distinctions. He was an invited speaker at the Nice ICM in 1970. He received the Bhatnagar Prize in 1972 and was elected a fellow of the Royal Society, London in 1988. He held distinguished positions in various centres of mathematics all over the world. In 2006, Seshadri was awarded the TWAS Science Prize along with Jacob Palis for his distinguished contributions to science. He received the National Professorship, the H. K. Firodia Award for Excellence in Science and Technology, Pune, 2008, the Rathindra Puraskar from Shantiniketan's Visva-Bharati University, Kolkata, 2008, and the Padma Bhushan by the President of India, 2009. He was elected a Foreign Associate of the US National Academy of Sciences, 2010. In 2013, Seshadri was awarded Docteur Honoris Causa of the Université Pierre et Marie Curie in Paris.

2. Seshadri's Mathematics in Brief Outline

Seshadri's work is extensive and I will limit myself to certain aspects of his contributions to moduli, GIT, and quotients.

Seshadri's doctoral thesis entitled "Generalised multiplicative meromorphic functions on a complex manifold" gave an independent proof of the so-called Birkhoff-Grothendieck theorem on vector bundles on the projective line. Grothendieck proved this for arbitrary principal bundles, while the result on vector bundles goes back to Dedekind and R. and H. Weber (1882) ("Theorie der algebraischen Functionen einer Veränderlichen," *J. Reine Angew. Math.* 92, 181–290; for a historical outline see W. Scharlau, "Some remarks on Grothendieck's theorem 'Sur la classification de fibrès holomorphes sur la sphere de Riemann'").

Seshadri shot to fame early during his visit to Paris. Serre had posed the following problem: *Over n -dimensional affine space, are there non-trivial vector bundles?* In other words, is the following statement true? Is any projective Noetherian module over $K[T_1, \dots, T_n]$, where K is a field, free? For $n = 1$, the ring $K[T]$ is an integral principal ideal ring. Therefore, any Noetherian torsion-free $K[T]$ -module (in particular, any projective Noetherian module) is free (see for example Lang's *Algebra*). For $n = 2$, there are no non-trivial bundles either and this theorem is due to Seshadri ("Triviality of vector bundles over the affine space K^n ," *Proc. Natl. Aca. Sci. USA* 44 (1958), 456–458).

The affirmative answer for all n (i.e., any projective Noetherian $K[T_1, \dots, T_n]$ -module is free) is due to A. Suslin ("Projective modules over polynomial rings are free," *Dokl. Akad. Nauk SSSR* (in Russian), translated in *Soviet Math.*, 17 (1976), no. 4, 1160–1164) and independently due to D. Quillen ("Projective modules over polynomial rings," *Invent. Math.* 36 (1976), no. 1, 167–171). Much work arose inspired by Seshadri's ideas, beginning with the work of Pavaman Murthy (Seshadri's first doctoral student; "Projective $A[x]$ -modules," *J. London Math. Soc.* 41 (1966), 453–456) and one by Murthy-Towber ("Algebraic vector bundles over \mathbb{A}^3 are trivial," *Invent. Math.* 24 (1974), 173–189). This subsequently led to a large body of very impressive work from the Tata Institute.

Around the time that Seshadri finished his doctoral work, algebraic geometry itself was undergoing a unique revolution in the hands of Alexander Grothendieck. Seshadri went to Paris in 1957 and very quickly entered the sanctum of this new temple of algebraic geometry. André Weil had published striking conjectures linking number theory to topology, and these were among the principal driving forces behind the renaissance in algebraic geometry in the hands of Chevalley, Serre, Grothendieck, and

several others. Seshadri gave three talks in the Chevalley seminar on Picard varieties, two of which were on divisors in algebraic geometry and the third on Cartier operations.

It is in this setting that we can view one of Seshadri's deepest contributions, which began in collaboration with his friend and colleague M. S. Narasimhan. This work had its roots in results of André Weil on "generalization of Abelian functions" (1938). I quote from the paper of Atiyah-Bott (1982):

The connection between holomorphic and unitary structures was already apparent in Weil's paper, and in the classical case of line bundles it is essentially equivalent to the identification between holomorphic and harmonic 1-forms, which in turn was the starting point for Hodge's general theory of harmonic forms.

The Narasimhan-Seshadri theorem sets up a correspondence between two basic classes of objects, namely "irreducible unitary representations of fundamental groups of Riemann surfaces" and "a class of geometric objects called stable vector bundles on algebraic curves." The first class of objects was what could be termed "topological" and the second "algebraic-geometric." Its foundations were subtly linked to the approach of Poincaré and Klein to the "uniformization theorem" by the so-called "method of continuity." Let me be a bit more precise.

The non-abelian version of the Jacobian needed to tackle several basic obstructions. The first paper of Narasimhan-Seshadri [NS64] handled the non-abelian structure of the fundamental group of X of genus bigger than one. They showed that the space of irreducible unitary representations of the fundamental group of a compact Riemann surface X was naturally a complex manifold.

On the side of bundles, the basic issue was that any decent topology on the space of bundles of a given degree and rank is necessarily *non-separated*. Moreover, such bundles are not "bounded," i.e., they cannot be parametrized by a finite number of varieties. Thus to obtain a "projective moduli space," one has to restrict oneself to a suitable subclass of bundles which would give separatedness.

In the early 60s, David Mumford had revived and newly built the Geometric Invariant Theory (GIT) of Hilbert into a superstructure, laying out strategies for construction of "compactifications of moduli problems." One of his examples where he applied GIT was in constructing moduli spaces of bundles. His GIT naturally gave him the precise subclass; he defined slope-stability of bundles and constructed the moduli of stable vector bundles of degree d and rank r on curves as a quasi-projective variety. Recall that for every vector bundle V on a smooth projective curve, Mumford defines the slope $\mu(V) := \deg(V)/\text{rank}(V)$

and calls a bundle *stable* if the slope μ strictly decreases when we restrict to a proper subbundle.

Narasimhan and Seshadri showed that irreducible unitary representations of the fundamental group of X correspond precisely to stable bundles of degree 0 on X [NS65]. These were two of the scripts of a trilingual inscription à la the Rosetta stone, the third script came up in the work of Donaldson in 1986. What followed was spectacular. Many subtle and beautiful aspects of differential geometry, topology, mathematical physics, and number theory got unravelled miraculously.

Narasimhan and Seshadri's paper does a bit more. They showed how this can be extended to the case when the degree need not be zero. This case was a precursor to "parabolic bundles" which Seshadri later developed firstly in "Moduli of π -vector bundles over an algebraic curve," (Proc. C.I.M.E. Session, Varenna, 1969) and then later along with Mehta [MS80]. Mehta and Seshadri prove the analogue of the Narasimhan-Seshadri theorem for unitary representations of more general Fuchsian groups by relating these to parabolic bundles on X . Recently, in a paper which appeared in 2015 [BS15], Seshadri and I completed the picture by setting up the correspondence for homomorphisms of these Fuchsian groups to the maximal compact subgroups of semisimple groups. Torsors under parahoric Bruhat-Tits group schemes are the new objects which replace parabolic bundles.

In one of Seshadri's not too well-known papers ("Desingularization of moduli varieties of vector bundles on curves," *Proceedings of the International Symposium on Algebraic Geometry*, Kyoto University, 1977, M. Nagata (ed.), Kinokuniya, Tokyo, 1978), we find a brilliant application of parabolic bundles. Seshadri constructed a moduli-theoretic desingularization of the moduli space of rank 2 and degree 0 bundles on curves of genus bigger than 1. Narasimhan and Ramanan had constructed a desingularization for the rank 2 case by an entirely different approach around the same time.

Seshadri's construction was sufficiently moduli theoretic since his space represented a nice moduli functor and could be defined over any characteristic and for all ranks. By an elegant formal smoothness argument, Seshadri reduced the smoothness question of his moduli space to one for a certain canonically defined subscheme of the variety of algebra structures on a vector space of dimension n^2 . The subscheme was the closure of all algebra structures isomorphic to the matrix algebra of $n \times n$ -matrices. In this paper, one can perceive his uncanny insight into moduli problems. His candidate for the desingularization was the moduli space of "isomorphism classes" of semistable vector bundles of rank n^2 whose endomorphisms lie in the scheme of algebra structures which are the limits of the

matrix algebra. In a somewhat mysterious fashion, he realizes that objects in this moduli space can be given natural parabolic structures and using this he proves the properness of his moduli functor. It seemed that parabolic structures revealed themselves to Seshadri almost mystically!

I now take up two papers of Seshadri, the first entitled "Some results on the quotient space by an algebraic group of automorphism" (*Math. Ann.* 149 (1963), 286–301, and the second being "Quotient spaces module reductive algebraic groups" (*Ann. of Math.* 95 (1972), no. 3, 511–556), to which I will return later. The aspect that I highlight here is somewhat general and does not really require the group to be reductive or even affine.

If X is a scheme on which a connected algebraic group acts properly, then does the geometric quotient X/G exist as an algebraic space? It is known that the question as stated above fails in general, but Seshadri gave some basic criteria under which it holds. He proved the following theorem. Let X be a normal scheme of finite type (or more generally a normal algebraic space of finite type over k) and G a connected affine algebraic group acting properly on X . Then the geometric quotient X/G exists as a normal algebraic space of finite type.

Recall that a G -morphism $f : X \rightarrow Y$ is called a *good quotient* if (1) f is a surjective affine G -invariant morphism, (2) $f_*(\mathcal{O}_X)^G = \mathcal{O}_Y$, and (3) f sends closed G -stable subsets to closed subsets and separates disjoint closed G -stable subsets of X . The quotient f is called a *geometric quotient* if it is a good quotient and moreover for each $x \in X$, the G -orbit $G \cdot x$ is closed in X . Observe that when the action is proper, a geometric quotient is simply a topological quotient with the property $f_*(\mathcal{O}_X)^G = \mathcal{O}_Y$.

Seshadri developed the important technique of *elimination of finite isotropies* which goes as follows. Let X be an irreducible excellent scheme over k and G an affine algebraic group acting properly on X . Then there is a diagram:

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ \downarrow p & & \\ Z & & \end{array}$$

where Y is irreducible and G acts properly on Y . Further, p is a Zariski locally trivial principal G -bundle and q a finite dominant G -morphism with Y/X Galois with Galois group Γ whose action on Y commutes with the G -action. In a sense completing the square or constructing a push-out is the goal. These ideas are central to the major developments by Kollar and Keel and Mori on "Quotients" in the 90s.

I now come to Seshadri's contributions to Geometric Invariant Theory (GIT). In his paper "Space of unitary vector bundles on a compact Riemann surface" (*Ann. of Math.*

85 (1967), 303–335) Seshadri relates unitary bundles to the natural compactification arising from GIT construction of the moduli space. This is important as much of the subsequent work revolves around the study of compact moduli spaces. Irreducible unitary bundles are the “simple” objects of this theory. Two bundles are S -equivalent if they have the same Jordan-Hölder decomposition. The points of the compact moduli space are then the S -equivalence classes of bundles of degree zero and rank r . The beautiful convergence of two lines of thought about these fundamental concepts of (semi)stable bundles was well expressed by Mumford on the occasion of Seshadri’s seventieth birthday:

“But I guess what thrilled both of us—it certainly thrilled me—was when our work on vector bundles on curves arrived at the same idea from two such different directions. What a strange thing it was that three people (you, me, M.S.) on *opposite* sides of the world (which, by the way, seemed a lot bigger in those days) using *totally* different techniques should construct the *same* compact moduli space.”

I will very briefly touch on a few other papers of Seshadri in the subject of GIT. This will give a feeling for the breadth and depth of his contributions. The first one was Mumford’s conjecture for $GL(2)$ which, apart from proving the conjecture, gave a restricted “valuative criterion” which predates the famous Langton criterion.

He needed to show what he called the “covariant” from R^{ss} (the open subset of semistable bundles in the Quot scheme) to a product of Grassmannians is an imbedding and the central issue was the properness of the covariant. Seshadri introduces a new amazing device, namely, a multiple-valued mapping, to prove the properness.

This approach of Seshadri’s became the standard prototype for all moduli constructions, the most general one being the one by Simpson in the early 90s.

Let me define geometric reductivity of a group G . Let G be a reductive algebraic group over an algebraically closed field k . Then G is geometrically reductive if, for every finite-dimensional rational G -module V and a G -invariant point $v \in V$, $v \neq 0$, there is a G -invariant homogeneous polynomial F on V of positive degree such that $F(v) \neq 0$.

Mumford’s conjecture on reductive groups states the following: *reductive algebraic groups are geometrically reductive*. This was first proved for the case of $SL(2)$ (hence $GL(2)$) in characteristic 2 by Tadao Oda, and in all characteristics by Seshadri (“Mumford’s conjecture for $GL(2)$ and applications,” Proc. Internat. Colloquium on Algebraic Geometry, Bombay, 1968, 347–371). W. Haboush proved the conjecture for a general reductive G in 1974 (“Reductive

groups are geometrically reductive,” *Ann. of Math. (2)* 102 (1975), 67–83). Haboush’s proof uses the irreducibility of the Steinberg representation in an essential way. A germ of this idea can perhaps be traced back to the appendix to Seshadri’s paper by Raghunathan!

There is also a different approach to the problem due to Formanek and Procesi (“Mumford’s conjecture for the general linear group,” *Adv. Math.* 19 (1976)), which is *à priori* for the full linear group, but the general reductive case can be deduced from this. In the late 70s, Seshadri finally extended geometric reductivity over general excellent rings, which is now a basic tool for constructing moduli in mixed characteristics.

Seshadri’s paper on “Quotients modulo reductive groups” which has already been referred to, has several beautiful ideas. He introduces the notion of “ G -properness” which under some simple conditions shows that quotients, if they exist, are “proper and separated.” One of the basic results in this paper is the following: *Let X be a projective variety on which there is given an action of a reductive algebraic group G with respect to an ample line bundle L on X . Let X^{ss} and X^s denote, respectively, the semistable locus and the stable locus of the action of G on (X, L) . Suppose that X is normal, $X^{ss} = X^s$, and G acts freely on X . Then the geometric quotient X^s/G exists as a normal projective variety*. Loosely put, this is Mumford’s conjecture when “semistable = stable.” Seshadri then gives a general technique to ensure the condition $X^{ss} = X^s$ can be made to hold. These have played a central role in several subsequent developments.

In the late 60s, Seshadri was keen to prove the general Mumford conjecture using a geometric approach. Loosely put, this amounted to showing that the set Y of equivalence classes of semistable points for a linear action of G on a projective scheme X gets a canonical structure of a projective scheme. The first difficulty is getting a natural scheme-theoretic structure on Y . The second one, which is more difficult, is to prove its projectivity. When “stable = semistable” Seshadri showed that Y is a proper scheme and the proof reduces to checking the Nakai-Moishezon criterion for L on Y . *This process led to Seshadri’s ampleness criterion and Seshadri constants*.

Sometime around 2009, Seshadri and Pramath Sastry completed Seshadri’s old argument [SS11]. The key new ingredient (which was the work of Sean Keel) was to prove that under some conditions, line bundles which are “nef” and “big” are semiample. It was a recursive property for “nef” line bundles to become semiample, in a sense a sort of “Nakai-Moishezon” for semiamplicity.

I will now very briefly touch on Seshadri’s important contributions to the study of bundles on stable curves and the problem of compactifications. In his long paper with Tadao Oda, Seshadri studies the problem of

compactifications of the Picard variety. The new insight in this work was the surprising fact that for compactifying the moduli space of line bundles on stable curves, one needs to introduce “polarizations” on the stable curve and this in turn compels the introduction of “semistability” of rank 1 torsion-free sheaves ([D’S79] and [OS79]). The paper of Oda-Seshadri has seen significant developments in the hands of Caporaso and others in the recent past. Seshadri then goes on to construct the compactification of vector bundles on singular curves in his *Asterisque* volume 96 of 1983. In collaboration with D. S. Nagaraj ([NS97], [NS99]), Seshadri made significant progress in the general problem of compactifications with “normal crossing singularities,” generalizing the work of Gieseker who had done it earlier for the rank 2 case. This approach also paved the way for solving the question of flat degenerations for G -torsors for general groups G .

3. Seshadri’s Contribution to Mathematics Education

The Chennai Mathematical Institute in its present form was founded in 1998, but its roots go back to 1989 when Seshadri founded a new institute, then called the School of Mathematics, SPIC Science Foundation. The Chennai Mathematical Institute (CMI) is a unique institution in India which attempts to integrate undergraduate education with research; it grew out of Seshadri’s vision that higher learning can be had only in an atmosphere of active research amidst the presence of masters in the subject. It was a brave venture in the face of extraordinary opposition and skepticism even from his very close friends and well-wishers. It was his dream to build a center of learning which could compare itself with the great centers such as the *École Normale* in Paris, Oxford and Cambridge Universities in England, and Harvard University in the US. It would open up opportunities for gifted students in India to learn in this unique academic atmosphere and also enable active researchers to participate in this experiment, which could have a lasting influence on the development of mathematics in India.

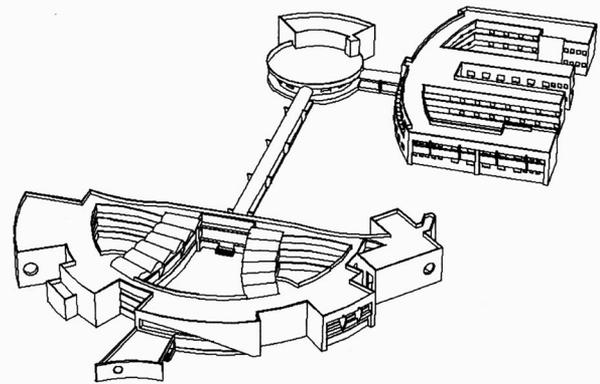
It would not be an exaggeration to say that the Chennai Mathematical Institute is now rated as one of the best schools in the world for undergraduate studies in mathematics. This is indeed a first big step, though much still needs to be done to fulfill Seshadri’s dream.

4. Seshadri, the Person

I have known Seshadri since 1984, first as his doctoral student, and later as his collaborator and colleague at the Chennai Mathematical Institute. He had a routine of coming to the office around noon, a habit coming from his early TIFR days. The first meeting every day with me (or for

Foundation Stone Ceremony

10.00 a.m., Friday, 6 August, 2004



cm_i

CHENNAI
MATHEMATICAL
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Figure 1. Foundation ceremony of CMI.

that matter anyone else) was a greeting smile and a customary one-word query “anything?,” uttered in a lilting south Indian accent. Loaded with multiple connotations, it invariably acted as a catalyst and pressure on the listener to come up with something meaningful. I have seen many who religiously avoided meeting his smiling eye and that dreaded one-worder.

I vividly remember his lectures. Notes were prepared with utmost meticulousness and the talks were quite spartan but always insightful. Every lecture had something as a take-away for an aspiring researcher. Getting a word of praise from him was something of a rarity. This used to come only as an award for something which he considered insightful and this was hard to come by for most of us. Many years later, when I was in my early forties, after I felt I had done something really significant, he came up to me and said in a rather matter of fact manner, “*There is meat in your work. Now I can say you are a mathematician!*”

A charming simplicity was the most prominent characteristic of his personality. As a mathematical personality, I saw someone unique in his vision and insight, an uncanny ability to consistently strike gold in a vast world of mathematics. He was extraordinarily generous with his ideas and shared his insights with one and all and this extreme generosity was his human side as well. His only caveat was



Figure 2. In Madrid in 2008, after some spirits and jokes.

that the listeners go back and pursue the ideas to the best of their abilities. There was a complete awareness of his own stature while being modest and humble at the same time. He had a unique sense of humour by which he unconsciously managed to “transform” even pedestrian jokes into memorable anecdotes. His interests ranged widely from mathematics and philosophy, to politics and music. He was confident of his insights and this made him unperturbed during several moments of crisis that the institute faced. I quote Professor K. Chandrasekharan, who in a letter to Seshadri on February 10, 2013, wrote “I cherish the values that inspired the creation of CMI and your unwavering commitment to those values.” Seshadri will be remembered for these values.

Seshadri was also an accomplished exponent of the Carnatic Music and till a few days before his passing, he continued to share his musical knowledge and insights with a young musical student Maitreyi from CMI. Seshadri was trained by his maternal grandmother who herself was a student of the well-known Kanchipuram Nainapillai. Seshadri showed the same traits in his musical discipline as in his mathematical ones. He meticulously did *riyaaz*, *musical practice*, and his repertoire of Muthuswamy Dikshitar’s and Shyama Sastry’s *kritis* was noteworthy. On several occasions I listened to his music, which could be described as a *royal gait profoundly suited to expressing Dikshitar’s kritis*. There were rare memorable occasions when he sang *jugalbandi* with his wife Sundari, who was a brilliant singer her-

self. When he was singing, a distinctly spiritual side would come to the fore. By a spiritual side, I do not mean anything religious, but a musical one which bore the stamp of an immense *sadhana*, *one-pointed pursuit*, where every nuance was expressed with a spiritual feeling which was way beyond religious emotion.



Figure 3. Sundari and Seshadri singing in Chennai, 2007.

On the 15th of July in 2020, I had an hour of mathematical discussion with him over the internet and he was very receptive and happy. It was a discussion “as usual,” and at the close I said we could continue the next day and he responded with a laugh that he cannot give me a guarantee for that!

He passed away on the 17th July in his home in Mandaveli, Chennai. Seshadri had been suffering from Parkinson’s for the past several months and after the passing of his wife Sundari in October 2019, his condition had been deteriorating.

I close with lines from W. H. Auden (“Hymn to the United Nations”):

*“Like music when
Begotten notes, New notes beget.
Making the flowing of time a growing.
T’is what it could be...
When even sadness, Is a form of gladness.”*



Vikraman Balaji

C. S. Seshadri

David Mumford

Seshadri, as I called him because this is his given name in the Indian tradition, was not only my mathematical colleague but one of my closest friends. Our connection began in the early 60s when I received a letter with exotic Indian stamps on it. The letter was from Seshadri, whose work on Serre's conjecture I knew, but India, little more than a dozen years after independence, still felt to me like an unknown and very distant place. The mathematical world was small in those days, largely concentrated in Cambridge, Princeton, Paris, and Moscow, and jet plane travel was just starting. I was thrilled to learn that he and M. S. Narasimhan, halfway around the world, had found exactly the same class of vector bundles on an algebraic curve as I had, a class for which a compact moduli space could be constructed. This was amazing because their method was totally different from mine, the sort of unexpected link that demonstrates the mysterious unity of math and confirms one's love for its unpredictable muse.



Figure 4. In a restaurant during the ICM in Zurich, in 1994. Left to right: Seshadri, D. Mumford, P. Murthy, S. Ramanan, A. Neeman, and V. Lakshmibai.

We got together, first when he visited Harvard in 1966, then in 1967 when I went with my family to Bombay, as the city was called then. The Tata Institute had just been built, standing like a mirage facing the Arabian Sea with manicured lawns. It was so seriously air conditioned that I had to keep a sweater in my office though it was 90° outside. The Institute had a world-class library, with vast

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stacks. At their far end, one day a year, an idol was set up to Saraswati, the Hindu goddess of learning and music, and I, like everyone there, professors and groundskeepers alike, performed puja (Hindu worship). But best of all, working at the Institute was a powerful group of algebraic geometers who were rapidly uncovering the secrets of moduli spaces. We would talk daily, at morning coffee, lunch, or afternoon tea, often in the West Canteen. The brass fittings in the elevator were “polished up so carefuller” just as in the Gilbert and Sullivan song. Fresh flowers were brought to my office every day from the abundant gardens that Homi Bhabha, the Tata Institute's founder, had considered an essential part of a modern science institute. Bhabha and the Institute were the 20th century version of Kubla Khan and the “stately pleasure dome” in Coleridge's poem. Bhabha was similarly revered but sadly recently deceased.

Curiously, Seshadri and I never wrote any joint papers but instead, our research intertwined for many years. I recently found a letter written in 1970 where I wrote “Thanks for your letter which is beautiful and very encouraging. I was personally getting quite discouraged about extending my result when your letter arrived. The blowing up trick is marvellous...” This may be an odd thing to say but the most wonderful thing Seshadri did, after many twists and turns, was to make my book *Geometric Invariant Theory* largely irrelevant! He went deeper than I had and realized that, because these moduli spaces clearly existed in the bigger category of algebraic spaces, the key thing you needed was a suitable criterion for ampleness that can be applied to the naturally occurring line bundles; well, you also needed a compactification if the space is not already complete. This is a much more natural and powerful way to show that moduli spaces are projective.

Seshadri's work on moduli is well described in detail in Balaji's contribution to this issue, but I'd like to add some words on the history as far as I know it. My approach had been based exclusively on representing the moduli spaces as quotients of varieties by reductive groups using invariant theory and then working out what “stability” (as defined in this geometric invariant theory) means in each case. Unfortunately, when extended from moduli of curves to moduli of surfaces, the method encountered baroque complexities, as shown in David Gieseker's tour-de-force computation of Hilbert scheme stability for surfaces of general type in his 1977 *Inventiones Mathematicae* paper. Seshadri meanwhile had shown in his 1972 *Annals of Math.* paper how geometric invariant theory could be avoided for certain vector bundle moduli spaces. Among many ingenious ideas, one was the “trick” (better called an “ingenious, startling idea”) referred to in the quoted letter above. I remember clearly my excitement when, walking

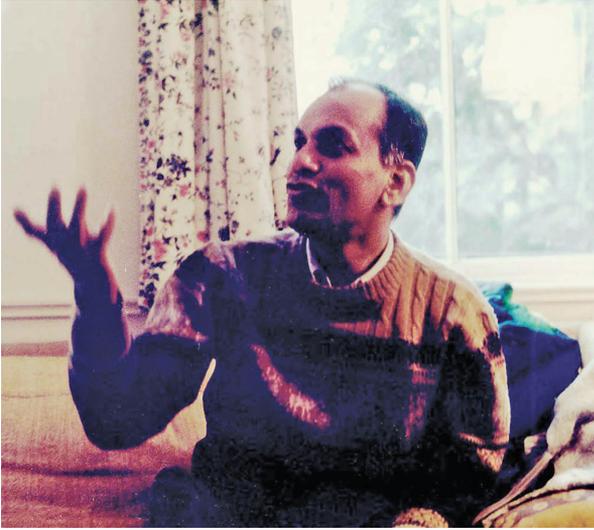


Figure 5. Seshadri in Cambridge in 1983.



Figure 6. Seshadri in Chennai in 1992.

with him to get lunch one day, he explained how he could circumvent “geometric reductivity” and still prove projectivity in the bundle case (and, in fact, geometric reductivity turned out later to follow from his method). And, as time went on, in the hands of János Kollár, using Mori’s theories, and with Viehweg, Shepherd-Barron, and many others, this approach led to the compactification and projectivity of almost all moduli spaces using their so-called tautological bundles. János puts it simply in his 1990 *Journal of Differential Geometry* paper, after Lemma 3.9: “This is the statement where we escape geometric invariant theory.” I take this as having been a huge breakthrough.

Seshadri and his family also intertwined with me and my family during all the decades since then. Our boys

played together, he became the godfather of my daughter Suchitra, and his family lived next door to us one year when he was visiting Harvard and Northeastern. I was truly honored when he came all the way from Chennai to Cambridge for my 80th birthday—as well as for the more significant Hindu milestone of my 1000th full moon. He opened his doors to all my children as they grew older and travelled on their own.

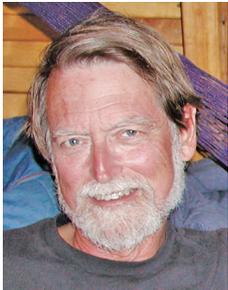
After retiring from the Tata Institute of Fundamental Research, Seshadri undertook something even more challenging than doing maths: creating from scratch an entirely new educational institution, its buildings, professors, funding, and student body. That a seemingly unassuming, intellectual, math & music loving, Tamil Brahmin could dream of creating an entire college and then succeed, boggles the mind. The Chennai Mathematical Institute is a blend of traditional Indian guru-shishya transmission of knowledge and ideas from American liberal arts colleges. He needed to pull all the strings contacting his friends in high places, allies in the exploding Chennai business community, his stature as “FRS” (Fellow of the Royal Society) and still, so much sleep was lost when money ran out, so many flights to Delhi were needed. All I can say is that it’s a miracle and it reveals that a surprisingly practical man was hidden inside an unassuming exterior. This is a good place to acknowledge the immense assistance given him by his Registrar S. Sripathy.



Figure 7. In 1967 in front of the Tata Institute. Left to right: C. Musili, Seshadri, M. S. Narasimhan, M. S. Raghunathan, and David Mumford.

But what I really want to describe, the thing that made the most profound impression on me, was the way

Seshadri integrated so fully and naturally his love of traditional Indian teachings and customs with his full awareness of the energy and force of Western culture. Unlike his fellow Tamilian mathematician S. Ramanujan who subsisted on boiled potatoes in Cambridge University, when Seshadri came to the West, Paris in his case, he sampled its food and drink and enjoyed the pleasures of French culture. In all his visits to the West, he was comfortable wearing Western garb at the office and relaxing in his lungis when he got home. He lived in the most unpretentious way, resisting the Western impulse for the latest gadgets and for collecting expensive ornaments. Yet he could fly to Delhi and argue effectively with high-up officials and ministers for funding for mathematics. His great passion, almost as strong as his love of mathematics, was singing classical karnatic (South Indian) ragas which he performed at a professional level. As the eldest son in his family, he could haul out his sacred thread when rituals demanded this symbol. Walking with me once in some quiet Maine woods, uncut for a century with no houses or roads nearby, he remarked that this experience helped him understand the early Indian sages whose woods are now long gone, replaced by villages and fields. I will miss him, his laughter, and his warmth for as long a time as is allotted to me.



David Mumford

Remembering Seshadri

William Haboush

What I noticed about Seshadri when I first met him was his sheer joy at a very ingenious turn in a proof or an application of something unexpected. He had a way of sort of jumping back a bit and saying “so you see...” as he showed you this marvellous turn of events on the way to proof. He would communicate a strong sense of the surprise and amusement that comes with a very clever sudden observation that changes things. The first time I experienced it was in my first formal exchange with him. He was

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visiting Harvard as was J. P. Serre in 1976 when I cracked the Mumford conjecture (“Reductive groups are geometrically reductive,” *Ann. of Math. (2)* 102 (1975), 67–83). I sent the proof to Armand Borel at the Institute and to David Mumford at Harvard. David Mumford, Serre, and Seshadri went through it carefully and then called me at my home to tell me it was ok. Seshadri got on the phone and enthusiastically explained how it came down to the orbit structure of the double product of the flag variety and the invariant divisors on that scheme. That was the first time I heard that “so you see!” chuckle of his. His enthusiasm caused me to rewrite my paper emphasizing that observation. I heard it again when he explained Peter Littelmann’s work to me and I saw it over and over in a series of seminar and conference talks. That “so you see...” moment is how I see him in my memory.

My first encounter with him a year or two before was less fortunate. It was at the Arcata conference of 1974 before I’d done anything interesting. It was held at a school that was quite distant from any grocery or liquor stores, so some attendees took long walks to buy beer at a Seven Eleven about a mile away. There were a couple of six-packs laboriously obtained by Seshadri and Narasimhan in the refrigerator of the common room of my dormitory. At that conference Douady discovered dirty limericks and was badgering everyone as he compiled a collection which he carefully recorded in a huge notebook. In the common lounge one night he got one or two from me and then rewarded me with a can of Heineken which I assumed was his. As I was drinking it, Seshadri and Narasimhan arrived and examined the now empty commons refrigerator. They came out of the room where the refrigerator was located and spotted my Heineken. With a certain tension in his voice Seshadri asked “Is that my beer you’re drinking?” That is how I met him.

Beyond mathematics, I saw this joy of discovery in his experience of food, art, and music. He had me spend some months in Chennai (then it was Madras). He guided me to experience the culinary riches of that city as well as the pleasures of the music festival. I remember a particular discussion of a sculpture of Shiva in the museum there. I really got a lot more than mathematics from him.

I enjoyed his presence at many conferences and at my visits to the Tata Institute and the SPIC institute in Chennai where he invited me for extended stays. He also visited the Institute for Advanced Study for one semester the year that I visited. At the IAS, I got to know his family pretty well. My wife and I had him to dinner a couple of times and he also invited us to enjoy his wife Sundari’s cooking which was phenomenal.

Although the theory of moduli would be how most mathematicians would remember Seshadri, most of my

interactions with him involved his work on the flag variety and representation theory. He had been fascinated by Kempf's work on $SL(n, k)$ (G. Kempf, *Schubert Methods with an Application to Algebraic Curves*, Stichting Mathematisch Centrum, Amsterdam, 1978), that is, the early preprints from Utrecht, and he decided to commit to standard monomial theory. Shortly after his paper with Lakshmibai and Musili in the *Annals* of the ENS (V. Lakshmibai, C. Musili, and C. S. Seshadri, "Cohomology of line bundles on G/B ," *Ann. Sci. Ecole Norm. Sup.* 7 (1974), 89–137), he began to pursue a structured program to find new proofs of such things as arithmetical normality and Cohen Macaulayness for ample line bundles on the flag variety. By 1980, this program had expanded considerably. Then Victor Kac noted an error in Demazure's proof [Dem74] that those line bundles were Cohen Macaulay. Thus the program received fresh impetus. Several of his goals in that program, including the vanishing theorem for higher cohomology of ample line bundles on the flag variety, were established independently. The program was just about completed by Seshadri and Lakshmibai, but for a last crucial step that was done by Peter Littelmann using Lusztig-type quantised enveloping algebras, and their representation theory, at a root of unity.

No remembrance of Seshadri would be complete without some mention of his last most sustained project, the establishment of the Chennai Mathematical Institute. This began as a way of continuing his work after his retirement from the Tata Institute in 1991, but it became clear that he was really motivated to establish a major center for advanced mathematical research in his home province. He somehow secured a grant for it and established it on a shoestring in the early 90s. He really had to fight rather substantial opposition on several occasions but against all odds he persisted and saw it through to its current form. The difficulty and uncertainty he faced can't be overemphasised, but he managed it. In a career punctuated by major accomplishments it is a stunning addition to his lifelong mission to build and enhance the mathematical project.



William Haboush

A Tribute to C. S. Seshadri. Standard Monomial Theory and Applications

Venkatramani Lakshmibai

Professor C. S. Seshadri was a many-faceted mathematician. In the write-ups by me and P. Littelmann, we have tried to portray his skills in the area of representation theory of algebraic groups, especially of semisimple algebraic groups, and its impact on the geometry of flag varieties and their Schubert subvarieties. In the 1950s, Hodge gave a basis for the homogeneous coordinate ring of the Grassmannian (as well as the Schubert subvarieties in the Grassmannian), for its canonical Plücker embedding, in terms of certain monomials in the Plücker coordinates, the so-called "standard monomials." With his amazing insight, Seshadri, along with his collaborators, was able to extend the "Standard Monomial Theory" (abbreviated SMT), to any "generalized" flag variety (and the Schubert varieties therein), and even to the "affine" flag variety (arising from an affine Kac-Moody algebra), and the Schubert varieties therein. The origin and the completion of SMT ran over roughly three decades, namely, from the 1970s through the 1990s. This theory has led to very many interesting geometric and representation-theoretic consequences. We tell the story of SMT, bringing out the reminiscence between the respective authors and Seshadri.

My association with Prof. C. S. Seshadri extends over five decades, first as my mentor, next as a collaborator, and then as a friend. Below I will briefly dwell upon these three phases!

Before taking this up, since the main discussion of the write-ups by me and P. Littelmann is on the story of SMT, I will tell the reader a bit about SMT for Schubert varieties in the Grassmannian.

1. The Flag Variety

The base field K is an algebraically closed field of arbitrary characteristic. For simplicity, one may take K to be \mathbb{C} .

Let us fix a positive integer, say, n , and let $V = K^n$. A *full flag* or just a *flag* is a chain or a sequence of subspaces $F = (V_0 \subset \cdots \subset V_i \subset \cdots \subset V_n = V)$, $\dim_K V_i = i$. For example, denoting the standard basis in V by e_i , $1 \leq i \leq n$, and taking V_i to be the span of $\{e_1, \dots, e_i\}$, we get the standard flag F_0 .

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Let $FL_n = \{\text{flags}\}$. We have a transitive action of GL_n on FL_n . Clearly, the stabilizer in GL_n at F_0 is $H_n := \{\text{upper triangular matrices}\}$. Thus we get a bijection:

$$FL_n \rightarrow GL_n/H_n.$$

Thus FL_n acquires a projective variety structure; FL_n , together with this projective variety structure, is called *the flag variety*.

The Weyl group. Let

$$T_n = \{\text{diagonal matrices in } GL_n\}.$$

We have

$$N := N_{GL_n}(T_n) = \{\text{permutation matrices}\}.$$

Hence we obtain

$$N/T = S_n, \text{ the symmetric group.}$$

One refers to the group N/T as the *Weyl group of GL_n* .

Identification of FL_n with SL_n/B . Let $G = SL_n$. Denoting the set of upper triangular matrices in SL_n by B , we have

$$GL_n/H_n \cong SL_n/B.$$

From the representation-theoretic viewpoint, it will be more convenient to work with SL_n . Hence, in the sequel, we shall identify FL_n with $SL_n/B (= G/B)$.

Schubert varieties. Under the T -action on G/B (given by left multiplication), it turns out that $(G/B)^T$, the set of T -fixed points, is finite, and in fact is given by $\{wB, w \in W\}$ (here, by wB we mean $n_w B$ for some lift in N of w).

It is easily checked that the coset wB is independent of the coset representative n_w . Let us denote the coset wB , considered as a point in G/B , by e_w .

For $w \in W$, $X(w) := \overline{Be_w}$, the Zariski closure (inside G/B) of the B -orbit Be_w through the T -fixed point e_w , with the canonical reduced scheme structure, is the *Schubert variety* associated to $w \in W$.

Schubert varieties inside partial flag varieties. More generally, let us fix an r , $1 \leq r \leq n-1$, and fix an r -tuple, say, $\underline{d} = (1 \leq d_1 < d_2 < \dots < d_r \leq n-1)$. Denote

$$Fl_{n,\underline{d}} = \{V_{d_1} \subset V_{d_2} \subset \dots \subset V_{d_r}\},$$

chains of vector spaces one in each of dimension d_1, \dots, d_r .

We shall refer to $Fl_{n,\underline{d}}$ as the set of *partial flags of type \underline{d}* . As above we get $Fl_{n,\underline{d}} \cong G/P_{\underline{d}}$, where $P_{\underline{d}}$ is a closed subgroup of G containing B . One refers to a closed subgroup of G containing B as a *parabolic subgroup*.

Schubert varieties in $G/P_{\underline{d}}$ are defined as above, namely, B -orbit closures through T -fixed points. Schubert varieties in $G/P_{\underline{d}}$ are indexed by $W/W_{P_{\underline{d}}}$, $W_{P_{\underline{d}}}$ being the Weyl group of $P_{\underline{d}}$. Note that $W_{P_{\underline{d}}}$ gets identified with $S_n/(S_{d_1} \times S_{d_2} \times \dots \times S_{d_r})$.

2. The Grassmannian

In the above discussion, as an extreme case, taking $r = 1$, $\underline{d} = d$, we get the celebrated *Grassmannian variety*:

$$G_{d,n} := Fl_{n,d} = \{d\text{-dimensional subspaces of } V = K^n\}.$$

The set $I_{d,n}$. The Schubert varieties in $G_{d,n}$ are indexed by W/W_{P_d} , where

$$P_d = \left\{ \left[\begin{array}{c|c} * & * \\ \hline 0_{N-d \times d} & * \end{array} \right] \in G \right\},$$

$$W/W_{P_d} = S_n/S_d \times S_{n-d}.$$

Denoting $I_{d,n} := S_n/S_d \times S_{n-d}$, we get an identification:

$$I_{d,n} = \{(i_1, \dots, i_d) \mid 1 \leq i_1 < i_2 < \dots < i_d \leq n\}.$$

3. The Grassmannian as a Projective Variety

Plücker embedding. We have a canonical projective embedding, called the *Plücker embedding*:

$$\begin{array}{ccc} G_{d,n} & \hookrightarrow & \mathbb{P}(\bigwedge^d K^n) \\ U = \begin{array}{c} | \\ u_1 \quad \dots \quad u_d \\ | \end{array} & \longmapsto & [u_1 \wedge \dots \wedge u_d]. \end{array}$$

Plücker coordinates. Consider $I_{d,n} := \{(i_1, \dots, i_d) \mid 1 \leq i_1 < \dots < i_d \leq n\}$. For $\tau = (i_1, \dots, i_d) \in I_{d,n}$ define

$$e_\tau := e_{i_1} \wedge \dots \wedge e_{i_d}.$$

Then $\{e_\tau\}_{\tau \in I_{d,n}}$ is a basis for $\bigwedge^d K^n$.

Define $\{p_\tau\}_{\tau \in I_{d,n}}$ to be the dual basis for $(\bigwedge^d K^n)^*$. These are the *Plücker coordinates*.

Projective variety structure for $G_{d,n}$.

The zero set. The Grassmannian is the zero set of a certain set of quadratic polynomials in the Plücker coordinates called the *Plücker relations*.

Example. Consider the Grassmannian $G_{2,4}$. We have

$$I_{2,4} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

and

$$P_{(1,2)}, P_{(1,3)}, P_{(1,4)}, P_{(2,3)}, P_{(2,4)}, P_{(3,4)}$$

are the coordinates of $\mathbb{P}(\bigwedge^2 K^4)$. Then $G_{2,4}$ is defined by the equation

$$P_{(1,2)}P_{(3,4)} - P_{(1,3)}P_{(2,4)} + P_{(2,3)}P_{(1,4)} = 0.$$

The Plücker relations. For $\underline{i} \in I_{d-1,n}$ and $\underline{j} \in I_{d+1,n}$, consider the following quadratic polynomial in the Plücker coordinates, called a Plücker polynomial:

$$\sum_{h=1}^{d+1} (-1)^h p_{i_1, \dots, i_{d-1}, j_h} p_{j_1, \dots, j_{h-1}, j_{h+1}, \dots, j_{d+1}}.$$

In the expression above, if \underline{a} has a repeated entry, $p_{\underline{a}}(A)$ is understood to be zero for any $n \times d$ matrix A . We shall denote the above system of polynomials by $\mathcal{P}_{d,n}$.

Theorem 3.1. *The image of $G_{d,n}$ in $\mathbb{P}(\bigwedge^d K^n)$ is precisely the zero set of the system of polynomials $\mathcal{P}_{d,n}$.*

Remark. We have, in fact, that as a subvariety of $\mathbb{P}(\bigwedge^d K^n)$, the Grassmannian $G_{d,n}$ is defined, scheme-theoretically, by the Plücker relations.

A presentation for $K[G_{d,n}]$. For the Plücker embedding $G_{d,n} \hookrightarrow \mathbb{P}(\bigwedge^d K^n)$, let R denote the homogeneous coordinate ring $K[G_{d,n}]$ of $G_{d,n}$, so that as a K -algebra, R is generated by Plücker coordinates. Let I be the defining ideal of $G_{d,n}$. Then, by the above Remark, we have that I is generated by the set of all (degree 2, homogeneous) Plücker polynomials.

4. Standard Monomial Basis

Standard monomials. Let $\tau_1, \dots, \tau_m \in I_{d,n}$. Then $p_{\tau_1} \cdots p_{\tau_m}$ is a *standard monomial* if $\tau_1 \geq \cdots \geq \tau_m$. Such a monomial is *standard on the Schubert variety $X(\tau)$* if in addition $\tau \geq \tau_1$.

For a Schubert variety $X(\tau)$ in $G_{d,n}$, let us denote the homogeneous coordinate ring of $X(\tau)$ for the Plücker embedding by $R(\tau)$.

Main theorem. *Monomials of degree m standard on $X(\tau)$ form a K -basis for $R(\tau)_m$, the m th graded piece of $R(\tau)$.*

5. Some Important Geometric Consequences

Let us denote the tautological bundle on $\mathbb{P}(\bigwedge^d K^n)$, as well as its restriction to any Schubert variety $X(\tau)$, by L . Let $R(\tau)$ be as above.

1. The natural inclusion $R(\tau)_m \subset H^0(X(\tau), L^m)$ is an equality for all $m \in \mathbb{Z}_+$. In particular, we have that $H^0(X(\tau), L^m)$ has a K -basis consisting of monomials of degree m that are standard on $X(\tau)$ for all $m \in \mathbb{Z}_+$.
2. We have that $H^i(X(\tau), L^m) = 0$, $i \geq 1, m \in \mathbb{Z}_+$.
3. Schubert varieties are arithmetically Cohen Macaulay and arithmetically normal.
4. Determination of the singular locus of $X(\tau)$.
5. Relationship with CIT (Classical Invariant Theory).
6. A Gröbner basis for $I(X(\tau))$.

Remark. In view of consequence (1), we find that the SMT consists in constructing explicit bases for $H^0(X(\tau), L^m)$ consisting of monomials of degree m , that are standard on $X(\tau)$, a Schubert variety, for all $m \in \mathbb{Z}_+$.

I will now describe the three phases of my association with Seshadri, highlighting the important events that took place during each of the phases.

I. Seshadri as my mentor. I joined TIFR in the fall of 1968, the iconic year which witnessed two important activities at the Institute. The first one is the “celebrated” International Symposium on Algebraic Geometry

in January 1968, which was marked by the presence of almost all of the then leading algebraic geometers including André Weil, Alexander Grothendieck, David Mumford, Mike Artin, Phillip Griffiths, Shreeram Abhyankar, Jun-ichi Igusa, and Teruhisa Matsusaka, among others. The second one is the “famous” year-long course on algebraic geometry given by Seshadri. Unfortunately, I couldn’t participate in either one of them for obvious reasons, namely, I joined the Institute in August 1968, after the two events were finished! After spending a couple of years learning the rudiments of the three basic courses in Algebra, Analysis, and Topology, during 1971, I formally started to work on my PhD, under the guidance of Seshadri. In the fall of 1972, Seshadri asked me and Musili to read the paper by G. Kempf, *Schubert Methods with an Application to Algebraic Curves* (Stichting Mathematisch Centrum, Amsterdam, 1978). In this paper, Kempf proves the Borel-Weil theorem for the flag variety $SL(n)/B$ in positive characteristics (let us recall that the celebrated Borel-Weil theorem is about the vanishing of higher cohomologies of ample line bundles on the generalized flag variety G/B , G a semisimple algebraic group and B a Borel subgroup of G , in characteristic 0). Musili and I concentrated for a whole month on reading Kempf’s paper and, among other things, we arrived at the Weyl group-theoretic description of a class of smooth Schubert varieties in $SL(n)/B$, constructed by Kempf in that paper. We then reported to Seshadri about our discoveries related to Kempf’s paper. Then the three of us had a serious discussion of Kempf’s paper for about three months, and we gave a proof of the Borel-Weil theorem for the generalized flag variety G/B , G being a group of classical type or type G_2 in all characteristics. This result was published in *Ann. Sci. Ecole Norm. Sup.* (V. Lakshmibai, C. Musili, and C. S. Seshadri, “Cohomology of line bundles on G/B ,” *Ann. Sci. Ecole Norm. Sup.* 7 (1974), 89–137). This was my first proud publication.

Kempf made use of a technical step in our paper, and gave a type-free proof of the Borel-Weil theorem for G/B , G a simple algebraic group and B a Borel subgroup in all characteristics, in 1976, now known as “Kempf’s vanishing theorem.”

II. Seshadri as my collaborator. My collaboration with Seshadri runs over a little more than four decades, starting from the mid 1970s through 2019. In the fall of 1976, when Seshadri had just returned from a two-year visit to Harvard University, he was all excited about a recent work of De Concini-Procesi: “A characteristic free approach to invariant theory” (*Advances in Math.*), where they present a characteristic-free approach to classical invariant theory. Their work essentially consists of a construction of a characteristic-free basis for the rings of invariants, appearing in Weyl’s *Classical Groups*. It’s more or less

around this time that Seshadri had just finished his work on “Geometry of G/P -I” (cf. [Ses78]), wherein he extends Hodge’s results—giving a natural basis (over \mathbb{C}) for the homogeneous coordinate ring of the Grassmannian (and its Schubert varieties) for the Plücker embedding, in terms of “standard monomials” in the Plücker coordinates—to G/P , where G is a simple algebraic group and P is a maximal, minuscule, parabolic subgroup of G , in all characteristics. We may describe Seshadri as the *inventor of modern standard monomial theory*. This work of Seshadri may be considered as the beginning of SMT! As soon as he got back to the Institute, he asked me to read the above-mentioned paper of De Concini-Procesi and at the same time he explained his work “Geometry of G/P -I” to me. After spending countless hours on mathematical computations and discussions with Seshadri, we figured out the relationship between the two papers (over a period of three months), and along the way, we also arrived at some important basic conjectures. Thus we have the birth of “Geometry of G/P -II” ([LS78]), which is to be considered as the *gateway* to SMT.

The conjectures, arrived at in [LS78], describe a nice conjectural basis, in all characteristics, for all G/P ’s, where G is a simple algebraic group and P is a maximal parabolic subgroup of *classical type*. In particular, noting that any maximal parabolic subgroup of a classical group G is of classical type, this would take care of the problem of developing an SMT for all G/P ’s, G being classical and P any maximal subgroup of G .

Musili and I started discussing proving the basic conjectures. As a first step, we were able to prove them for the case of G/P ’s, where G is simple and P is a maximal parabolic subgroup of *quasi-minuscule type*, in “Geometry of G/P -III,” in 1978. These are the parabolic subgroups that are considered by Kempf, while proving his vanishing theorems, in 1976. The point here is that the class of the quasi-minuscule parabolic subgroups is wider than the class of minuscule parabolic subgroups, in that every simple algebraic group admits at least one maximal parabolic subgroup of quasi-minuscule type, while one may note that there aren’t any minuscule parabolic subgroups if G is of type E_8 , F_4 , or G_2 .

At this time M. Demazure was visiting TIFR, and gave lectures on his “celebrated” paper “Désingularisation des variétés de Schubert généralisées” [Dem74]. In this paper, among other things, Demazure proves a character formula for the space of sections of ample line bundles on G/B as well as their restrictions to the Schubert subvarieties in G/B ; this character formula is used in a crucial way in “Geometry of G/P -III.” In addition, the normality of Schubert varieties in all characteristics (a consequence of the results in the paper of Demazure mentioned above) was also used in a crucial way in “Geometry of G/P -III.”

The three of us—Musili, Seshadri, and I—continued our investigations of the extension of the results for G/P , P being a maximal parabolic subgroup of classical type to any G/Q , where G is a simple algebraic group and Q is an intersection of maximal parabolic subgroups of classical type—such a parabolic subgroup is called a *parabolic subgroup of classical type*. After an intense discussion by the three of us for four months, we did arrive at a set of conjectures for developing an SMT for the case of G/Q , where Q is a parabolic subgroup of classical type. Then after a six-month period of intense/serious discussions by the three of us, we wrote “Geometry of G/P -IV” (cf. [LMS79]), in which we gave the complete proof of the conjectures of [LS78], we stated the conjectures for the case of G/Q , and we outlined a proof of the same. I have very good reminiscences of our discussion: we used to meet around noon and go on until 7:00 p.m., with a lunch break for an hour and then a couple of tea breaks. It should be remarked that most of the time, the crucial idea would strike us while we were having our tea!

Procesi visited the Tata Institute in January and February of 1978. Upon his invitation, I was a visiting professor at the University of Rome, for the academic year 1978–79. Procesi, De Concini, and I had very many interesting mathematical discussions. The following two years, at the invitation of Kempf, I was a visiting professor at Johns Hopkins. The following three years, I spent at the University of Michigan, Ann Arbor. In the summer of 1982, on my way to India, I made a stop in Germany to attend a week-long conference on “Algebraic Groups” in Oberwolfach, organized by Springer and Tits. During the conference, Victor Kac pointed out a serious error in the paper of Demazure mentioned above. We got very worried, since, as mentioned above, we had used the results of Demazure’s paper in our papers “Geometry of G/P -III, IV” in quite a non-trivial way. Once again, we got into serious mathematical discussions. Thankfully, Seshadri and I were able to fix our proof by taking a totally different approach than that found in [LMS79]. Thus, we wrote the paper “Geometry of G/P -V” ([LS86]).

We were still thinking about the extension of SMT to other exceptional groups. In the process, I arrived at a set of conjectures (see [LS91] for a statement of these conjectures). Our goal was to extend the SMT to exceptional groups, but to our surprise, the conjectures seem to include the Kac-Moody groups also. As a first step towards proving the conjectures, Seshadri and I showed that the conjectures hold for $\widehat{SL}(2)$ (*Standard Monomial Theory for \widehat{SL}_2* , Advanced Series in Mathematical Physics, vol. 7, 1989).

Thanks to his ingenuity, Littellmann completed the SMT even for the Kac-Moody groups by proving the above-mentioned conjectures (cf. [Lit98]). Littellmann’s proof

makes a clever use of the representation theory of quantum groups at a root of unity, as developed by Lusztig in the 1990s.

Once the SMT was complete, Seshadri and I were looking at the problem of relating the cotangent bundle to G/B to some suitable Schubert variety in the affine flag variety. To make this a little more precise: in his seminal paper "Canonical bases arising from quantized enveloping algebras" (*JAMS*), Lusztig relates certain orbit closures arising from the type A cyclic quiver \widehat{A}_h to certain affine Schubert varieties. On the other hand, in the case $h = 2$, in her paper, "On the conormal bundle of the determinantal variety" (*J. Algebra*), Strickland relates such orbit closures to conormal varieties of determinantal varieties; furthermore, any determinantal variety can be canonically realized as an open subset of a Schubert variety in the Grassmannian (cf. [LS78]). Inspired by these results, we were interested in finding a relationship between affine Schubert varieties and conormal varieties to Schubert varieties in the Grassmannian. As a first step, I was able to show in "Cotangent bundle to the Grassmannian variety" (*Transformation Groups*) that the compactification of the cotangent bundle to the Grassmannian is canonically isomorphic to a Schubert variety in a two-step affine partial flag variety. This result was extended to cominuscule Grassmannians by me (together with V. Ravikumar and W. Slofstra) in "Cotangent bundle to cominuscule Grassmannian varieties" (*Michigan Math. J.*).

Then in 2017, together with Rahul Singh, we (Seshadri and I) were able to extend these results to $SL(n)/B$ in "Cotangent bundle to the flag variety-I" (*Transformation Groups*, 2017). This is my last publication together with Seshadri, and his last publication also!

III. Seshadri as a friend. After spending nearly three decades at TIFR, Seshadri moved to Chennai in the mid-1980s, and joined IMSc, as the head of the Mathematics Department. At more or less the same time, I moved to Boston to join the math faculty at Northeastern University.

After spending a brief time at IMSc, Chennai, Seshadri branched out and started an institution with the mission of training undergraduates, besides undertaking research. This new institution was formed, in October 1989, with the help of Parthasarathy, who was then working at the petrochemical company SPIC in Chennai, and the new institution was called the SPIC Mathematical Institute.

After going through some initial financial difficulties, this institution eventually evolved into the Chennai Mathematical Institute. CMI has emerged internationally as one of the most well-recognized Indian institutions for mathematics. The undergraduates from CMI are accepted into PhD programs in such top-notch universities as Harvard, MIT, Brandeis, Northeastern, Princeton, Emory, University

of Chicago, Caltech, and UCLA in the US, I.H.E.S., Ecole Normale, and Université Paris 7 in France, and Max-Planck Institute and Universität Hamburg in Germany.



Figure 8. Lakshmibai and Seshadri in Rome in 2001.

Since the birth of CMI, in 1989, I have been a regular visitor during the winter months of every year, which enabled our collaboration to continue! Having known Seshadri for five decades, I would describe him as a very genteel person, patient to the core, who brought nothing but joy & pleasure to the people around him.

I could say that just by observing him, I have learnt so many things about life, which have polished my nature/character unknowingly!

Another trait that cannot be missed by anyone who might have had just an acquaintance with him is his modesty. In spite of being the recipient of so many glorious/prestigious awards, he remained so modest which only added to his personality! I would like to end this article by quoting Seshadri's general PHILOSOPHY OF LIFE:

"NOTHING IS THE END OF THE WORLD"!



Venkatramani
Lakshmibai

Further Development of Standard Monomial Theory and Applications

Peter Littelmann

I remember very well the first time I met Seshadri in fall 1983. He was at Brandeis University as a visiting professor for one year, and I had just arrived there as an exchange PhD student, also for a year. Seshadri's course *Introduction to Standard Monomial Theory* seemed to me a good opportunity to learn some algebraic geometry related to algebraic groups. Pradeep Shukla and I agreed to take notes for the course. After being back in Europe, I used what I just had learned about standard monomial theory to prove a combinatorial tensor product decomposition rule for all groups for which standard monomial theory was available at that time; this was part of my PhD thesis. This rule generalizes the classical Littlewood-Richardson rule. Later, I formulated an effective version of this rule in positive characteristic, meaning standard monomial theory can be used to construct a good filtration of the tensor product $H^0(G/B, \mathcal{L}_\lambda) \otimes H^0(G/B, \mathcal{L}_\nu)$, and the multiplicities are calculated by the same rules.

While writing up the notes for Seshadri's course at Brandeis, I had many discussions with him. Seshadri was an extremely good and very patient teacher. He was not only explaining mathematics to me, I was also going with him to listen to a lot of Indian music. Of course, there were some concerts we went to in Boston, together with Lakshmibai, as well later in Chennai, when I visited him. But also while thinking about a question, he suddenly started to hum a raga, a very soothing habit.

There were two points coming up during these discussions with Seshadri, which never left my mind: the role of the extremal weight vectors in the theory, and the problem of finding a "useful" indexing system for a basis of a representation.

When Kashiwara developed the theory of crystal bases, one was looking for combinatorial models for this theory. The tableaux defined by Lakshmibai, Musili, and Seshadri turned out to be perfectly adapted to form such a model. Meanwhile, Lakshmibai had a conjectural version of tableaux for integrable highest weight representations of arbitrary Kac-Moody algebras. The tableaux consisted of a linearly ordered sequence of Weyl group elements and an increasing sequence of rational numbers satisfying

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certain conditions. To circumvent the intricate Weyl group combinatorics and to add more flexibility to the combinatorics, I had the idea to view the tableaux as a special class of piecewise linear paths, the weight of a path π being the endpoint $\pi(1)$. The crystal operators act in this language as local affine reflection operators, leading to the "path model" of a representation. In this language, Lakshmibai's pairs of sequences provide an explicit parametrization of a special path model, now called the Lakshmibai-Seshadri paths, or for short *LS paths*. This combinatorial model provides character formulae for Demazure modules and Weyl modules. Other consequences are a Littlewood-Richardson rule for tensor products of integrable highest weight representations of arbitrary Kac-Moody algebras, and a decomposition formula for restrictions of representations to Levi subalgebras.

The starting point of a standard monomial theory for the ring of sections $\bigoplus_{n \in \mathbb{N}} H^0(G/B, \mathcal{L}_{n\lambda})$ is the construction of a basis for $H^0(G/B, \mathcal{L}_\lambda)$, indexed by LS paths. Seshadri pointed out the importance of the role of the extremal weight vectors for the standard monomial theory, and the path model is very suggestive in this direction. For an LS path π of shape λ and an appropriate $m > 0$, the stretched path $m\pi$ is just a concatenation of straight lines joining the origin with extremal weights. This suggests taking the product of the corresponding extremal weight vectors, and defining the basis vector $p_\pi \in H^0(G/B, \mathcal{L}_\lambda)$ as the " m th root" of this product. The problem of finding an m th root of a section could be solved with the help of Lusztig's Frobenius map for quantum groups at a root of unity. This approach provides a type-independent construction of a standard monomial basis. The basis is compatible with Schubert varieties, opposite Schubert varieties, and Richardson varieties. The quadratic straightening relations generate the vanishing ideal of the embedded flag variety. In fact, these relations provide a Groebner basis. Other consequences: Schubert varieties $X_\tau \subset G/Q$ are defined linearly, the ring of sections $\bigoplus_{n \geq 0} H^0(X_\tau, \mathcal{L}_{n\lambda})$ is a Koszul ring, and one can construct a good filtration of tensor products. Chirivì used this theory to construct a flat deformation of an embedded Schubert variety into a union of toric varieties, and he and Maffei lifted the standard monomial theory to complete symmetric varieties.

The explicit construction of the bases of fundamental representations for the classical groups makes it possible to use the Jacobian matrix to determine the dimension of the tangent space of a Schubert variety at a T -fixed point. This point of view was taken in a series of articles by Seshadri, Lakshmibai, Sandhya, and others in order to establish explicit formulae for the dimension of tangent spaces, and to derive smoothness criteria for Schubert varieties (see for example the article by Lakshmibai and Seshadri on the

“Singular locus of a Schubert variety”). Similar arguments apply to varieties which can be realized as an open subset of a Schubert variety. As an example, let me mention the irreducible components of the variety of complexes discussed in “Schubert varieties and the variety of complexes” by Musili and Seshadri.

When Seshadri was visiting me in Strasbourg, we started to discuss an article by Pittie and Ram. Suppose G is a semisimple algebraic group. They consider in the T -equivariant K -theory of a flag variety the product $[\mathcal{O}_{X_\tau}]_T \cdot [\mathcal{L}_\lambda]_T$ of the class of the structure sheaf of a Schubert variety with the class of a line bundle associated to a dominant weight. The product is expressed as an explicit linear combination of classes of structure sheaves $[\mathcal{O}_{X_{e(\pi)}}]_T e^{\pi(1)}$, where the sum is running over a certain subset of LS paths, and the Schubert variety $X_{e(\pi)}$ is determined by the “final direction” $e(\pi)$ of π . Pittie and Ram were mostly using the combinatorial properties of the LS paths. Out of our discussion grew a joint article [LS03]. We use standard monomial theory to construct an explicit filtration of the $\mathcal{O}_{G/B}$ -sheaf $\mathcal{O}_{X_\tau} \otimes \mathcal{L}_\lambda$, stable under the action of a Borel subgroup B , such that the associated graded sheaf is a direct sum of structure sheaves of Schubert varieties, and the B -action is twisted by a character as in the formula by Pittie and Ram. In addition, we present indications that to have such an effective version of a Pieri-Chevalley formula is roughly equivalent to standard monomial theory. Indeed, similar filtrations have been considered in [LMS79, LS86], where they play a crucial role in developing a standard monomial theory for G/Q , where Q is a parabolic subgroup of classical type.

Another beautiful basis of representations is the one given by the Mirković-Vilonen cycles. This approach uses the realization of the representation of the group G as the intersection cohomology of an appropriate Schubert variety in the affine Grassmannian \mathcal{G} of its Langlands dual group G^\vee . The connection with LS paths, or rather LS galleries in this case, was established in a joint work with Gaussent. These galleries can be used to parametrize an open and dense subset of a Mirković-Vilonen cycle, and hence the LS-galleries form a natural indexing set for the cycles. Another consequence of this geometric construction: these galleries can be used to count points (over a finite field) in a certain intersection of orbits in the affine Grassmannian, and thus provide beautiful formulas to calculate Hall-Littlewood polynomials.

Despite all this, the LS paths remain somewhat mysterious. A new input came recently with the development of the Newton-Okounkov theory. Many polytopes known in representation theory, like Gelfand-Tsetlin polytopes and the string polytopes, got a new algebraic-geometric interpretation as Newton-Okounkov bodies. The integral

points in these polytopes index a basis of the representation. In a joint work with Chirivì and Fang, we get a corresponding interpretation of the LS paths. A standard way to get a \mathbb{Z}^{N+1} -valued valuation ($N = \dim G/B$) on the ring of sections $R_\lambda = \bigoplus_{n \in \mathbb{N}} H^0(G/B, \mathcal{L}_{n\lambda})$ is to fix a maximal chain of Schubert varieties. Roughly speaking, the valuation of a section is given by the degree and the successive vanishing multiplicities. Instead of choosing one fixed maximal chain, we take the minimum of the valuations arising from all possible maximal chains of Schubert varieties. This is just a quasi-valuation, but one can still associate to it a Newton-Okounkov body: it turns out to be the generalized polytope with integral structure studied by Dehy. The integral points are the LS paths. The standard monomials in R_λ form a perfect section to the filtration of the ring induced by the quasi-valuation. This provides an explanation of the LS path indexing a standard monomial p_π : the linearly ordered sequence of Weyl group elements corresponds to a maximal chain for which the minimal valuation is attained, the sequence of rational numbers corresponds to successive renormalized vanishing multiplicities of p_π with respect to this chain, and the sum of these renormalized vanishing multiplicities is the degree of the monomial. This can be viewed as a first step towards Seshadri’s conjectural interpretation of the LS paths, formulated by him in Volume 2 of his *Collected Papers*, as follows:

I have felt that a good understanding of SMT would be via a cellular Riemann-Roch formula as the definition of LS paths could be formulated geometrically in terms of the canonical cellular decomposition of G/B .

We will miss the stimulating, humorous, and encouraging discussions with him.



Peter Littelmann

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