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The cover design is based on imagery from "Making Journeys of Black Mathematicians," *page 240.*

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FEATURES

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INVITATIONS FROM THE AMS

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A WORD FROM...

Asamoah Nkwanta, Professor of Mathematics, Morgan State University

The opinions expressed here are not necessarily those of the Notices *or the AMS.*

Photo of Asamoah Nkwanta is courtesy of Charly Woodruff-White.

Hello mathematics community. The February issue of the *Notices* observes the nation's celebration of Black History Month. Three feature articles highlight the research of some Black mathematicians involving topics from compressed sensing for COVID-19 testing, graph optimization, and category theory and linear algebra in machine learning. The articles are: "Compressed sensing-based SARS-CoV-2 pool testing" by Bah, Petersen, and Jung, "The many face(t)s of zero forcing" by Hicks and Brimkov, and "The structure of meaning in language: parallel narratives in linear algebra and category theory" by Bradley, Gastaldi, and Terilla.

The Early Career section contains the following articles. Michael Young's article "The mathematicians of color alliance," which introduces us to a Black student support organization at Iowa State University. A brief history of the organization is given and four areas of how students are supported in the community are explained. Marissa Kawehi Loving's article, "How can we know what we deserve when toxicity is the norm?" covers harmful norms that marginalized people encounter in academic spaces, especially Black women. Strategies on how to challenge toxic experiences are also mentioned in the article. In India White's article, "Three steps for achieving equity and access in the math classroom," she states "Achieving equity in the math classroom is not only a matter of fairness but also crucial for ensuring that all students have equal opportunities to excel." The article explores strategies for departments to consider to create inclusion and diverse representation, collaborative learning, and safe learning spaces.

The Memorial section contains the following articles. Peter Eley's memorial article, "What I know for sure," remembers Dr. Lee Stiff, who was a nationally recognized mathematics educator and served for several years as a scholar and administrator at North Carolina State University. He also served as the National Council of Teachers of Mathematics (NCTM) President from 2000–2002, was on the NCTM Board of Directors from 1990–1993, and was an NCTM Lifetime Achievement Award winner in 2019. Johnny Houston's memorial article, "The legacy of Evelyn Boyd Granville (1924–2023)," honors the memory of Dr. Granville, who was the second African American woman to earn a PhD in mathematics. She received her PhD degree from Yale University in 1949. The National Association of Mathematicians (NAM) recognized her with a NAM 50th Anniversary Legacy Award in 2019.

The Math Education section contains the following articles. Luis Leyva's article, "Queer of color justice in undergraduate mathematics education," proposes a vision for undergraduate mathematics classrooms as spaces for queer of color justice. Daniel Zaharopol's article, "Nobody majors in STEM to fail," presents concrete ideas for how to create environments that support students' successes and see them as individuals.

In the Book Review section there is a review by Deanna Haunsperger of the book *Aspiring and Inspiring: Tenure and Leadership in Academic Mathematics* by Rebecca Garcia, Pamela E. Haris, Dandrielle Lewis, and Shanise Walker.

George Csicsery's Communication article, "Making *Journeys of Black Mathematicians*," describes his film project and gives a summary of some of the Black mathematicians interviewed and featured in his film.

The Black History Month *Notices* articles mentioned promote inclusivity and diversity in the mathematical sciences. In closing, I am again honored to serve as an associate editor to share with the *Notices* readers these fantastic articles.

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Compressed Sensing-Based SARS-CoV-2 Pool Testing

Bubacarr Bah, Hendrik Bernd Petersen, and Peter Jung

Introduction

In most countries, it seems that the population will have to live with COVID-19 for at least a while. We hope that most readers of this article will require very little convincing of the need to continually monitor the level of infection in the population. However, the monitoring has financial implications, in particular the cost of test kits. This results in a strong desire to reduce the number of tests required to identify infected individuals. One approach is pool testing, where test samples of individuals are pooled together and tested as one sample. If this sample turns out negative, then every individual whose sample is in the pool is declared negative. Otherwise, the group of samples can be subdivided and retested. This has the potential to significantly reduce the number of tests required. It has been intensively studied for COVID-19 (see for example [SHB+20, MNB+20]), and the United States Federal Drug Agency (FDA) granted pooled testing methods an emergency use authorization in July 2021.

The mathematical field of *group testing* is concerned with this pooling problem and has therefore also gained interest recently. It was invented by Dorfman [Dor43]. The methods from classical group testing often suffer from several drawbacks including slowness due to adaptivity of the tests and sensitivity to errors [DH06]. More on group testing in the next section. Compressed sensing, on the other hand, is a mathematical field concerned with recovering a vector with many zero entries from as few nonadaptive linear measurements as possible [Don06]. Compressed sensing has achieved several theoretical goals including achieving the optimal number of measurements, independence of the measurements from each other, robustness to noise, and sometimes even error-correcting properties [Don06, CRT06, LDB09]. A more detailed discussion on compressed sensing follows after the discussion on group testing.

In this manuscript, we revisit the use of compressed sensing for pool testing. We propose to use compressed sensing instead of classical group testing for viral detection since it tackles some weak points in the method of group testing mentioned above. We by no means claim originality in this; one of the first works is [GIS08]. However, we advocate for the use of a practical pooling strategy out of the forest of strategies proposed in the literature and the use of a new algorithm with comparable performance to the state-of-the-art in the field but having additional desirable properties that give it an advantage over other algorithms for pool testing.

The use of compressed sensing for viral detection of pools is premised on the fact that it is possible to measure the viral loads of patients. Presently, a standard test for the detection of SARS-CoV-2 is the real-time or quantitative polymerase chain reaction (qPCR).[1] The qPCR measurement are fluorescence intensities at every DNA amplification cycle, and potentially every one of those measurements can be used to compute viral loads.

Remark 1. The main reason for pooling, be it using group testing or compressed sensing, is to reduce the number of tests needed to identify infected individuals. This is why we focus on the number of pools (directly or indirectly in evaluating performance of polling designs and algorithms), typically given in complexity ("Big-O") notation.

[1] *This should not be confused with a reverse transcription PCR which is often being abbreviated RT-PCR* [SLB+06].

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This article is organized as follows. The next two sections give a more detailed introduction to group testing and compressed sensing. Then we discuss the modeling of the pooling process and the identification of infections. We end with a discussion and conclusion.

Group testing. Group testing is concerned with discovering a class of interest from a larger set containing nonmembers of the class of interest without testing each member of the set. This is done by subdividing the set into subsets and testing each subset as one element. The goal is that with the number of subset/groups being far smaller than the total number of elements in the set, one is able to successfully identify the class of interest. More precisely, group testing seeks to identify k members of a class of interest out of a set of n elements by performing m tests of m groups, where $m \ll n$. In the disease identification setting, we would like to identify k infected individuals from a population of n individuals performing m group (pool) tests, again we require that $m \ll n$. The ratio k/n (denoted as p) is known as the prevalence. Usually, group testing works well when p is small. It can be shown by a combinatorial argument that we need $\mathcal{O}\left(k \log_2(n/k)\right)$ tests for pool testing to work [Hwa72]. Figure 1 shows an example of pool testing to identify an infected individual. Here each individual participates in only one pool.

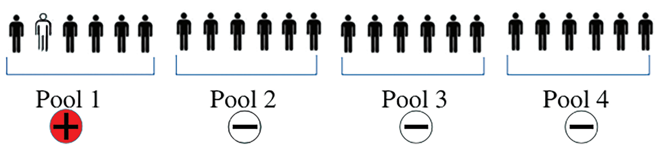

Figure 1. Testing a population of size $n = 24$ with only 1 infected individual (in white), i.e., prevalence of $\approx 4\%$. If each individual is tested, it would required 24 tests (i.e., 24 test kits). Using group testing with pool sizes of 6, members of pools 2, 3, and 4 are declared negative; while those in pool 1 are positive. Each individual in pool 1 is then retested, making it a total of 10 tests for group testing as opposed to 24 tests for testing each individual.

We can have pool testing setups where individuals participate in more than one pool. Figure 2 illustrates this with each individual participating in 2 pools. This can be represented in a two-dimensional (2D) grid; while the example in Figure 1 can be considered one-dimensional (1D). Going from 1D to 2D may reduce the need to retest. Moreover, this can be extended to a d-dimensional (dD) setting, where each individual will participate in d pools. This is what the successfully applied *hypercube* method is about, see [MNB$^+$20].[2] The 1D, 2D, ..., dD pool testing are instances of the so-called *array testing*, see [PS94].

Mathematically, we can represent the population of size n by a binary vector, $\mathbf{x} \in \{0,1\}^n$ where there are k ones

[2] *This is the approach being used in Rwanda.*

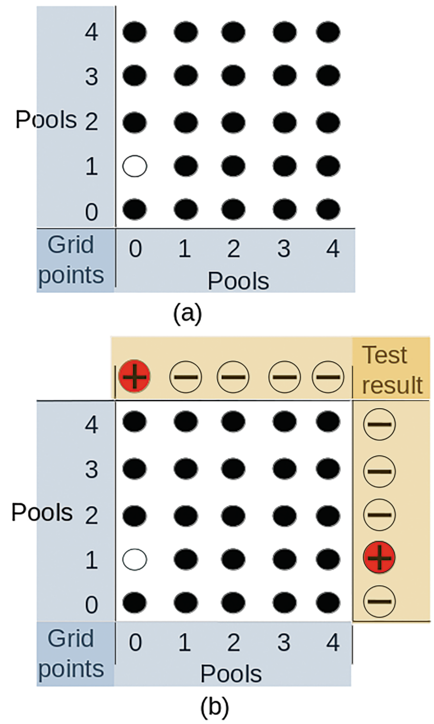

Figure 2. Testing a population of size $n = 25$ with only 1 infected individual (white dot), i.e., prevalence of 4%. Individual testing would require 25 tests (i.e., 25 test kits). Using group testing with pool sizes of 5 and each individual participating in two pools, the infected individual is identified by the row and column pools which contain their sample. This pooling strategy requires only 10 tests.

for the infected individuals and $n-k$ zeros for the non-infected individuals. Pooling samples of a group together is equivalent to evaluating

$$\mathbf{a} \circ \mathbf{x} := \max_{j=1,\dots,n} \{a_j x_j\}, \tag{1}$$

for some appropriate binary vector $\mathbf{a} \in \{0,1\}^n$. For a matrix $\mathbf{A} \in \{0,1\}^{m \times n}$ whose rows $\mathbf{a}_i$ represents the m groups, we set

$$\mathbf{y} := \mathbf{A} \circ \mathbf{x} := [\mathbf{a}_i \circ \mathbf{x}]_{i=1,\dots,m}\,. \tag{2}$$

The vector $\mathbf{y}$ is referred to as the *measurement/observation* vector.

The recovery of the infected individuals, equivalently $\mathbf{x}$, from observing $\mathbf{y}$ and knowing $\mathbf{A}$ would lead to solving the following binary system of equations, since both $\mathbf{A}$ and $\mathbf{x}$ are binary.

$$\mathbf{A} \circ \mathbf{x} = \mathbf{y}. \tag{3}$$

The setting described above where the groups are fixed is known as the *nonadaptive* case of group testing. Alternatively, one may form a group after knowing the outcome of the preceding test(s). In that case we perform *adaptive* group testing. Nonadaptive approaches are faster but they require more measurements than adaptive approaches. Adaptive methods achieve the optimal $\mathcal{O}\left(k \log_2(n/k)\right)$

measurements [Hwa72]; while non-adaptive methods need $\mathcal{O}\left(k^2 \log_k(n)\right)$ measurements [AA13].

Many recovery algorithms for group testing have been proposed. One popular such algorithm for nonadaptive group testing is Combinatorial Orthogonal Matching Pursuit (COMP). In the noiseless setting the COMP algorithm assigns a sample as positive (i.e., infected) if and only if all the k tests containing the sample satisfy $y_m > 0$. COMP will work perfectly, if the pooling strategy is such that we have "k-disjunct" sets of measurements (see [PBJ20, Definition 4.1]). Another algorithm for the adaptive case, is the binary splitting algorithm by [Hwa72].

Compressed sensing. Mathematically, compressed sensing is concerned with the construction of a linear operator and the solution of an underdetermined system of linear equations resulting from the application of the linear operator. More precisely, let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be our linear operator, let $\mathbf{x} \in \mathbb{R}^n$ be the unknown variable vector and let the application of $\mathbf{A}$ result in $\mathbf{y} \in \mathbb{R}^m$. Therefore, with $m \ll n$, we have an underdetermined linear system (note both $\mathbf{A}$ and $\mathbf{x}$ are real):

$$\mathbf{A}\mathbf{x} = \mathbf{y}. \tag{4}$$

In compressed sensing parlance, $\mathbf{A}$ is called the sensing/measurement matrix, $\mathbf{y}$ is the measurement vector, and $\mathbf{x}$ is the signal of interest. Linear algebra tells us that there are infinitely many solutions to (4). However, it is possible to obtain a unique solution if we make some assumptions on $\mathbf{A}$ and $\mathbf{x}$. The main assumption on $\mathbf{x}$ is that it has some simplicity/redundancy. The simplicity of $\mathbf{x}$ is either *sparsity* or *compressibility*. By sparsity of $\mathbf{x}$ we mean it has few nonzero elements. Let $\mathcal{S}$ be the *support* of $\mathbf{x}$, i.e., $\mathcal{S} = \{j \;:\; x_j \neq 0, \forall j \in 1,\dots,n\}$ and let $|\mathcal{S}| \leq k$, then $\mathbf{x}$ is said to be k-sparse. Conventionally, the $\|\mathbf{x}\|_0$ (the ℓ_0-norm of $\mathbf{x}$)[3] is used to denote the sparsity of $\mathbf{x}$. Precisely,

$$\|\mathbf{x}\|_0 := \sum_{j=1}^{n} \mathbb{I}_{\mathcal{S}}(j), \tag{5}$$

where $\mathbb{I}$ is the indicator/characteristic function. On the other hand, $\mathbf{x}$ is said to be k-compressible when $\mathbf{x}$ can be approximated quite well by a k-sparse vector. Furthermore, $\mathbf{x} = \Psi\mathbf{z}$ is k-sparse (k-compressible) in a basis Ψ when $\mathbf{z}$ is k-sparse (k-compressible). We denote the restriction of $\mathbf{x}$ on its support as $\mathbf{x}_{\mathcal{S}} \in \mathbb{R}^k$ and the restriction of $\mathbf{x}$ on the complement of the support as $\mathbf{x}_{\mathcal{S}^c} \in \mathbb{R}^{n-k}$.

The assumption on $\mathbf{A}$ is that it is an information-preserving projection or a bi-Lipschitz linear metric space embedding of all k-sparse vectors into $\mathbb{R}^m$ (henceforth referred to as *a stable linear embedding*). The conditions for $\mathbf{A}$ to fulfill the information preservation requirement include the following.

(i) **Restricted Isometry Property (RIP).** A more general definition of the RIP, i.e., ℓ_p-norm-restricted isometry property (RIP_p) for $p \geq 1$ is the following.

Definition 2. Matrix $\mathbf{A}$ has RIP_p of order k, with constants $\delta_k < 1$, if for all k-sparse $\mathbf{x}$, it satisfies

$$(1-\delta_k)\|\mathbf{x}\|_p^p \leq \|\mathbf{A}\mathbf{x}\|_p^p \leq (1+\delta_k)\|\mathbf{x}\|_p^p. \tag{6}$$

For small δ_k, we have $\mathbf{A}$ being a near isometry and this implies that it is information preserving. Note that $p = 2$ is typically written as RIP (without the subscript 2).

(ii) **Nullspace Property (NSP)**, here defined in the noiseless settings.

Definition 3. Matrix $\mathbf{A}$ has the nullspace property of order k, if for any $\mathbf{v} \in \text{Null}(\mathbf{A})$ and any set $\mathcal{S} \subset \{1,\dots,n\}$ with $|\mathcal{S}| \leq k$, we have

$$\|\mathbf{v}_{\mathcal{S}}\|_1 \leq \|\mathbf{v}_{\mathcal{S}^c}\|_1. \tag{7}$$

If $\mathbf{A}$ satisfies this property, then there exists a unique k-sparse vector solving (4).

(iii) M^+ **criterion** is defined as follows.

Definition 4. Matrix $\mathbf{A}$ obeys the M^+ criterion with vector $\mathbf{u}$ and constant $\varkappa$, if $\mathbf{A}^T\mathbf{u} > 0$ and $\varkappa := \max_{i\in[n]} \left|(\mathbf{A}^T\mathbf{u})_i\right| \times \max_{i\in[n]} \left|\left((\mathbf{A}^T\mathbf{u})_i\right)^{-1}\right|$.

Note that $\varkappa$ is actually a condition number of the diagonal matrix with diagonal $\mathbf{A}^T\mathbf{u}$.

An important goal of compressed sensing is to make projections by $\mathbf{A}$ to map to spaces with very small dimensions (i.e., small m) such that recovery is possible to a certain error tolerance. This translates to finding matrices $\mathbf{A}$ with small m, which have a good restricted isometry property or a good nullspace property. It is known that matrices can have RIP_p with $p = 1$ or $p = 2$ or the nullspace property of order k, if the number of measurements, $m = \mathcal{O}(k \log(n/k))$ [CDD09]. This is referred to as an *optimal sampling rate*.

The second part of the compressed sensing problem is the reconstruction of $\mathbf{x}$ from the projection $\mathbf{A}\mathbf{x}$. The system in (4) is underdetermined, which means $\mathbf{A}$ is not invertible. The aim of getting the sparsest solution that is faithful to the data leads to what is called the ℓ_0 problem, i.e.:

$$\hat{\mathbf{x}} = \operatorname{argmin}_{\mathbf{z}} \|\mathbf{z}\|_0 \quad \text{such that} \quad \mathbf{A}\mathbf{z} = \mathbf{y}. \tag{8}$$

This is a combinatorial and nonconvex problem, which has been shown to be NP-hard to solve. However, many discrete (also referred to as "greedy") algorithms have been proposed for this problem with provable recovery guarantees. These include iterative hard thresholding (IHT), orthogonal matching pursuit (OMP), and compressive sampling matching pursuit (CoSaMP) [FR13].

[3] *This is not a norm because $\|\lambda\mathbf{x}\|_0 = \|\mathbf{x}\|_0$ for all scalars $\lambda \neq 0$ and all vectors* $\mathbf{x}$.

On the other hand, the ℓ_0 problem can be relaxed to a convex one. A typical case is the Basis Pursuit (BP) algorithm that solves

$$\hat{\mathbf{x}} = \operatorname{argmin}_{\mathbf{z}} \ \|\mathbf{z}\|_1 \quad \text{such that} \quad \mathbf{Az} = \mathbf{y}. \tag{9}$$

It has been well established that the solution of BP coincides with the solution of the ℓ_0 problem under most of the conditions on $\mathbf{A}$ and $\mathbf{x}$ stated above. The recovery of all compressed sensing algorithms are expected to be stable and obey an instance optimality (ℓ_p/ℓ_q-approximation) guarantee according to which any solution $\hat{\mathbf{x}}$ satisfies

$$\|\hat{\mathbf{x}} - \mathbf{x}\|_p \leq C \frac{\sigma_k(\mathbf{x})_q}{\sqrt{k}}, \quad \text{for } 1 \leq q \leq p \leq 2, \tag{10}$$

where C is an absolute constant independent of $\mathbf{x}$ and

$$\sigma_k(\mathbf{x})_q \coloneqq \min_{k-\text{sparse } \mathbf{x}'} \|\mathbf{x} - \mathbf{x}'\|_q, \tag{11}$$

that is the best k-term approximation of $\mathbf{x}$. There is a more general definition of the instance optimality encompassing the noise setting too [FR13].

We conclude this section by mentioning compressed sensing with non-negativity constraints. This problem is typically formulated in the following way.

$$\hat{\mathbf{x}} = \operatorname{argmin}_{\mathbf{z} \geq 0} \ \|\mathbf{Az} - \mathbf{y}\|. \tag{12}$$

In the case relevant to the focus of this manuscript, which is pool testing, the norm in (12) is taken to be $\|\cdot\|_1$ (i.e., the ℓ_1-norm). In this setting (12) is known as the *non-negative least absolute deviation* (NNLAD) problem. The authors of [PBJ21] proposed an efficient and tuning-free algorithm (dubbed NNLAD) for this problem.

Viral Detection in Pooled Tests

Testing design. Suppose we want to find k individuals infected with a virus among n individuals. According to the information theoretic lower bound we require at least $\log\left(\binom{n}{k}\right)$ tests to find the infected individuals. The binary splitting algorithm proposed by [Hwa72] finds the k infected individuals with a number of tests in $\mathcal{O}\left(k \log_2(n/k)\right)$. However, since the tests are adaptive, each subsequent test is designed depending on the outcome of previous tests, and thus each test has to wait for the result of the previous one. If it takes a long time to perform a test, it might be desirable to perform multiple tests at once. In such cases nonadaptive methods are preferable. There are many deterministic nonadaptive methods for group testing. Most of these prove their results using disjunct matrices. For instance, in [AA13] a nonadaptive group testing method is presented whose number of tests is in $\mathcal{O}\left(k^2 \log_2(n)/\log_2(k)\right)$.

We propose to use compressed sensing with additional nonnegativity constraints to solve this problem. We collect specimens from n individuals arranged in a vector $\mathbf{x} \in \mathbb{R}^n$, and the amount of virus in the specimen of the jth individual is denoted by the non-negative quantity $x_j \in \mathbb{N} \cup \{0\} \subset \mathbb{R}$. We assume that viruses are evenly distributed in each specimen, meaning that if we take $\alpha \in [0,1]$ of the volume of the specimen of the ith individual, it will contain roughly αx_j viruses. Since, k individuals are infected we have $\|\mathbf{x}\|_0 = k$. Let the sample of the ith test contain a fraction a_{ij} of the amount of specimen of the jth individual. The sample of the ith test thus contains, up to rounding errors, the amount of viruses

$$\sum_{j \in [n]} a_{ij} x_j = (\mathbf{Ax})_i \eqqcolon y_i,$$

where $\mathbf{A}$ is an $m \times n$ matrix with entries $a_{ij} \in [0,1]$ with column sums of at most one. The number of tests is thus m. Often one makes the assumption that $\mathbf{x}$ is a random vector with i.i.d. elements, for instance a Poisson random variable multiplied by a Bernoulli random variable with parameter $p = k/n$. Instead we take it here as a deterministic unknown. It is assumed that qPCR can be used to generate an estimate y_i of the amount of virus in the ith test $(\mathbf{Ax})_i$. This procedure is not accurate and errors $e_i \coloneqq y_i - (\mathbf{Ax})_i$ might occur. We try to recover the amount of virus in the specimen of the individuals according to

$$\mathbf{y} = \mathbf{Ax} + \mathbf{e}, \tag{13}$$

with a possibly small $\|\mathbf{x}\|_0$ and m as small as possible. As discussed above, this is exactly a compressed sensing problem where, due to the nature of the problem, $\mathbf{x}$ is non-negative and exactly k-sparse (compressed sensing guarantees often also work for compressible vectors which are well approximated by sparse vectors). The theory of compressed sensing states that there exist matrices and efficient decoders that allow recovery of $\mathbf{x}$ if m is $\mathcal{O}\left(k \log(n/k)\right)$ [Don06, CRT06]. It remains to design a suitable measurement/sensing matrix and a reconstruction algorithm.

Measurement matrices. Recall from the discussion on compressed sensing above that there exist matrices that allow recovery of $\mathbf{x}$ from (13) if m is in $\mathcal{O}\left(k \log(n/k)\right)$, referred to as the *optimal scaling*. The testing design described above is a special compressed sensing task where the elements of the matrix $\mathbf{A}$ and the unknown sparse vector $\mathbf{x}$ are non-negative. Non-negative $\mathbf{A}$ does not have RIP_2 in the optimal regime and instead one has to resort to RIP_1 or use tools like the NSP (7). For example, it is known that matrices with independent and uniformly distributed entries on $\{0,1\}$ achieve optimal scaling [KJ18] using the NSP, while adjacency matrices of expander graphs have the optimal scaling using RIP_1. However, deterministic (i.e., nonrandom) construction of such matrices is an open problem.

As far as we know, the best deterministic construction (in terms of optimal scaling) of binary compressed sensing matrices (expander matrices) is [GUV09]. This construction has a near optimal scaling but is difficult to implement. Precisely, for any $\alpha > 0$ there exists a constant C_α such that the number of rows of the matrix is:

$$m \leq C_\alpha k^{1+\alpha} \left(\log_2(n) \log_2(k)\right)^{2+\frac{2}{\alpha}}. \tag{14}$$

The downside of this result is that the constant C_α is rather large for small α.

For ease of implementation (i.e., practical reasons), we propose using the sub-optimal matrices with explicit constructions in [LV20] and [KS64], i.e., adjacency matrices of *low-girth left-regular bipartite graphs* and *k-disjunct* matrices respectively, in which the number of measurements m is in the order of $k\sqrt{n}$. Actually, we established the equivalence of the two constructions in [PBJ20, Theorem 4.3]. Such k-disjunct matrices do not achieve the optimal rate for fixed k as $n \to \infty$ according to [DR82]. However, for viral detection we have a fixed prevalence $p = k/n$ in mind. The prevalence might be larger in the beginning of a pandemic and smaller when healthcare professionals are testing regularly and asymptotically, but for each of these applications we consider a fixed $p = k/n$. In this scenario our result achieves the rate

$$m = pn^{\frac{3}{2}} + n^{\frac{1}{2}}, \tag{15}$$

which can outperform results that achieve optimality according to [DR82]. For instance the construction of [AA13, Corollary 1] achieves a number of measurements

$$m \leq 2k^2 \frac{\log_2(n)}{\log_2(k)} = 2p^2n^2 \frac{\log_2(n)}{\log_2(n) + \log_2(p)}. \tag{16}$$

For a suitably chosen n, this upper bound and other constructions can be outperformed by the construction we propose.

Determining infections. The classical decoding procedure for disjunct matrices (known as COMP) iterates over all $j \in [n]$ and, if for a fixed j and for all i with $a_{ij} \neq 0$, the ith test is positive, it declares the jth individual as infected. This process is simple and, if $\mathbf{A}$ is scaled to be k-disjunct with column sums of 1, there is no noise and there are no more than k infected individuals, then it is guaranteed to find all infected individuals [KS64]. However, every false negative test will result in at least one individual that is falsely flagged as not infected. Thus, this decoding procedure is incredibly sensitive to noise. There are extensions to these matrices which tolerate a fixed number of errors under restrictive conditions [AA13].

Compressed sensing, on the other hand, not only detects infected individuals but also estimates the viral load, which may have further benefits to the medical practitioners. The noise in (13) is nonzero in general, since the estimate is affected by some errors including, rounding errors and inaccuracy of the qPCR. In [GAR+20] the noise is modeled as a heavy-tailed random variable depending on the unknown $\mathbf{Ax}$.

Therefore, it is difficult to apply recovery methods from compressed sensing for independent additive noise out of the box. Parameter tuning, using, e.g., cross validation, is often crucial for most of these methods in these noise settings. Interestingly, non-negativity helps with such noise models. Combining the heavy-tailed noise model with the non-negativity of the viral load data (i.e., $\mathbf{x}$), we recommend using the parameter tuning free *non-negative least absolute deviation* (NNLAD), proposed in [PBJ21], for recovery which is any minimizer

$$\hat{\mathbf{x}} = \operatorname*{argmin}_{\mathbf{z} \geq 0} \|\mathbf{Az} - \mathbf{y}\|_1. \tag{NNLAD}$$

In [PBJ21], the authors showed that for certain matrices $\mathbf{A}$, for example what they refer to as random walk matrices of lossless expander graphs (which can be considered to include disjunct matrices), the convex NNLAD approach is indeed sparsity promoting and allows for an estimate of the form

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_1 \leq C \|\mathbf{e}\|_1, \tag{17}$$

for some constant C independent of $\mathbf{e}, \mathbf{x}, \mathbf{y}$, and $\hat{\mathbf{x}}$.

Finally, given a certain threshold ϵ, one declares the individual n to be infected if $\hat{x}_j > \epsilon$, for small $\epsilon > 0$. If the noise level $\|\mathbf{e}\|_1$ is small enough, this method will guarantee that these are exactly the infected individuals. Thus, in this sense compressed sensing gives a nonadaptive testing procedure which provably finds k infected individuals among n with the scaling-optimal number of tests, and small errors would have no effect on the test result. The guarantees in [PBJ21] follow from a self-regularization feature in the non-negative case which has been worked out already for other cases, like the non-negative least squares in [KJ18]. Note that a comprehensive comparative study of the performance of NNLAD vis-a-vis other traditional compressed sensing algorithms was conducted in [PBJ21].

The take-home message of this section is that we suggest a quasi-optimal pooling procedure with an efficient noise-robust recovery algorithm. This is a more practical setup that can be implemented in a straight forward way at a medical lab doing COVID testing. In particular, for the measurement matrix/design we trade-off optimal scaling with ease of use; while for the decoder, i.e., NNLAD, we trade-off ease of decoding for noise robustness (when compared to COMP) and for no parameter tuning (when compared to many compressing decoders).

Discussion

Now we try to put the above theoretical results into the right perspective by doing a bit more detailed comparison of our proposed approach to other approaches proposed for group/pool testing. We also discuss some empirical results and then draw conclusions.

Nonadaptive methods. Consider the nonadaptive method from [AA13] with number of tests less than $p^2 n^2 \frac{\log_2(n)}{\log_2(n)+\log_2(p)}$, see (16). Some empirical example comparisons are made in Table 1. In many scenarios, our method requires a similar number of tests as other methods from nonadaptive group testing and sometimes even requires fewer.

Work	p (k/n)	n	Q	m/n
[AA13]	0.01	900	*unknown*	0.5573
Ours	0.01	900	30	0.3333
[AA13]	0.001	10000	*unknown*	0.0800
Ours	0.001	10000	100	0.1100

Table 1. A comparison of the method of [AA13] to ours looking at different prevalences and population sizes. We denote prevalence by p, number of infected individuals by k, population size by n, pool size by Q and number of tests by m. Note that the rates (m/n) are based on approximations of the exact expected number of tests.

Adaptive methods. Adaptive methods outperform nonadaptive methods in general. Although nonadaptive, we compare our method to other adaptive methods in Table 2 below. Many adaptive group testing methods require half as many tests as our method, but ours has the advantage that it only requires one stage and can thus be performed faster.

Work	p (k/n)	n	Q	m/n
[Dor43]	0.01	*indep. of n*	10	0.2000
[MNB+20]	0.01	*indep. of n*	35	0.1168
Ours	0.01	900	30	0.3333
[Dor43]	0.001	*indep. of n*	32	0.0633
[MNB+20]	0.001	*indep. of n*	350	0.0179
Ours	0.001	10000	100	0.1100

Table 2. A comparison of our method to other methods including [Dor43, MNB+20] for different prevalences and population sizes. We denote prevalence by p, number of infected individuals by k, population size by n, pool size by Q and number of tests by m. Note: (i) that the rates (m/n) are based on approximations of the exact expected number of tests; (ii) where these approximations can be computed without involving n, we put *independent (abbreviated indep.) of n* under the n column.

Error correcting properties. We demonstrate an advantage of the NNLAD approach over other compressed sensing decoders in a short experiment with synthetic data. One common error model is to use multiplicative noise, i.e., $y_m = g_m \times (\mathbf{A}\mathbf{x})_m$ for some random variable g_m. This yields that the magnitude of certain noise components are significantly larger than others and you end up with a peaky noise. This motivates us to model the additive noise vector $\mathbf{e} := \mathbf{y} - \mathbf{A}\mathbf{x}$ approximately as a sparse random vector. Following [PBJ20, Theorem 4.5] for group sizes of 31 and $k = 7$, we are guaranteed to detect up to 7 infected people among 961 by using 248 tests. However, empirically the identification will succeed even if more than 7 individuals are infected and even if multiple measurements are corrupted. In Figure 3, we vary the prevalence $p = \frac{\|\mathbf{x}\|_0}{n}$ and the fraction of corrupted measurements $p_e = \frac{\|\mathbf{e}\|_0}{m}$ and plot the probability that the NNLAD estimator $\hat{\mathbf{x}}$ is sufficiently close to the true signal $\mathbf{x}$ in the ℓ_1-norm.

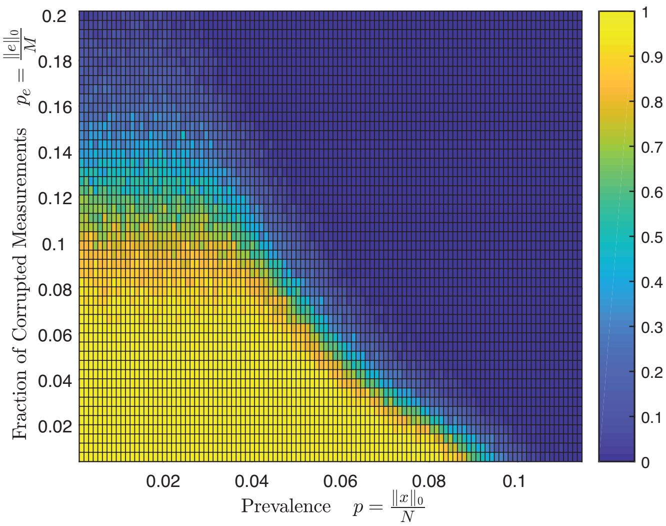

Figure 3. A phase-transition plot of the probability of recovery as a function of the prevalence and the number of corrupted measurements for pool sizes of 31 and $k = 7$.

We see that as guaranteed by [PBJ20, Theorem 4.5] for $p = \frac{\|\mathbf{x}\|_0}{n} \leq \frac{7}{31^2} \approx 0.0073$ and $p_e = 0$, the recovery succeeds. But empirically the recovery also succeeds for $p \leq 0.08$ and $p_e = 0$, i.e., 10 times higher than guaranteed. This suggests that k and thus m could be reduced and still recover whenever $p \leq 0.02$ for instance, which might be sufficient for a prevalence of $p = 0.01$.

Further the NNLAD seems to successfully recover even in the presence of sparse noise, i.e., when p_e is small. This is not surprising as the NNLAD minimizes the ℓ_1 norm of all possible noises and ℓ_1-minimization is sparsity promoting [CT05]. Recovery seems to succeed whenever $p_e \leq 0.06$ and $\frac{4}{3}p_e + p \leq 0.08$. This error correcting property gives the NNLAD advantages over other compressed sensing decoders in the presence of heavy outliers.

However, the error correcting properties cannot be guaranteed uniformly for all $\mathbf{e}$ with $p_e \leq 0.06$. If the noise components are active exactly on the support of a column of $\mathbf{A}$ where the signal $\mathbf{x}$ is nonvanishing, the measurement might correspond to a different signal with the same support. Hence, recovery might fail as soon as $\|\mathbf{e}\|_0$ is at least as large as the number of nonzero entries in a column of $\mathbf{A}$, in this case $k+1=8$. However, there are $\binom{m}{k+1}=\binom{248}{8}\approx 3.16\cdot 10^{14}$ different possible supports of a $k+1$-sparse noise, but only $n=961$ of these appear as columns of the matrix. Thus, if the noise support is drawn uniformly at random, such an event is highly unlikely. Thus, we see that the NNLAD is empirically correcting more errors than expected.

Conclusion. We have explained how compressed sensing can be used to solve the viral detection problem. It generates a nonadaptive testing procedure. Further, we have presented a construction of design matrices that can be used for classical nonadaptive group testing and compressed sensing based viral detection. The construction requires roughly as many tests as other methods from nonadaptive group testing and can possibly outperform those. Adaptive group testing methods still require fewer tests but can only be performed sequentially which is a critical problem in a pandemic with an exponential growth rate. We have proposed to use the NNLAD as a compressed sensing decoder, since, compared to other compressed sensing decoders, it is robust against heavy-tailed noise and does not requires knowledge of noise level. In particular, it also has certain error correcting properties. Lastly our method also computes the viral load of infected individuals.

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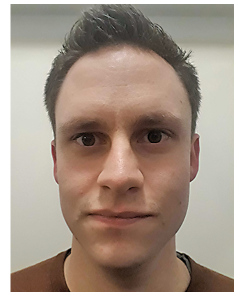
Bubacarr Bah



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Credits

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The Many Face(t)s of Zero Forcing

Illya V. Hicks and Boris Brimkov

Introduction

What does getting good movie recommendations from Netflix have in common with monitoring the electrical power grid, searching for a fugitive who is trying to evade capture, and controlling a quantum system? All these tasks, and several others, can be modeled as the same graph optimization problem called *zero forcing*, and can be approached with the same battery of computational techniques. As an analogy, the zero forcing process can be thought of as a sudoku puzzle where through the limited information given, the rest of the missing information in the grid can be inferred. In this article, we outline several different settings where the zero forcing problem arose independently, and then discuss some of the solution techniques and remaining challenges related to the problem.

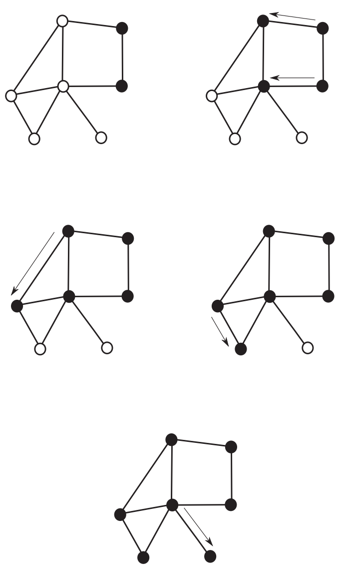

Figure 1. *Top Left:* A minimum zero forcing set of the graph is marked by filled vertices (which represent blue vertices). The color changes are illustrated in each step from left to right, top to bottom.

Zero forcing. We will begin with a purely graph-theoretic definition of zero forcing. Let $G = (V, E)$ be a graph and $S \subset V$ be a set of vertices *initially colored* blue, all other vertices being colored white. The *color change rule* dictates that at each timestep, a blue vertex u with a single white neighbor v may change the color of (or *force*) that neighbor to blue. Note that when several color changes are possible in a given timestep, they can be performed in any order or all at once. The *closure* of S is the set of blue vertices obtained after the color change rule is applied iteratively until no more color changes are possible. A *zero forcing set* is a set whose closure is all of V and the *zero forcing number* of G, denoted $Z(G)$, is the minimum cardinality of a zero forcing set. See Figure 1 for an illustration.

Minimum rank. A *recommender system* is a system that provides recommendations to users that best suit their preferences. Such systems are used by online retailers to suggest related products, by streaming platforms to suggest new songs or movies, and by dating sites to suggest potential matches. One of the key techniques used by recommender systems is based on the following idea: The recommender system has available a partially known preference matrix whose (i, j) entry represents the preference of user i toward product j; see Table 1 for an example.

	Bridesmaids	*Inception*	*Star Wars*	*Jaws 2*
Alice	0.85		0.91	
Bob	0	0.33		0.79
Caleb			0.84	0.95
Daisy	0.67	0.45		0

Table 1. A preference matrix for a movie recommender system.

The system assumes that the users can be grouped into a relatively small number of categories where users within each category have similar preferences for a given product and that products have a relatively small number of attributes that determine whether or not they are preferred by a given user. Then, in order to make good suggestions to the users, the recommender system tries to guess the

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unknown entries in the matrix so that the resulting matrix has minimum rank. The rank of this completed matrix can roughly be thought to correspond to the number of attributes of products or categories of users.

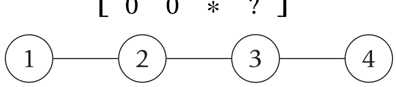
Due to its importance and prevalence, this matrix completion problem has received considerable attention and funding (such as the million-dollar Netflix prize [Kor09]). Many different variants of the minimum rank matrix completion problem have been studied. One variant, which is particularly attractive because it allows a graph theoretic representation of the problem, requires that the preference matrix be symmetric and that the only available information about its entries be whether they are zero or nonzero. Thus, such a preference matrix can be thought of as a "matrix pattern" encompassing all possible completions that fit the pattern.

More formally, let I be a set of ordered pairs (i,j) such that $i,j \in \{1,\dots,n\}$, $i \neq j$, and if $(i,j) \in I$ then $(j,i) \in I$. A *symmetric matrix pattern indexed by I* is the set of all real symmetric matrices whose (i,j) entry is

$$\begin{cases} \text{nonzero} & \text{if } (i,j) \in I, \\ \text{zero} & \text{if } (i,j) \notin I \text{ and } i \neq j, \\ \text{zero or nonzero} & \text{if } i = j. \end{cases}$$

We will say that a real symmetric matrix A_0 *fits the pattern of A* if $A_0 \in A$. Moreover, since A is symmetric, we can also think of it as a weighted adjacency matrix of a graph; we will say that a graph G *corresponds to A* if the adjacency matrix of G fits the pattern of A. Below is an example of a symmetric matrix pattern, the graph corresponding to that pattern, and two matrices that fit that pattern. In the matrix pattern, 0 denotes zero entries, * denotes nonzero entries, and ? denotes entries that could be zero or nonzero.

$$A = \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix},$$

$G =$ 

$$A_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & -5 & 0 & 0 \\ -5 & \sqrt{2} & 99 & 0 \\ 0 & 99 & \pi/3 & 0.2 \\ 0 & 0 & 0.2 & 7 \end{bmatrix}.$$

We will note here that in the literature, symmetric matrix patterns are more commonly defined as $S(G) = \{A \in S_n : \mathcal{G}(A) = G\}$, where G is a graph of order n, S_n is the set of all real symmetric $n \times n$ matrices, and $\mathcal{G}(A)$ is the graph with vertices $\{1,\dots,n\}$ and edges $\{\{i,j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. That way, the set of matrices is described by a graph, not the other way around. However, in the context of preference matrices for recommender systems, the graph is only a technical tool rather than the primary object of interest. Thus, we will proceed with our version of the definition.

With that, we define the *minimum rank* of an $n \times n$ symmetric matrix pattern A as the minimum rank over all matrices that fit the pattern, denoted $\mathrm{mr}(A) = \min\{\mathrm{rank}(A_0) : A_0 \in A\}$. Similarly, the *maximum nullity* of A is defined as $M(A) = \max\{\mathrm{null}(A_0) : A_0 \in A\}$. By the rank-nullity theorem, $\mathrm{mr}(A) = n - M(A)$.

Since by definition the minimum rank is computed over an uncountably infinite family of matrices, direct computation of this parameter is very difficult. There are direct computation approaches (for example by finding Gröbner bases of several systems of polynomials [Bri19]), but they are prohibitively slow. Thus, research has instead focused on approximating or bounding the minimum rank of a matrix pattern by combinatorial or algebraic means. It was this goal of finding an accessible bound to the minimum rank that first gave rise to zero forcing [AMRSGWG08].

We will illustrate this idea by computing the minimum rank of the matrix pattern A shown above. Let

$$A^\star = \begin{bmatrix} a_1^\star & a_2^\star & 0 & 0 \\ a_2^\star & a_3^\star & a_4^\star & 0 \\ 0 & a_4^\star & a_5^\star & a_6^\star \\ 0 & 0 & a_6^\star & a_7^\star \end{bmatrix}$$

be a matrix which fits the pattern of A and realizes $\mathrm{mr}(A)$, i.e., $\mathrm{rank}(A^\star) = \mathrm{mr}(A)$. Note that $a_2^\star, a_4^\star, a_6^\star$ are nonzero real numbers, and $a_1^\star, a_3^\star, a_5^\star, a_7^\star$ are arbitrary real numbers. Let $x = [x_1\ x_2\ x_3\ x_4]^T$ be a null vector of $A^\star$. Then, we have the following equations:

$$a_1^\star x_1 + a_2^\star x_2 = 0, \tag{1}$$
$$a_2^\star x_1 + a_3^\star x_2 + a_4^\star x_3 = 0, \tag{2}$$
$$a_4^\star x_2 + a_5^\star x_3 + a_6^\star x_4 = 0, \tag{3}$$
$$a_6^\star x_3 + a_7^\star x_4 = 0.$$

Suppose $x_1 = 0$; then, in (1), x_2 is forced to be zero; in (2), x_3 is forced to be zero (since x_1 and x_2 are zero), and in (3), x_4 is forced to be zero (since x_2 and x_3 are zero). Thus, if $A^\star x = 0$ and $x \neq 0$, it follows that $x_1 \neq 0$. Now, let $X = \{x \in \mathbb{R}^4 : x_1 = 0\}$ and $K = \ker(A^\star)$. Clearly, $\dim(X) = 3$; moreover, since we found that every nonzero x in $\ker(A^\star)$ has $x_1 \neq 0$, it follows that $\dim(X \cap K) = 0$. By the well-known formula for the dimension of the intersection and sum of finite-dimensional subspaces, $\dim(X+K) + \dim(X \cap K) = \dim(X) + \dim(K)$. Thus,

$$\begin{aligned} M(A) &= \mathrm{null}(A^\star) = \dim(K) \\ &= \dim(X+K) + \dim(X \cap K) - \dim(X) \\ &= \dim(X+K) + 0 - 3 \leq 4 - 3 = 1, \end{aligned}$$

where the last inequality follows from the fact that

$\dim(X+K) \leq \dim(\mathbb{R}^4) = 4$. On the other hand, the matrix

$$A_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

fits the pattern of A and has nullity 1, so $1 = \text{null}(A_0) \leq M(A) \leq 1$. Thus, $M(A) = 1$ and hence $\text{mr}(A) = 4 - M(A) = 3$.

What allows us to form an upper bound on $M(G)$ in the example above is the choice of an appropriate set of indices Z, such that when $x_i = 0$ for all $i \in Z$, all other entries of x are also forced to be zero. In particular, the principle by which a set of zero entries of x forces other entries to be zero is precisely the color change rule in the graph corresponding to A: when all-but-one terms in the left-hand-side of $A_i^\star x = 0$ are zero for some row $A_i^\star$ of $A^\star$, the remaining term (and the corresponding entry of x) must also be zero. For example, this is the case in (1) when choosing x_1 to be zero forces x_2 to be zero, which in turn forces x_3 and x_4 to be zero. Equivalently, selecting x_1 to be zero corresponds to choosing the end-vertex of the path P_4 as a zero forcing set of P_4. This process is the source of the nomenclature of "zero forcing"! The same principles can be applied to any symmetric matrix pattern and the corresponding graph to obtain upper and lower bounds on $M(A)$ (and hence on $\text{mr}(A)$); this is stated formally below.

Theorem 1 ([AMRSGWG08]). *Let A be a symmetric matrix pattern, G be the graph corresponding to A, and A_0 be any matrix that fits the pattern of A. Then, $null(A_0) \leq M(A) \leq Z(G)$.*

Thus, the zero forcing number can be used as a combinatorial bound on the minimum rank of a symmetric matrix pattern. Similar procedures have been developed for other types of matrix patterns (e.g., ones that are positive-semidefinite [EEH+13] or skew-symmetric [IIrgomr10]) using other types of graphs and color change rules; nonsymmetric versions of zero forcing such as signed zero forcing have also been considered [GB14]. While both the graph-based parameters and the matrix-based parameters are difficult to compute, the former are generally much more tractable than the latter. Roughly speaking, in the worst case, evaluating a graph-based parameter such as $Z(G)$ for a graph with n vertices requires 2^n operations while evaluating a matrix-based parameter such as $\text{mr}(A)$ for an $n \times n$ matrix pattern requires 2^{2^n} operations. Thus, working with the graph-based parameters gives a significant computational advantage.

Power domination. Electric power networks consist of energy-producing generators, energy-consuming loads, transmission lines connecting the generators and loads, and busses where transmission lines intersect. An electric power company must constantly monitor the state of its network in order to detect system failures and assure that demands are being met. To this end, phase measurement units (PMUs) are placed at select locations around the network; these devices measure the voltage at the bus where they are placed, and the phase angle at the transmission lines incident with the bus. The PMU readings are then synchronized in processing stations, where the data from multiple PMUs is leveraged with physical laws governing the behavior of electrical circuits such as Ohm's and Kirchoff's laws in order to gain information about parts of the network which are not being directly monitored. Thus, the state of the entire network can be determined from partial information measured at appropriate locations. Because of the high cost of the equipment, labor, and communication infrastructure associated with installing and maintaining PMUs, electric power companies aim to use the smallest number of PMUs necessary to maintain full control of the network.

Electrical power networks can naturally be represented as graphs, where the generators, loads, and busses are vertices, and the transmission lines are edges. The problem of interest is then to select a minimum set of vertices, corresponding to the locations of PMUs, from which the entire graph can be observed. The rules by which vertices and edges can be observed are listed below (introduced in [HHHH02]); these rules reflect information gained by direct measurements of PMUs, as well as information gained through Ohm's and Kirchoff's laws about locations in the network which are not directly monitored by PMUs.

1. All vertices at which a PMU is placed are observed.
2. All edges incident to a vertex at which a PMU is placed are observed.
3. A vertex incident to an observed edge is observed.
4. An edge joining two observed vertices is observed.
5. If a vertex is incident to $k > 1$ edges and $k-1$ of these edges are observed, then all k are observed.

This process is identical to zero forcing (where observed vertices are colored blue), with the exception that in the first timestep, by rules 2 and 3, every neighbor of an observed vertex is also observed. In other words, the set of vertices initially colored blue performs a "domination" step, and then the color change rule is applied iteratively as in zero forcing. Due to this domination step and the application to power network monitoring, this problem is referred to as *power domination*. A *power dominating set* is a set of vertices which causes the entire graph to be observed under the rules above, and the *power domination number* of a graph G, denoted $\gamma_P(G)$, is the minimum cardinality of a power dominating set of G. Note that in a power dominating set, eventually all edges are observed, and in a non-power dominating set, the set of observed edges is the edge set of the graph induced by the observed vertices. Therefore, it is sufficient to only work with the observed vertices. See Figure 2 for an illustration.

Since a set is contained in its closed neighborhood, it follows that any zero forcing set is also a power dominating set. Thus, for any graph $G = (V, E)$, $\gamma_P(G) \leq Z(G)$. Moreover, $S \subset V$ is a power dominating set of G if and only if $N[S]$ is a zero forcing set, where $N[S]$ is S together with all neighbors of S. Thus, if S is a minimum power dominating set of G and $\Delta(G)$ is the maximum degree of G,

$$\begin{aligned}\gamma_P(G) \leq Z(G) \leq |N[S]| \leq |S|(\Delta(G)+1) \\ = \gamma_P(G)(\Delta(G)+1).\end{aligned}$$

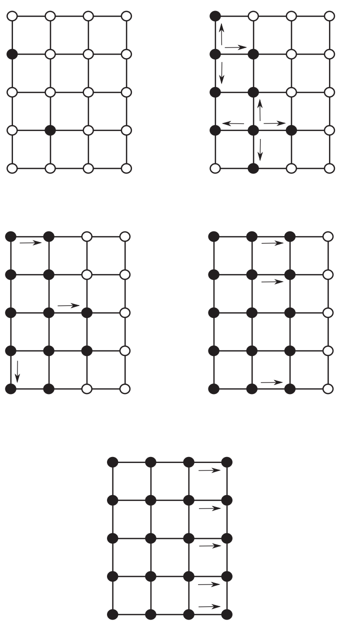


Figure 2. In the first panel, a minimum power dominating set is marked by shaded vertices. The second panel illustrates the domination step, and the remaining panels illustrate the other color changes.

Due to these relations, zero forcing has been used as a technical tool in the derivation of various results about power domination.

Graph searching. Zero forcing also arose independently in the context of graph search problems. The objective of graph search problems is to capture a fugitive hidden on the vertices or edges of a graph while using a limited number of guards, search actions, or other resources. The earliest graph search problems that have received considerable attention are the edge search and node search problems, introduced by Megiddo et al. [MHG+88] and Kirousis and Papadimitriou [KP86], respectively. In these problems, the aim is to capture an invisible fugitive using the smallest number of guards, where search actions consist of placing a guard at a vertex, removing a guard from a vertex, or sliding a guard along an edge. Bienstock and Seymour [BS91] introduced the mixed search problem which combines the edge and node search problems, and focuses on minimizing the number of guards used at any step. Dyer et al. [Dye08] introduced the fast search problem, which focuses on minimizing the number of search actions in which the fugitive is captured.

Yang [Yan13] introduced the fast-mixed search problem, which combines the fast search and the mixed search models. In the fast-mixed search problem, an invisible fugitive can freely move at great speed from one vertex to another along a guard-free path. A vertex or edge where the fugitive may hide is *contaminated*, otherwise, it is *cleared*; a vertex is *occupied* if it has a guard on it. The search actions include *placing* a guard on a contaminated vertex and *sliding* a guard along a contaminated edge uv from u to v if v is contaminated and all edges incident to u except uv are cleared. A contaminated edge becomes cleared if both endpoints are occupied by guards or if a guard slides along it from one endpoint to the other. The *fast-mixed search number* of a graph G, denoted $\text{fms}(G)$ is the minimum number of guards required to clear all edges of G and therefore capture a fugitive hiding in G. See Figure 3 for an illustration.

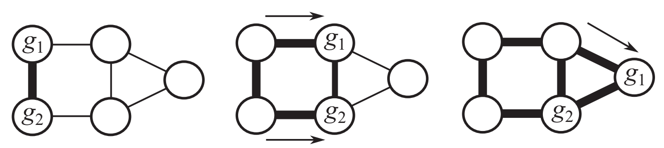


Figure 3. Guards g_1 and g_2 clearing a graph. Cleared edges are indicated in bold. Note that the movement of the guards is identical to zero forcing color changes initiated by two blue vertices placed at the guards' initial positions.

Unexpectedly, it was later realized that the movements of the guards in the fast-mixed search model precisely coincide with the color changes in the zero forcing process initiated by blue vertices at the guards' initial positions. Hence, fast-mixed searching and zero forcing are essentially the same problem, which yields the following identity.

Theorem 2 ([FMY16]). *For any graph G, $Z(G) = \text{fms}(G)$.*

This relation allowed results and techniques developed for graph search algorithms to be applied to zero forcing and minimum rank problems, and vice versa [BFH19].

Quantum control. Finally, zero forcing was also studied in quantum control theory under the name *propagation*, where it was introduced as a scheme for controlling large quantum systems by acting on small subsystems satisfying certain conditions. More precisely, a network C of coupled spin 1/2 quantum particles with Hamiltonian H can be represented as a graph whose vertices are the spins of C and whose edges are the non-Ising components of H. The protocol for operating on this system through local quantum transformations can be described in graph terminology as follows: each vertex in an initially selected set has a packet of information that has to be diffused among all the vertices of the graph; a vertex v can pass its packet to an adjacent vertex w only if w is the only neighbor of v which still does not have the information [Sev08]. It is easy to see that this propagation protocol is identical to the zero forcing color change rule and that the smallest number of particles sufficient to control the system is the zero forcing number of the corresponding graph. This equivalence has enabled the use of techniques and results from zero forcing in quantum control theory, paving the way for applications like control of quantum hard drives and quantum RAM [Bur07].

Learning the Ways of the Force

The zero forcing problem is difficult to solve directly, as it belongs to the class of NP-Hard optimization problems for which there are no efficient algorithms. Zero forcing appears to be difficult even when compared to other NP-Hard problems, since all the leading computational methods in the literature can only solve it for relatively small instances. Nevertheless, in light of its many applications, solving the zero forcing problem for moderately sized instances is an important practical task.

An obvious way to find a minimum zero forcing set of a graph is by brute force: simply consider all sets of vertices of size i for $i \geq 1$ and check whether the closure of each set is the whole graph. As soon as such a set is found, it is guaranteed to be a minimum zero forcing set since sets are checked in order of increasing size. However, since a graph on n vertices could have zero forcing number as high as $n-1$ (e.g., if the graph is complete), this approach could require as many as $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} \approx 2^n$ sets to be checked.

The force is strong with this one. An improvement on the brute force method is the wavefront algorithm. This algorithm stores candidate vertex sets together with their closures, starting from singleton sets. Then, for each candidate set S of a given size, the algorithm adds to S one vertex that is outside the closure of S and recomputes the closure. This is repeated until the closure of some candidate set is the entire graph; this candidate set must then be a minimum zero forcing set. The advantage of the wavefront algorithm [BGH15, BFH19] (whose name comes from the idea that closures are checked in "waves" according to the cardinalities of the underlying sets), is that it avoids checking a set of vertices if some subset of the set has the same closure. Thus, the algorithm tends to perform well in graphs where a small subset of vertices can force many other vertices. A downside of the wavefront algorithm is that it requires the simultaneous storage of a large number of closures. Thus, while the wavefront algorithm is often faster than other methods, it sometimes causes the computer to run out of memory.

Use the forts, Luke. A third approach to finding minimum zero forcing sets is to model the problem as an integer linear program (ILP) or a Boolean satisfiability (SAT) problem and use powerful ILP and SAT solvers to find a solution. An advantage of the ILP- and SAT-based algorithms is that they are more flexible and can accommodate additional constraints to the zero forcing problem. For example, they can ensure that the zero forcing set is connected or that the zero forcing set minimizes the number of forcing steps, which are problems of independent interest. Another advantage of ILP- and SAT-based algorithms is that they often give good upper and lower bounds on the zero forcing number even if they are stopped before they have time to find the exact solution.

Several different ways to model the zero forcing problem using ILP and SAT have been explored. One of the more robust models is based on the concept of forts. A *fort* of G is a nonempty set $F \subset V(G)$ such that every vertex that is outside F and adjacent to F has at least two neighbors in F. See Figure 4 for an example. The name "fort" comes from the design of bastion forts, which made it possible for any nearby invader to be visible from at least two walls of the fort.

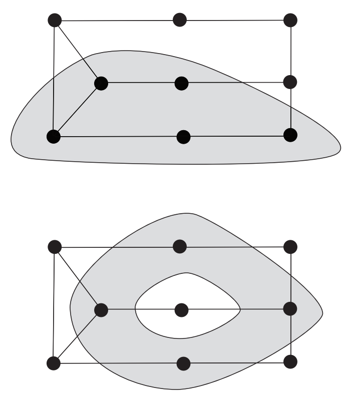


Figure 4. Two of the forts of a graph indicated as gray areas. No vertex outside the fort has exactly one neighbor in the fort.

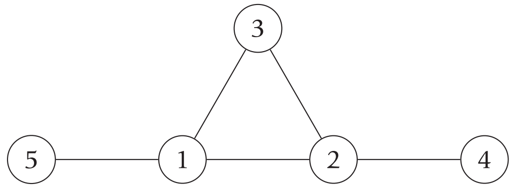
Forts are an important concept in zero forcing because they indicate barriers that the forcing process must overcome. In particular, if all vertices outside a fort are colored blue and all vertices inside it are white, no vertex can penetrate the fort and force a vertex inside to turn blue. Therefore, in order for a set to be zero forcing, it must contain at least one vertex from every fort of the graph (a saboteur who opens the gates and lets the invaders in). The converse is also true.

Theorem 3 ([CFaIVH18]). *S is a zero forcing set of G if and only if S intersects every fort of G.*

This theorem suggests the following simple ILP model to find a minimum zero forcing set (here $x_v = 1$ if vertex v is in the set and $x_v = 0$ if vertex v is not in the set):

$$\begin{array}{rll} \min & \sum_{v \in V} x_v & \\ \text{so that} & \sum_{v \in F} x_v \geq 1 & \text{for every fort } F \\ \text{and} & x_v \in \{0,1\} & \text{for every } v \in V. \end{array}$$

A downside of this model is that a graph can have exponentially many forts. Thus, the fort constraints could require exponential time and space even simply to be written down. One way to overcome this problem is that instead of solving the ILP with all the fort constraints, we can instead solve a sequence of simpler ILPs each of which has only a small subset of the fort constraints. If chosen appropriately, one of these small subsets of constraints will subsume all the omitted constraints, and will yield an optimal solution. To illustrate this idea, let G be the following graph:



In the first iteration, we will solve the ILP model without *any* fort constraints. The optimal solution to min $x_1 + x_2 + x_3 + x_4 + x_5$ is clearly $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0$, which means our candidate set is empty. Since this set is not zero forcing, we find one single fort of G that is not intersected by this set, for example, $\{3, 4, 5\}$. In the next iteration, we will solve the ILP with just that fort constraint: $x_3 + x_4 + x_5 \geq 1$. An optimal solution to that ILP is $x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 0$, which means our new candidate set is $\{3\}$. Since this set is also not zero forcing, we find a single fort of G that is not intersected by this set, for example, $\{1, 2, 4, 5\}$. In the next iteration, we add the fort constraint $x_1 + x_2 + x_3 + x_5 \geq 1$ to the ILP. An optimal solution to this new ILP is $x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1, x_5 = 0$, which means our new candidate set is $\{3, 4\}$. Since this set is in fact a zero forcing set, it satisfies every fort constraint of the ILP and minimizes the objective function. It is therefore a minimum zero forcing set and we terminate the algorithm. Thus, instead of having to find all the forts of G, we only found two forts which subsumed all the others.

There are several ways of finding individual forts which are not intersected by a given candidate set. One way is to simply take the complement of the closure of the candidate set. Another way is to use a secondary ILP. The first way is the fastest, but may yield "low quality" forts that require a large number of iterations of the main ILP. The second way can be calibrated to find either the "highest quality" forts at a high computational cost, or "medium quality" forts at a medium computational cost. Finding a fort generation subroutine which results in the best tradeoff between the number of iterations and the cost per iteration is an active area of research.

While the solution methods discussed above can compute the zero forcing numbers of graphs with several hundred vertices, other optimization problems such as the maximum clique problem, the minimum vertex cover problem, and the traveling salesman problem can be solved for graphs with thousands and sometimes hundreds of thousands of vertices. One simple reason is that these problems have been around for many decades longer than zero forcing, and have been attacked by many more researchers who have continuously improved the available solution approaches. A more nuanced and technical reason could be explained by the polyhedral structure of the zero forcing problem. The constraints in the model above (and indeed in any ILP model) can be viewed as hyperplanes in $\mathbb{R}^n$. The intersection of those hyperplanes is a convex set in $\mathbb{R}^n$ called a *polyhedron*. The "sides" of this surface are called *facets*; see Figure 5 for an illustration. Knowing the facets of a polyhedron helps ILP solvers find an optimal solution. On the other hand, if a polyhedron has a large number of facets, it can be inherently difficult to solve problems on that polyhedron with integer programming and other methods. In [BMH21] it was shown how the facets of the zero forcing polyhedron corresponding to certain commonly occurring forts in the graphs can be characterized. It was also shown that the number of these facets can be exponential (around $1.3247^{n/3}$ for a graph of order n). This suggests that the zero forcing problem may be inherently difficult, at least in graphs whose forts have those structures.

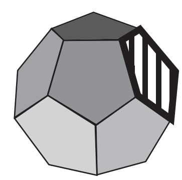


Figure 5. A polyhedron in $\mathbb{R}^3$ with one facet shown in white with a bold border.

Forts are interesting not just from a computational point of view but also from an algebraic point of view. For example, the following result shows that the forts of a graph have a connection to the kernel of its adjacency matrix:

Theorem 4 ([HBD$^+$22]). *Let $G = (V, E)$ be a graph of order n and let $x \in \mathbb{R}^n$. There exists a matrix A that fits the same matrix pattern as the adjacency matrix of G such that $Ax = 0$ if and only if the indices of the nonzero entries of x form a fort of G.*

In addition, forts can characterize the circuits of matroids linked to the minimum rank of a matrix pattern [HBD$^+$22]. Hence, forts are very interesting structures and should be explored further.

Conclusion

Zero forcing is a multifaceted problem with connections to many areas of mathematics and many practical applications. The wide interest in zero forcing has created a rapidly growing and diverse research community that has generated a large volume of work in the last few years. There are many remaining computational and theoretical challenges to resolve, as well as deeper connections to linear algebra and matroid theory that may be just the tip of an iceberg. The zero forcing community is excited to welcome a new generation of mathematicians to this problem and see what new discoveries and advancements we can forge together.

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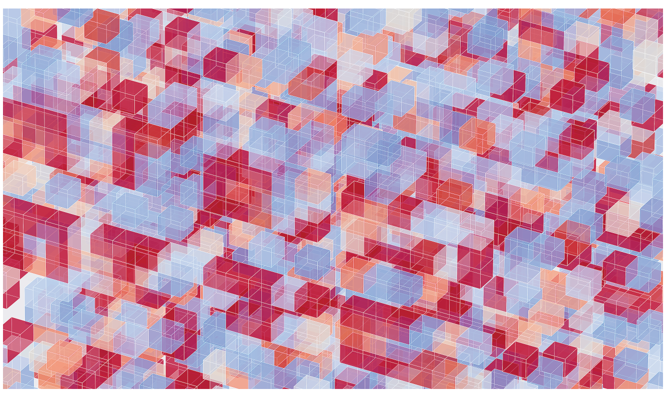
The Structure of Meaning in Language: Parallel Narratives in Linear Algebra and Category Theory

Tai-Danae Bradley, Juan Luis Gastaldi, and John Terilla



Introduction

Categories for AI, an online program about category theory in machine learning, unfolded over several months beginning in the fall of 2022. As described on their website `https://cats.for.ai`, the "Cats for AI" organizing committee, which included several researchers from industry including two from DeepMind, felt that the machine learning community ought to be using more rigorous compositional tools and that category theory has "great potential to be a cohesive force" in science in general and in artificial intelligence in particular. While this article is by no means a comprehensive report on that event, the popularity of "Cats for AI" — the five introductory lectures have been viewed thousands of times — signals the growing prevalence of category theoretic tools in AI.

One way that category theory is gaining traction in machine learning is by providing a formal way to discuss how learning systems can be put together. This article has a different and somewhat narrow focus. It's about how a fundamental piece of AI technology used in language modeling can be understood, with the aid of categorical thinking, as a process that extracts structural features of language from purely syntactical input. The idea that structure arises from form may not be a surprise for many readers — category theoretic ideas have been a major influence in pure mathematics for three generations — but there are consequences for linguistics that are relevant for some of the ongoing debates about artificial intelligence. We include a section that argues that the mathematics in these pages rebut some widely accepted ideas in contemporary linguistic thought and support a return to a structuralist approach to language.

The article begins with a fairly pedantic review of linear algebra which sets up a striking parallel with the relevant category theory. The linear algebra is then used to review how to understand word embeddings, which are at the root of current large language models (LLMs). When the linear algebra is replaced, *Mad Libs* style, with the relevant category theory, the output becomes not word embeddings but a lattice of formal concepts. The category theory that gives rise to the concept lattice is a particularly simplified piece of enriched category theory and suggests that by simplifying a little less, even more of the structure of language could be revealed.

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Objects Versus Functions on Objects

When considering a mathematical object X that has little or incomplete structure, one can replace X by something like "functions on X" which will have considerably more structure than X. Usually, there is a natural embedding $X \to \mathrm{Fun}(X)$ so that when working with $\mathrm{Fun}(X)$, one is working with all of X and more.

The first example that comes to mind is k^X, the set of functions on a set X valued in a field k, which forms a vector space. The embedding $X \to k^X$ is defined by sending $x \in X$ to the indicator function of x defined by $y \mapsto \delta_{xy}$. When X is finite, there is a natural inner product on k^X defined by $\langle f|g\rangle = \sum_{x\in X} f(x)g(x)$ making k^X into a Hilbert space. The physics "ket" notation $|x\rangle$ for the indicator function of $x \in X$ nicely distinguishes the element $x \in X$ from the vector $|x\rangle \in k^X$ and reminds us that there is an inner product. The image of X in k^X defines an orthonormal spanning set and if the elements X are ordered, then the vectors $\{|x\rangle : x \in X\}$ become an ordered basis for k^X and each basis vector $|x\rangle$ has a *one-hot coordinate vector*: a column vector consisting of all zeroes except for a single 1 in the x-entry. This, by the way, is the starting point of quantum information theory. Classical bits $\{0,1\}$ are replaced by quantum bits $\{|0\rangle, |1\rangle\}$ which comprise an orthonormal basis of a two-dimensional complex Hilbert space $\mathbb{C}^{\{0,1\}}$. There might not be a way to add elements in the set X, or average two of them for example, but those operations can certainly be performed in k^X. In coordinates, for instance, if $x \neq y$, then the sum $|x\rangle + |y\rangle$ will have all zeroes with 1s in both the x- and y- entries and the sum $|x\rangle + |x\rangle$ has all zeroes and a 2 in the x-entry.

When the ground field is the field with two elements $k = \{0,1\}$, the vector space structure seems a little weak. Scalar multiplication is trivial, but there are other notable structures on $\{0,1\}^X$. Elements of $\{0,1\}^X$ can be thought of as subsets of X, the correspondence being between characteristic functions and the sets on which they are supported. So $\{0,1\}^X$ has all the structure of a Boolean algebra: the join $v \vee w$ and the meet $v \wedge w$ of two vectors correspond to the union and intersection of the two subsets defined by v and w, and neither the meet nor the join coincide with vector addition. Every vector has a "complement" defined by interchanging $0 \leftrightarrow 1$, the vectors in $\{0,1\}^X$ are partially ordered by containment, every vector has a cardinality defined by the number of nonzero entries, and so on.

Another closely related example comes from category theory. By replacing a category C by $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$, the set-valued functors on C, one obtains a category with significantly more structure. Here the "op" indicates the variance of the functors in question (contravariant, in this case), a technical point that isn't very important here, but is included for accuracy. It's common to call a functor F in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ a *presheaf* on C. The Yoneda lemma provides an embedding $\mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ of the original category as a full subcategory of $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$. Given an object x in C, Grothendieck used h^x to denote a *representable presheaf* $h^x := \mathsf{C}(-,x)$ which is defined by mapping an object y to the set $\mathsf{C}(y,x)$ of morphisms from y to the object x represting the presheaf. In this notation, the Yoneda embedding is defined on objects by $x \mapsto h^x$. And just as the vector space k^X has more structure than X, the category $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ has more structure than C. For any category C, the category $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ of presheaves is complete and cocomplete, meaning that the categorical limits and colimits of small diagrams exist in the category, and more. It is also an example of what is called a *topos* which is a natural place in which to do geometry and logic.

As an illustration, consider a finite set X, which can be viewed as a discrete category X, that is, a category whose only morphisms are identity morphisms. In this case $\mathsf{X} = \mathsf{X}^{\mathrm{op}}$, and a presheaf F on X assigns a set to every object x in X. If the elements of X are ordered, then F can be thought of as a column vector whose entries are sets, with the set $F(x)$ in the x-entry. The representable functor h^x can be thought of as a column vector whose entries are all the empty set except for a one-point set $*$ in the x-entry. Using 0 for $\emptyset$ and 1 for $*$ produces the same arrays as the one-hot basis vectors that span the vector space k^X. Notably, the categorical coproduct $x \coprod y$ does not exist in the category X, but the coproduct $h^x \coprod h^y$ of the representable functors h^x and h^y *does* exist in the category of presheaves on X. If $x \neq y$ then $h^x \coprod h^y$ is a column consisting of empty sets except for a one-point set $*$ in the x- and y-entries; the coproduct $h^x \coprod h^x$ consists of all empty sets except for a two point set $* \sqcup *$ in the x-entry. And just as the indicator functions form a basis of the vector space k^X, every functor $\mathsf{X}^{\mathrm{op}} \to \mathsf{Set}$ is constructed from representable functors. When X is a finite set, every vector in k^X is a linear combination of basis vectors, and analogously every presheaf in $\mathsf{Set}^{\mathsf{X}^{\mathrm{op}}}$ is a colimit of representables.

In this article, it will be helpful to consider *enriched category theory*, which is the appropriate version of category theory to work with when the set $\mathsf{C}(y,x)$ of morphisms between two objects is no longer a *set*. That is, it may be a partially ordered set, or an Abelian group, or a topological space, or something else. So, enriched category theory amounts to replacing Set with a different category. This is analogous to changing the base field of the vector space k^X, and if the new base category has sufficiently nice structure, then most everything said about replacing C by $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ goes over nearly word-for-word. For example, replacing Set by the category $\mathsf{2}$, which is a category with two objects 0 and 1 and one nonidentity morphism $0 \to 1$, results in the category $\mathsf{2}^{\mathsf{C}^{\mathrm{op}}}$ of 2-valued presheaves on C. In the case when C is a set X viewed as a discrete category $\mathsf{X} = \mathsf{X}^{\mathrm{op}}$, the presheaves in 2^X are exactly the same as $\{0,1\}$-valued functions on X, which are the same as subsets of X. The structure on 2^X afforded by it being a category of 2-enriched presheaves is the Boolean algebra structure on the subsets of X previously described. The categorical coproduct of 2-enriched presheaves f and g is the join (union) $f \vee g$, and the categorical product is the meet (intersection) $f \wedge g$. So for any set X, the set of functions $\{0,1\}^X$ can either be viewed as a vector space over $F_2 = \{0,1\}$, the field with two elements, or as enriched presheaves on X valued in $2 = \{0,1\}$, depending on whether we think of $\{0,1\}$ as a field, or as a category $0 \to 1$, and we get different structures depending on which point of view is taken.

Now, before going further, notice that replacing an object by a free construction on that object can't immediately reveal much about the underlying object. Whether it's the free vector space on a finite set X resulting in k^X or the free cocompletion of a category C resulting in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$, the structures one obtains are *free* and employ the underlying object as little more than an indexing set. The structures on the "functions" on X are owed, essentially, to the structure of what the functions are valued *in*. For example, the source of the completeness and cocompleteness of $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ is the completeness and cocompleteness of the category of sets. Similarly, vector addition and scalar multiplication in k^X arise from addition and multiplication in the field k. The point is that passing to a free construction on X provides some extra room in which to investigate X, and in the theory of vector spaces, for instance, things become interesting when linear transformations are involved. As another example, passing from a finite group G to the free vector space $\mathbb{C}^G$ doesn't tell you much about the group, until that is, you involve the elements of the group as operators $\mathbb{C}^G \to \mathbb{C}^G$. The result is the regular representation for G, which among its many beautiful properties, decomposes into the direct sum of irreducible representations with every irreducible representation of G included as a term with meaningful multiplicity.

This brings us back to the strategy suggested in the first line of this section. When only a little bit is known about the internal structure of an object X, an approach to learn more is to replace X by something like functions on X and study how the limited known structure of X interacts with the freely defined structures on the functions of X. A choice is required of what, specifically, to value the functions in and how mathematically that target is viewed. The remaining sections of this article can be interpreted simply as working through the details in an example with linguistic importance for a couple of natural choices of what to value the functions in.

Embeddings in Natural Language Processing

In the last decade, researchers in the field of computational linguistics and natural language processing (NLP) have taken the step of replacing words, which at the beginning only have the structure of a set, ordered alphabetically, by vectors. One gets the feeling that there is structure in words — words appear to be used in language with purpose and meaning; dictionaries relate each word to other words; words can be labelled with parts of speech; and so on — though the precise mathematical nature of the structure of words and their usage is not clear. Language would appear to represent a significant real-world test of the strategy to uncover structure described in the previous section. While the step of replacing words by vectors constituted one of the main drivers of current advances in the field of artificial intelligence, it is not readily recognizable as an instance of replacing a set by functions on that set. This is because replacing words by vectors is typically performed implicitly by NLP tools, which are mathematically obscured by their history — a history which we now briefly review.

Following the surprisingly good results obtained in domains such as image and sound processing, researchers working to process natural language in the 2010s became interested in the application of deep neural network (DNN) models. As a reminder, in its most elementary form, a DNN can be described as a function $f : \mathbb{R}^n \to \mathbb{R}^m$ explicitly expressed as a composition:

$$f : \mathbb{R}^n \xrightarrow{f_1} \mathbb{R}^{n_1} \xrightarrow{f_2} \mathbb{R}^{n_2} \xrightarrow{f_3} \cdots$$
$$\cdots \xrightarrow{f_K} \mathbb{R}^{n_K} \xrightarrow{g} \mathbb{R}^m \quad (1)$$

$$f_i(\mathbf{x}) = a(\mathbf{M}_i\mathbf{x} + b_i), \quad (2)$$

where the $\mathbf{M}_i$ are $n_i \times n_{i-1}$ matrices, the $b_i \in \mathbb{R}^{n_i}$ are "biases", the a is a (nonlinear) "activation" function, and g is an output function. After fixing the activation and output functions, a DNN lives in a moduli space of functions parametrized by the entries of the matrices M_i and the bias vectors b_i. Training a DNN is the process of searching through the moduli space for suitable M_i and b_i to find a function that performs a desired task, usually by minimizing a cost function defined on the moduli space. In the particular case of natural language tasks, this typically requires feeding linguistic data into the model and setting the optimization objective to the minimization of the error between the actual and intended outputs, with the optimization performed by a form of gradient descent.

Significantly, in this setting, linguistic data (typically words) would be represented as vectors—the domain of a DNN is a vector space. A natural first choice for practitioners was then to represent words as one-hot vectors. Thus, if one has a vocabulary D consisting of, say, 30,000 words, then $D \to \mathbb{R}^D$ embeds words as the standard basis vectors in a 30,000-dimensional real vector space.

$$\begin{aligned}
\texttt{aardvark} &\mapsto (1,0,0,0,\dots,0,0)\\
\texttt{aardwolf} &\mapsto (0,1,0,0,\dots,0,0)\\
&\vdots\\
\texttt{zyzzyva} &\mapsto (0,0,0,0,\dots,0,1).
\end{aligned}$$

Likewise, if the output is to be decoded as a word, then the output function g is to be interpreted as a probability distribution over the target vocabulary, which is usually the same $\mathbb{R}^D$, though it could certainly be different in applications like translation.

At the time, a DNN was thought of as an end-to-end process: whatever happens between the input and the output was treated as a black-box and left for the optimization algorithm to handle. However, a surprising circumstance arose. If the first layer (namely, f_1 in Equation (1)) of a model trained for a given linguistic task was used as the first layer of another DNN aimed at a different linguistic task, there would be a significant increase in performance of the second task. It was not long thereafter that researchers began to train that single layer independently as the unique hidden layer of a model that predicts a word given other words in the context. Denoting that single layer by σ, one can then obtain

$$D \hookrightarrow \mathbb{R}^D \xrightarrow{\sigma} \mathbb{R}^{n_1}, \tag{3}$$

which embeds D in a vector space of much lower dimension, typically two or three hundred. Take, for instance, the vector representations made available by [ea], where the word `aardvark` is mapped to a vector with 200 components:

$$\texttt{aardvark} \mapsto (0.632, 0.370, -0.620, \dots, -0.475).$$

In this way, the images of the initial one-hot basis vectors under the map σ could be used as low-dimensional dense word vector representations to be fed as inputs across multiple DNN models. The word "dense" here is not a technical term but is used in contrast to the original one-hot word vectors in $\mathbb{R}^D$, which are "sparse" in the sense that all entries were 0 except for a single 1. On the other hand, the word vectors in $\mathbb{R}^{n_1}$ generally have nonzero entries.

The set of word vector representations produced in this way — also known as "word embeddings" — were found to have some surprising capabilities. Not only did the performance of models across different tasks increase substantially, but also unexpected linguistic significance was found in the vector space operations of the embedded word vectors. In particular, the inner product between two vectors shows a high correlation with semantic similarity. As an example, the vector representations for `tubulidentata`, `orycteropus`, `anteaters`, `shrews`, and `pangolins` are among the ones with the largest inner product with that of `aardvark`. Even more surprisingly, addition and subtraction of vectors in the embedding space correlate with analogical relations between the words they represent. For instance, the vector for `Berlin` minus the vector for `Germany` is numerically very near the vector for `Paris` minus the vector for `France` [MSC+13], all of which suggests that the word vectors live in something like a *space of meanings* into which individual words embed as points. Subtracting the vector for `Germany` from the vector for `Berlin` does not result in a vector that corresponds to any dictionary word. Rather, the difference of these vectors is more like the concept of a "capital city", not to be confused with the vector for the word `capital`, which is located elsewhere in the meaning space.

Word vector representations such as the one described are now the standard input to current neural linguistic models, including LLMs which are currently the object of so much attention. And the fact, as suggested by these findings, that semantic properties can be extracted from the formal manipulation of pure syntactic properties — that meaning can emerge from pure form — is undoubtedly one of the most stimulating ideas of our time. We will later explain that such an idea is not new but has, in fact, been present in linguistic thought for at least a century.

But first it is important to understand why word embeddings illustrate the utility of passing from X to $\mathrm{Fun}(X)$ introduced in the previous section. The semantic properties of word embeddings are not present in the one-hot vectors embedded in $\mathbb{R}^D$. Indeed, the inner product of the one-hot vectors corresponding to `aardvark` and `tubulidentata` is zero, as it is for any two orthogonal vectors, and the vector space operations in $\mathbb{R}^D$ are not linguistically meaningful. The difference between the one-hot vectors for `Germany` and `Berlin` is as far away from the difference between the one-hot vectors for `France` and `Paris` as it is from the difference between the one-hot vectors for `salami` and `therefore` or any other two one-hot vectors. Linguistically significant properties emerge only after composing with the embedding map $\sigma\colon \mathbb{R}^D \to \mathbb{R}^{n_1}$ in (3) that was obtained through neural optimization algorithms, which are typically difficult to interpret.

In the specific case of word embeddings, however, the algorithm has been scrutinized and shown to be performing an implicit factorization of a matrix comprised of information about how words are used in language. To elaborate, the optimization objective can be shown to be equivalent to factorizing a $|D|\times|D|$ matrix M, where the i-j-entry is a linguistically relevant measure of the term-context association between words w_i and w_j. That measure is based on the pointwise mutual information between both words, which captures the probability that they appear near each other in a textual corpus [LG14]. The map σ is then an optimal low-rank approximation of M. That is, one finds σ' and σ of sizes $|D| \times d$ and $d \times |D|$ respectively, with $d \ll |D|$, such that $\|M - \sigma'\sigma\|$ is mimimal. The upshot is that neural embeddings are just low-dimensional approximations to the columns of M. Therefore, *the surprising properties exhibited by embeddings are less the consequence of some magical attribute of neural models than the algebraic structure underlying linguistic data found in corpora of text.* Indeed, it has since been shown that one can obtain results comparable to those of neural word embeddings by directly using the columns of M, or a low-dimensional approximation thereof, as explicit word-vector representations [LGD15]. Interestingly, the other factor σ', which is readable as the second layer of the trained DNN, is typically discarded although it does contain relevant linguistic information.

In summary, the math story of word embeddings goes like this: first, pass from the set D of vocabulary words to the free vector space $\mathbb{R}^D$. While there is no meaningful linguistic information in $\mathbb{R}^D$, it provides a large, structured setting in which a limited amount of information about the structure of D can be placed. Specifically, this limited information is a $|D| \times |D|$ matrix M consisting of rough statistical data about how words go with other words in a corpus of text. Now, the columns of M, or better yet, the columns of a low-rank factorization of M, then interact with the vector space structure to reveal otherwise hidden syntactic and semantic information in the set D of words. Although a matrix of statistical data seems more mathematically casual than, say, a matrix representing the multiplication table of a group, it has the appeal of assuming nothing about the structure that D might possess. Rather, it is purely a witness of how D has been used in a particular corpus. It's like a set of observations is the input, and a more formal structure is an output.

So, if word embeddings achieved the important step of finding a linguistically meaningful *space* in which words live, then the next step is to better understand what is the *structure* underlying that space. Post facto realizations about vector subtraction reflecting certain semantic analogies hint that even more could be discovered. For this next discussion, it is important to understand that there is an exact solution for the low-rank factorization of a matrix M using the truncated singular value decomposition (SVD), which has a beautiful analogue in category theory. To fully appreciate the analogy, it will be helpful to review matrices from an elementary perspective.

From the Space of Meanings to the Structure of Meanings

In this section, keep in mind the comparison between functions on a set X valued in a field k and functors on a category C valued in Set or another enriching category. Now, let's consider matrices.

Given finite sets X and Y, an X-Y matrix valued in a field k is a function $m\colon X \times Y \to k$. By simple currying, m defines functions $X \to k^Y$ and $Y \to k^X$ defined by $x \mapsto m(x, -)$ and $y \mapsto m(-, y)$. Ordering the elements of X and Y, the function m can be represented as a rectangular array of numbers with $|X|$ rows and $|Y|$ columns with the value $m(x, y)$ being the number in the x-th row and y-th column. The function $m(x, -) \in k^Y$ is then identified with the x-th row of the matrix, which has as many entries as the elements of Y and defines a function on Y sending y to the y-th entry in the row. Similarly, the y-th column of m represents the function $m(-, y) \in k^X$. Linearly extending the maps $X \to k^Y$ and $Y \to k^X$ produces linear maps $M^*\colon k^X \to k^Y$ and $M\colon k^Y \to k^X$, which of course are the linear maps associated with the matrix M and its transpose M^*. Here is a diagram:

$$\begin{tikzcd} & k^Y \\ X \arrow[ur] \arrow[r, hook] & k^X \arrow[u, dashed, "M^*"'] \end{tikzcd} \qquad \begin{tikzcd} k^Y \arrow[d, dashed, "M"'] & Y \arrow[l, hook'] \arrow[dl] \\ k^X & \end{tikzcd}$$

Now, the compositions $MM^*\colon k^X \to k^X$ and $M^*M\colon k^Y \to k^Y$ are linear operators with special properties. If we fix the ground field k to be the real numbers $\mathbb{R}$, we can apply the spectral theorem to obtain orthonormal bases $\{u_1, \dots, u_m\}$ of k^X and $\{v_1, \dots, v_n\}$ of k^Y consisting of eigenvectors of MM^* and M^*M respectively with shared nonnegative real eigenvalues $\{\lambda_1, \dots, \lambda_r, 0, \dots, 0\}$, where $r = \min(m, n)$. This data can be refashioned into a factorization of M as $M = U\Sigma V^*$. This is the so-called singular value decomposition of M. The $\{u_i\}$ are the columns of U, the $\{v_j\}$ are the rows of V^*, and Σ is the $m \times n$ diagonal matrix whose i-th entry is $\sigma_i = \sqrt{\lambda_i}$. The matrices U and V satisfy $U^*U = I$ and $V^*V = I$. In SVD terminology, the nonnegative real numbers σ_i are the *singular values* of M, and the vectors $\{u_1, \dots, u_r\}$ and $\{v_1, \dots, v_r\}$ are the *left and right singular vectors* of M. In other words, we have pairs of vectors $\{(u_1, v_1), \dots, (u_r, v_r)\}$ in $k^X \times k^Y$ related to each other as

$$M^*u_j = \sigma_j v_j \text{ and } Mv_j = \sigma_j u_j.$$

Moreover, these pairs are ordered with $(u_i, v_i) \leq (u_j, v_j)$ if the corresponding singular values satisfy $\sigma_i \leq \sigma_j$. Finally, it is not difficult to show that the matrix M' with rank at most s that is closest in Frobenius norm to the matrix M is given by $M' = U\Sigma'V^*$ where Σ' is the $m \times n$ diagonal matrix containing only the s greatest nonzero singular values on the diagonal. By eliminating the parts of U, Σ', and V^* that do not, because of all the zero entries in Σ', participate in the product $U\Sigma'V^*$, one obtains a factorization $M' = U'\Sigma''V'^* \approx M$, where U' is an $m \times s$ matrix, V'^* is an $s \times n$ matrix, and Σ'' is an $s \times s$ diagonal matrix with the s largest singular values of M on the diagonal. Principal component analysis (PCA) employs this approximate factorization of a matrix into low-rank components for dimensionality reduction of high-dimensional data.

Moving from linear algebra to category theory, one finds a remarkably similar story. Given two categories C and D, the analogy of a C-D matrix is something called a *profunctor*, which is a set-valued functor $f : \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$. As before, the "op" here and in what follows is used to indicate the variance of functors and is needed for accuracy, but can on first reading be ignored. Experts will surely know of situations in which this op is involved in interesting mathematical dualities, but for the analogy with linear algebra described here, it can be thought of as indicating a sort of transpose between rows and columns. If both domain categories are finite sets viewed as discrete categories, then a profunctor is simply a collection of sets indexed by pairs of elements — that is, a matrix whose entries are sets instead of numbers. Again, by simple currying, a profunctor defines a pair of functors $\mathsf{C} \to \left(\mathsf{Set}^{\mathsf{D}}\right)^{\mathrm{op}}$ and $\mathsf{D} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ defined on objects by $c \mapsto f(c, -)$ and $d \mapsto f(-, d)$. As in the linear algebra setting, the functor $f(c, -)$ can be pictured as the c-th row of sets in the matrix f, which defines a functor $\mathsf{D} \to \mathsf{Set}$ where the j-th object of D is mapped to the j-th set in the row $f(c, -)$. The functor $f(-, d) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ can be similarly be pictured as the d-th column of f. Thinking of a category as embedded in its category of presheaves via the Yoneda (or co-Yoneda) embedding, the functors $\mathsf{C} \to \left(\mathsf{Set}^{\mathsf{D}}\right)^{\mathrm{op}}$ and $\mathsf{D} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ can be extended in a unique way to functors $F^* : \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \to (\mathsf{Set}^{\mathsf{D}})^{\mathrm{op}}$ and $F_* : (\mathsf{Set}^{\mathsf{D}})^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ that preserve colimits and limits, respectively.

$$
\begin{tikzcd}
 & (\mathsf{Set}^{\mathsf{D}})^{op} \\
\mathsf{C} \arrow[ur] \arrow[r, hook, "Yoneda"'] & \mathsf{Set}^{\mathsf{C}^{op}} \arrow[u, dashed, "F^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathsf{Set}^{\mathsf{D}})^{op} \arrow[d, dashed, "F_*"'] & \mathsf{D} \arrow[l, hook', "Yoneda"'] \arrow[dl] \\
\mathsf{Set}^{\mathsf{C}^{op}} &
\end{tikzcd}
$$

Now, just as the composition of a linear map M and its transpose M^* define linear maps with special properties, the functors F^* and F_* are adjoint functors with special properties. This particular adjunction $F^* : \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \leftrightarrows (\mathsf{Set}^{\mathsf{D}})^{\mathrm{op}} : F_*$ is known as the *Isbell adjunction*, which John Baez recently called "a jewel of mathematics" in a January 2023 column article in this publication [Bae23]. Objects that are fixed up to isomorphism under the composite functors F^*F_* and F_*F^* are called the *nuclei* of the profunctor f and are analogous to the left and right singular vectors of a matrix. One can organize the nuclei into pairs (c_i, d_i) of objects in $\mathsf{C}^{\mathrm{op}} \times \mathsf{D}$, where

$$F^*c_i \cong d_i \text{ and } F_*d_i \cong c_i.$$

The nuclei themselves $\{(c_i, d_i)\}$ have significant structure — they organize into a category that is complete and cocomplete. The pairs $\{(u_i, v_i)\}$ of singular vectors of a matrix have some structure — they are ordered by the magnitude of their singular values, and the magnitudes themselves are quite important. The nuclei $\{(c_i, d_i)\}$ of a profunctor have a different, in some ways more intricate, structure because one can take categorical limits and colimits of diagrams of pairs, allowing the pairs to be combined in various algebraic ways. In the context of linguistics, this is significant because the nuclei are like abstract units and the categorical limits and colimits provide ways to manipulate those abstract units. This is illustrated concretely in the next section. For now, interpret word embeddings obtained from the singular vectors of a matrix as a way to overlay the structure of a vector space on meanings. For certain semantic aspects of language, like semantic similarity, a vector space structure is a good fit, but overlaying a vector space structure could veil others. The Isbell adjunction provides a different structure that may help illuminate other structural features of language.

For flexibility, it is useful to look at the Isbell adjunction in the enriched setting. If the base category is $\mathsf{2}$ instead of Set, then a profunctor r between two finite sets X and Y, viewed as discrete categories enriched over $\mathsf{2}$, is just a function $r : X \times Y \to \{0, 1\}$, which is the same as a relation on $X \times Y$. The functors $R^* : \mathsf{2}^X \to \mathsf{2}^Y$ and $R_* : \mathsf{2}^Y \to \mathsf{2}^X$ are known objects in the theory of formal concept analysis [GW99]. The function R^* maps a subset $A \subseteq X$ to the set $R^*(A) = \{y \in Y : R(x, y) = 1 \text{ for all } x \in A\}$ and R_* maps a subset $B \subseteq Y$ to the set $R_*(B) = \{x \in X : R(x, y) = 1 \text{ for all } y \in B\}$. The fixed objects of R^*R_* and R_*R^* are known as *formal concepts*. They are organized into pairs $(A_i, B_i) \subset X \times Y$ with

$$R^*(A_i) = B_i \text{ and } R_*(B_i) = A_i$$

and the set of all formal concepts $\{(A_i, B_i)\}$ is partially ordered with $(A_i, B_i) \leq (A_j, B_j)$ if and only if $A_i \subseteq A_j$ which is equivalent to $B_i \supseteq B_j$. Moreover, $\{(A_i, B_i)\}$ forms a complete lattice, so, like the singular vectors of a matrix, there is a least and a greatest formal concept, and more. The product and coproduct, for example, of formal concepts are defined by $(A_i, B_j) \wedge (A_j, B_j) := \left(A_i \cap A_j, R^*R_*(B_i \cup B_j)\right)$ and $(A_i, B_j) \vee (A_j, B_j) := \left(R_*R^*(A_i \cup A_j), B_i \cap B_j\right)$. The point is that limits and colimits of formal concepts have simple, finite formulas that are similar to, but not exactly, the union and intersection of sets and give an idea of the kind of algebraic structures one would see on the nucleus of a profunctor.

Structures in the Real World

If there's any place where neural techniques have indisputably surpassed more principled approaches to language, it is their capacity to exhibit surprisingly high performance on empirical linguistic data. Whatever the nature of formal language models to come, it will certainly be

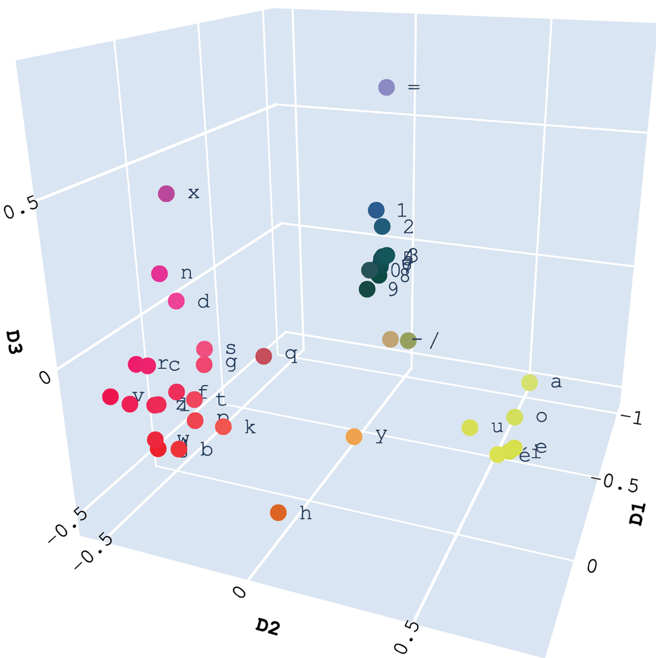

Figure 1. Characters of the Wikipedia corpus as three-dimensional embeddings obtained through SVD.

decisive to judge their quality and relevance in the real world. In this section, we illustrate how the tools of linear algebra and enriched category theory work in practice, and in the conclusion we will share how the empirical capabilities of the enriched category theory presented here can be used to do more.

To start, consider the English Wikipedia corpus comprising all Wikipedia articles in English as of March 2022 [Wik]. If we consider this corpus as a purely syntactic object without assuming any linguistic structure, then the corpus appears as a long sequence of a finite set of independent tokens or characters. To simplify things, let's restrict ourselves to the 40 most frequent characters in that corpus (excluding punctuation), which account for more than 99.7% of occurrences. So, our initial set X contains the following elements:

$$\begin{aligned} X = \{&\texttt{-}, \texttt{/}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}, \texttt{=}, \texttt{a}, \texttt{b}, \texttt{c}, \texttt{d}, \texttt{e}, \texttt{f}, \\ &\texttt{g}, \texttt{h}, \texttt{i}, \texttt{j}, \texttt{k}, \texttt{l}, \texttt{m}, \texttt{n}, \texttt{o}, \texttt{p}, \texttt{q}, \texttt{r}, \texttt{s}, \texttt{t}, \texttt{u}, \texttt{v}, \texttt{w}, \texttt{x}, \texttt{y}, \texttt{z}, \texttt{é}\}. \end{aligned} \quad (4)$$

Now let $Y = X \times X$ and, in line with what we have seen in the previous sections, consider a matrix $m : X \times Y \to \mathbb{R}$ representing some linguistically relevant measure of the association between the elements of X and Y. A straightforward choice for m is the empirical probability that the characters $(y_l, y_r) \in Y$ are the left and right contexts of the character $x \in X$. For instance, we have that $m(\texttt{h}, (\texttt{t}, \texttt{e})) \approx 0.3836$ while $m(\texttt{h}, (\texttt{p}, \texttt{o})) \approx 0.0037$ reflecting that it is over a hundred times more probable to see the sequence `the` than `pho`, given `h` as the center character.

Considered as elements of X, each character is independent and as different as it can be from all the others. However, embedding them $X \to \mathbb{R}^Y$ via the matrix m and leveraging the relationships they exhibit in concrete linguistic practices as reflected by a corpus brings out revealing structural features. Indeed, if we perform an SVD on the induced operator $M^* : \mathbb{R}^X \to \mathbb{R}^Y$ we can obtain a vector representation of each character.[1] Figure 1 shows a plot of all characters in X as points in a three-dimensional space, where the coordinates are given by the singular vectors corresponding to the three largest singular values, scaled by those singular values. We can see how what were originally unrelated elements now appear organized into clusters in the embedding space with identifiable linguistic significance. Namely, the elements are distinguished as vowels, consonants, digits, and special characters.

What's more, the dimensions of the embedding space defined by the singular vectors of M^* have a little bit of natural structure — there is a canonical order given by the singular values that is endowed with linguistic significance. Looking at the first three singular vectors, we see that the first one discriminates between digits and letters, the second one distinguishes vowels from the rest, and the third one identifies special characters (see Figure 2). The rapid decay of the successive singular values indicates that dimensions beyond three adds only marginal further distinctions (Figure 3).

While the decomposition into singular values and vectors reveals important structural features — each singular vector discriminates between elements in a reasonable way, and the corresponding singular values work to cluster elements into distinct types — the linear algebra in this narrow context seems to run aground. However, as discussed in the previous sections, we can gain further and different structural insights by considering our sets X and Y as categories or enriched categories. As a primitive illustration, view X and Y as discrete categories enriched over 2. For the next step, a $\{0,1\}$-valued matrix $r : X \times Y \to \{0,1\}$ is required. A simple and rather unsophisticated choice is to establish a cutoff value (such as 0.001) to change the same matrix M used in the SVD above into a $\{0,1\}$-valued matrix. All the entries of M less than the cutoff are replaced with 0, and all the entries above the cutoff are replaced with 1.

Then, extend to obtain functors $R^* : 2^X \to 2^Y$ and $R_* : 2^Y \to 2^X$ and look at the fixed objects of R_*R^* and R^*R_*, which are organized as described earlier into formal concepts $\{(A_i, B_i)\}$ that form a highly structured and complete lattice. Visualizing the lattice in its entirety is challenging in a static two-dimensional image. To give the idea in these pages, one can look at a sublattice defined by selecting a single character and looking at only those

[1] *For reasons beyond the scope of this paper, it's convenient to take the square roots of the entries and center the matrix around 0 before performing the SVD.*

Figure 2. Top three left singular vectors of the X-Y matrix of characters in the Wikipedia corpus, scaled by the corresponding singular values.

concepts $\{(A, B)\}$ for which A contains the selected character. For Figure 4, the characters a and 3 are selected and to further reduce the complexity of the images, only nodes representing a large number of contexts, $|B| \geq 20$, are drawn.

Right away the lattices make clear the distinction between digits and letters, but there is also more. Each set of characters A_i is associated to an explicit dual set of contexts B_i suggesting a principle of compositionality — namely, the elements of the corresponding classes can be freely composed to produce a sequence belonging to the corpus. Such composition reveals relevant features from a linguistic viewpoint. While digits tend to compose with other digits or special characters, vowels compose mostly with consonants. Strictly speaking, a similar duality was present in the SVD analysis, since left singular vectors are canonically paired (generically) in a one-to-one fashion with right singular vectors, which discriminate between contexts in Y based on the characters for which they are contexts. The formal concepts, on the other hand, as fixed objects of $R_* R^*$ and $R^* R_*$, display dualities between large but discrete classes of characters. Numbers, consonants, and letters are all distinguished, but finer distinctions are also made. Moreover, the collection of such dual classes is not just a set of independent elements but carries the aforementioned operations $\vee$ and $\wedge$ allowing one to perform algebraic operations on the concept level.

Is It Really Meaning? Content from Form in Linguistic Thought

Upon reflection, it may not be surprising that syntactic features of language can be extracted from a corpus of text, since a corpus — as a sequence of characters — is a syntactic object itself. After all, from a linguistic viewpoint, consonants, vowels, and digits are purely syntactic units devoid of any meaning per se. What is true for characters, however, is also true for linguistic units of higher levels. This can be illustrated by letting the set X be the 1000 most frequent words in the British National Corpus [BNC07]. Use the empirical probabilities that the words y_l, y_r are the left and right contexts of word x in that corpus to make an X-Y-matrix M and repeat the same calculations done before. The singular vectors of M corresponding to the ten largest singular values capture all manner of syntactic and semantic features of words, such as nouns, verbs (past and present), adjectives, adverbs, places, quantifiers, numbers,

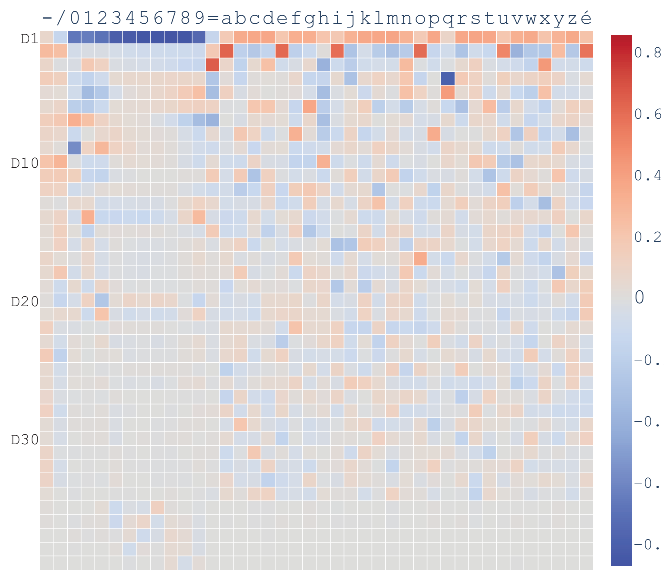

Figure 3. Left singular vectors of the X-Y matrix of characters in the Wikipedia corpus, scaled by their corresponding singular values.

countries, and so on. These ten singular vectors are pictured in descending order in Figure 5 where, for readability, only eight of the 1000 entries are displayed, namely, the four greatest and four least.

Further information about the terms appearing in the singular vectors can be obtained by choosing a cutoff (here, 0.01) to create a Boolean matrix M, just as was done for the character matrix, and a lattice of formal concepts can be extracted for these 1000 words. To illustrate, a few words (`france`, `could`, `10`) have been selected from the entries of the most significant singular vectors pictured in Figure 5, and the corresponding sublattices of formal concepts are shown in Figure 6. The linear algebra highlights these words as significant and goes some of the way toward clustering them. Choosing a cutoff and passing to the formal concepts reveals the syntactic and semantic classes these words belong to and manifests interesting and more refined structural features.

The broader and more philosophical question remains, though. Is it really *meaning* that has been uncovered, and if so, how is it possible that important aspects of meaning emerge from pure form? In the wake of recent advances in LLMs, this question has become increasingly important. One idea that's often been repeated is that language models with access to nothing but pure linguistic form (that is,

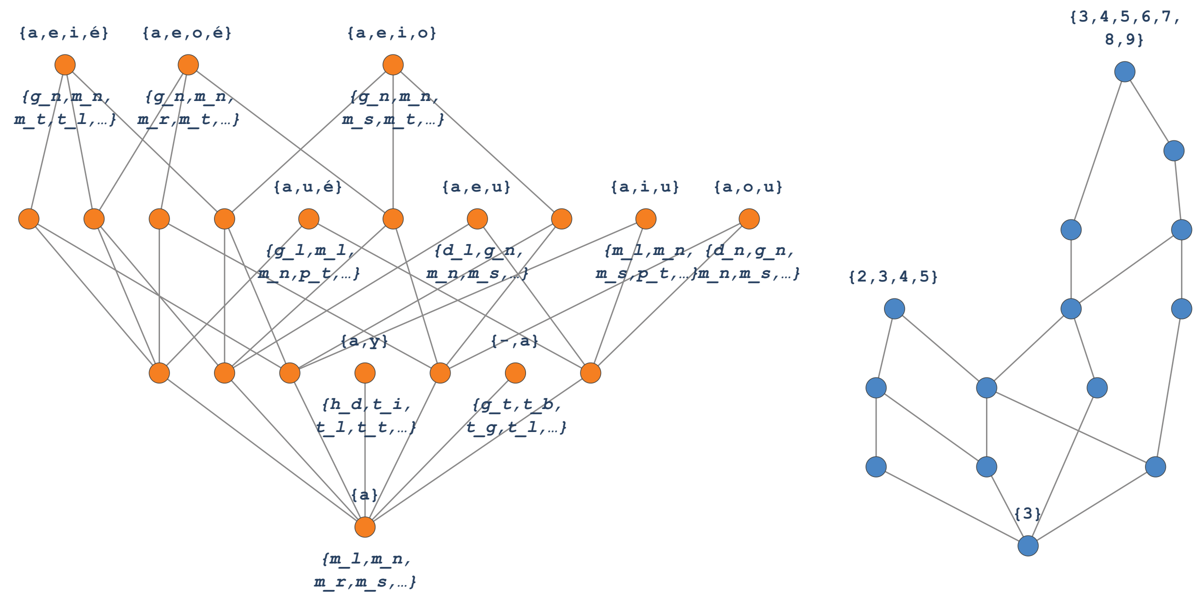

Figure 4. Sublattices of formal concepts for characters in the Wikipedia corpus for the characters a and 3 for which there are at least 20 contexts. Only the minimal and maximal nodes are labeled. Contexts not shown for the lattice for 3.

	-1	-2	-3	-4	...	4	3	2	1
D1	church	university	field	house	...	used	offered	found	made
D2	use	leave	keep	buy	...	club	sun	uk	hotel
D3	show	boy	project	move	...	size	interests	activities	nature
D4	used	expected	made	considered	...	europe	scotland	france	england
D5	used	expected	food	water	...	couple	under	series	lot
D6	perhaps	indeed	under	during	...	bit	series	couple	lot
D7	difficult	hard	easy	necessary	...	gave	started	saw	took
D8	europe	scotland	england	france	...	want	needs	army	could
D9	wish	tried	seem	seemed	...	established	won	published	produced
D10	10	15	20	30	...	on	black	into	through

Figure 5. Words in the BNC corpus corresponding to the four greatest and four least values for the first 10 singular vectors in decreasing order.

raw text) do not and can not have any relation to meaning. This idea rests upon an understanding of meaning as "the relation between a linguistic form and communicative intent" [BK20, p. 5185]. While these pages are not the place to provide a substantial philosophical treatment of this question, it seems important to point out that the mathematics presented here supports the idea that meaning is inseparable from the multiple formal dimensions inherent in text data.

The idea that meaning and form are inseparable is not new, it just is not prevalent in the current philosophical debates around AI. From a strictly philosophical standpoint, Kant and Hegel's influential work stood on the principle that form and content are not exclusive, an idea that one can also find at the core of Frege's thought, the father of analytic philosophy. More importantly, the perspective that form and meaning are not independent became central in linguistics with the work of Ferdinand de Saussure [Sau59] and the structuralist revolution motivating the emergence of modern linguistics. The key argument is that both form and meaning, signifier and signified, are simultaneously determined by common *structural* features — structural differences on one side correlate with structural differences on the other. Significantly, one of the main tools to infer structure in the structuralist theory is the *commutation test*, which tries to establish correlations between pairs of linguistic units at different levels. For example, substituting "it" by "they" requires substituting "is" by "are" in the same context, while substituting "it" by "she" does not, although it might necessitate substitution

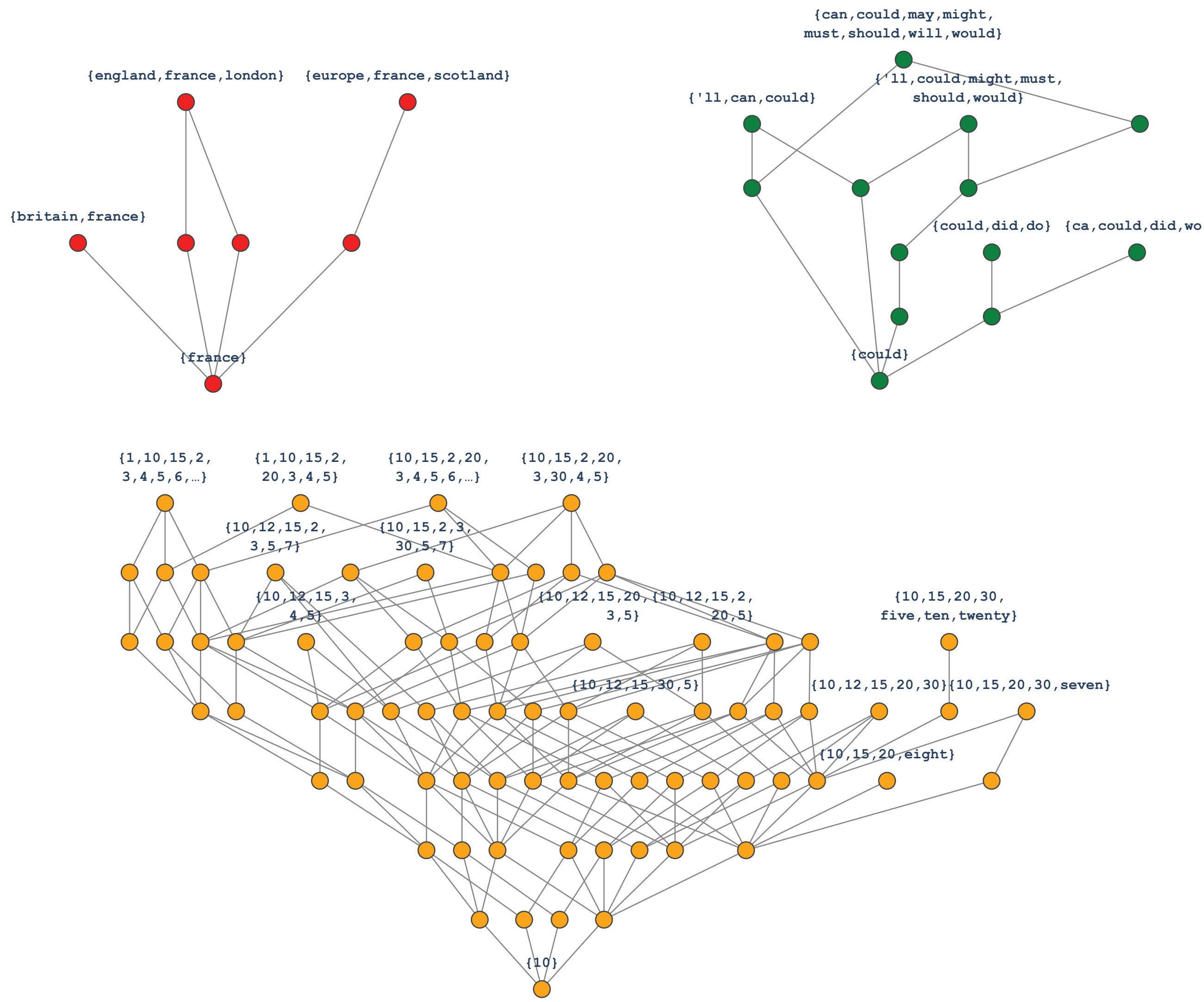

Figure 6. Sublattices of formal concepts for words in the British National Corpus for the words `france`, `could`, and `10` for which there are significant contexts (at least ten for the words `france` and `10`, and at least five for the word `could`). Only the minimal and maximal nodes are labeled. Contexts are not shown.

in other units. This phenomenon is neatly addressed by the mathematical approach presented here.

Halfway through the 20th century, Saussure's idea that linguistic form and meaning are intimately related, like two sides of the same sheet of paper, was dominant in the field of linguistics. While the introduction of Chomsky's novel generative linguistics in the late 1950s brought a dramatic slowdown to the structuralist program, structural features continued to be taken as defining properties of language under Chomsky's program. The return of empirical approaches to linguistics toward the end of the 20th century represented a new change of course in this evolution. Connectionism, corpus linguistics, latent semantic analysis, and other approaches to language learnability represented renewed efforts to draw all kinds of structural properties from empirical data, providing a myriad of conceptual and technical means to intertwine semantics and syntax (see [CCGP15], and references within).

Although it may come as a surprise that semantics is at stake in language models with access to linguistic form only, the point here is that a theory of the emergence of meaning from form is part of an extensive and well-established tradition of linguistic thought. And what such a tradition tells us, in particular in its structuralist version, is that, if meaning is at stake in the analysis of syntactic objects, it is entirely due to structural features reflected in linguistic form.

It is at this precise point, however, where current neural language models fall short since they do not *reveal* the structural features that are necessarily at work as they perform their tasks. The mathematical discussion in this article suggests that this is not an insurmountable issue but rather is a fascinating research subject, squarely contained in a mathematical domain, and independent of the architectures of the language models.

Conclusion: Looking Forward

By understanding word embeddings through well-known tools in linear algebra and by framing formal concept analysis in categorical terms, one finds parallel narratives to unearth structural features of language from purely syntactical input. More specifically, using a real-valued matrix that encodes syntactical relationships found in real world data, one can use linear algebra to pass to a space of meanings that displays some semantic information and structure. By introducing cutoffs to obtain a $\{0,1\}$-valued matrix, one can use formal concept analysis to reveal semantic structures arising from syntax. While there is nothing new about the well-known tools from linear algebra and enriched category theory used in this article (principal component analysis and formal concept analysis), the parallel narratives surrounding both sets of tools is less well known. More important than communicating the narrative, however, is the possibility that the framework of enriched category theory can provide new tools, inspired by linear algebra, to improve our understanding of how semantics emerges from syntax and to study the structure of semantics.

One approach that immediately comes to mind is a way to bring the linear algebra and formal concept analysis closer together. The extended real line $[-\infty,\infty]$ or the unit interval $[0,1]$ can be given the structure of a closed symmetric monoidal category making it an appropriate base category over which other categories can be enriched. So, $[-\infty,\infty]^X$, the functions on a set X valued in the extended reals, can be viewed as a category of presheaves enriched over $[-\infty,\infty]$, much in the same way that 2^X can be viewed as a category of presheaves enriched over $\{0 \to 1\}$. Then, the matrix M or some variation of it can be regarded as a profunctor enriched over the category of extended reals. The structure of the nucleus could be studied directly in a way comparable to the formal concept analysis without introducing cutoffs to obtain a $\{0,1\}$-matrix and also would involve the order on the reals that arranges singular vectors in order of importance. This idea has been around in certain mathematical circles for about a decade. See [Pav12, Pav21, Wil13, Ell17, Bra20, BTV22] and the references within. One might think of this approach as a way to unify the lattices of formal concepts for all cutoff values into one mathematical object, formal concepts intricately modulated by real numbers.

Another important point is that the linear algebra discussion began with sets X and Y having no more structure than an ordering on the elements. To keep the categorical discussion as parallel as possible, X and Y were considered discrete categories with no nonidentity morphisms. However, one can introduce morphisms or enriched morphisms into X and Y and the presence of those morphisms will be carried throughout the constructions described and ultimately reflected in the nucleus of any X-Y profunctor. There is no obvious way to account for such information with existing tools in linear algebra. The recent PhD thesis [dF22] recasts a number of linguistic models of grammar—regular grammars, context-free grammars, pregroup grammars, and more—in the language of category theory, which then fits in the wider context of Coecke et. al's compositional distributional models of language [CCS10]. In these models, which date back to the 2010s, the meanings of sentences are proposed to arise from the meanings of their constituent words together with how those words are composed according to the rules of grammar. This relationship is modeled by a functor from a chosen grammar category to a category that captures distributional information, such as finite-dimensional vector spaces. Such models may also be thought of as a passage from syntax to semantics, though they rely heavily on a choice of grammar. The point here, however, is that if one would like to begin with X having more structure than a set, then enriched category theory provides a way to do so without disrupting the mathematical narrative described in this article.

One place where the linear algebra tools have developed further than their analogues in enriched category theory is in multilinear algebra. For example, factorizing a tensor in the tensor product of vector spaces $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ into what is called a *tensor train*, or *matrix product state*, can be interpreted as a sequence of $n-1$ compatible truncated SVDs. We are not aware of any similar theory of "sequences of compatible nuclei" for a functor on a product of categories $\mathsf{C}_1 \times \mathsf{C}_2 \times \cdots \times \mathsf{C}_n$. Given that text data is more naturally regarded as a long sequence of characters than mere term-context pairs, it is reasonable to think an enriched categorical version of such an object could be the right way to understand how multi-layered semantic structures emerge from syntactical ones in language.

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EARLY CAREER

The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Krystal Taylor and Ben Jaye serve as the editors of this section. Next month's theme will be Women's History Month.

Black History Month

The Mathematicians of Color Alliance

Michael Young

Where are all the Black mathematicians? Why do so few Black math majors go on to graduate school? And when they do, why do so many of them drop out before they finish? In an era when equity and access are on the minds of many people, these are the kinds of questions that mathematicians need to ask.

In 2016, nearly 1000 mathematical science PhDs were awarded to US citizens. Yet only 29 went to Black mathematicians. In a country where 13.4% of the population is Black, only 3% of math PhDs go to Black people. Clearly there is room for improvement in the profession.

In 2012, only two people of color (both women) were in the Iowa State University Mathematics PhD program working on passing the required qualifying exams. Both were struggling and uncertain about how to go about getting help. Their challenges led to the creation of the Mathematicians of Color Alliance (MOCA), an organization at Iowa State University founded to give people of color the community a person needs to succeed in mathematics.

Most of our members are not only the first in their families to pursue an advanced degree, but they are also the first to earn a college degree. Even if their families support their decisions, their parents and other family members often do not know how to help their students navigate the administrative aspects of higher education. Students who don't know the steps involved in earning an advanced degree may feel like they don't belong. Mathematics should be accessible to everyone. We strive to provide support so that the students who are willing to put in the work can enjoy the fruits of their labors.

Since it began, the group has grown and the purpose has shifted to do more than provide support for those trying to gain access to and find success in an advanced degree program. Today, we work toward offering a more complete support system from recruitment through graduation and beyond.

Identifying a Need

Shanise, a Georgia native then working on passing her qualifying exams at Iowa State, would become one of the original MOCA members back in August of 2012. She was still new to the entire system and was trying to figure things out as she went. As a Black student in a state where less than four percent of the population is Black, she already felt a bit isolated. There were many things that she had to get accustomed to, and sometimes she felt like Iowa State was the wrong place for her to continue her education.

Scarlitte, a current MOCA member who grew up in the diverse and agricultural community of Salinas, California,

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also faced challenges. While her family was incredibly supportive of her choice to pursue a graduate degree in mathematics, they did not entirely understand it. When she was accepted and preparing to leave for Iowa State, she told them how long it would take to get her degree, and to this day, they still ask when she will come home.

Many MOCA members have discussed feeling isolated from their White peers at graduate school, in part because of differing family backgrounds. Many Black students also report feeling that they do not belong in graduate school. For Mike, a MOCA graduate who grew up in South Chicago, there was always a feeling of being an "impostor," as though he couldn't possibly earn a master's degree, let alone a doctorate. When Mike found himself sitting in his first class as a graduate student, he was nearly certain that it was a fluke and that he did not belong because there was no way he was smart enough to be working toward a graduate degree.

For students whose parents and family members don't have experience with higher education, navigating graduate school can seem daunting. This means that students have to look elsewhere for support. For Black students, the lack of Black peers can make it difficult to relate to the people around them. MOCA helps students find a supportive community.

A Brief History of MOCA

Iowa State is actively seeking to advance underrepresented students, so we have created a fellowship to help draw in a more diverse group of students from around the country. We do a lot to help set up graduate students for success, including a stipend and tuition assistance. However, there are many other types of support that students need. They need mentorship and guidance to navigate a system that is accustomed to a different kind of student. While Iowa State could help provide the financial assistance to ensure that students could work toward their degrees without the distraction of being in financial peril, MOCA could provide other kinds of support that Black students needed to be successful.

For many people of color coming from the East or West Coasts, Iowa can be quite a culture shock. Though the people in the mathematics department were always incredibly responsive and supportive, dealing with the wider community could be difficult. Mike, the South Chicago native, found the lack of diversity and the kinds of activities available in a more rural area a bit difficult to get accustomed to. And for Scarlitte, though she came from a community primarily based on agriculture, arriving in Iowa came as a shock. In Salinas, different races work together, and she thought that was how most communities functioned. When she moved to Iowa State in the fall of 2016 to work on her graduate degree, she quickly noticed that, instead of the migrants and other people out working the fields, machines managed most of the work in the fields.

In 2012, I approached the director of diversity and chair of the department to request support to create a community for these students. I knew that on their own these students would not likely seek others who could understand their background. By starting a community for Black students to come together and talk, I knew that they would start to open up, feel like they belong, and would be able to help each other when they had similar problems. These students faced challenges that no one else would be able to help them with, such as finding a place where women could have their hair done. Mundane tasks that were easy to resolve back home were suddenly more of a problem because the community simply wasn't built with typical Black students' lives in mind.

In 2013, we had three more Black women join the PhD program in mathematics. We discussed the good experiences because it helps reinforce the idea that they do belong and that there are always good things to consider. Of course, we also discussed the problematic aspects, particularly about race.

The Four Areas of Support

MOCA mainly provides four types of support: recruitment, personal, mentoring, and post-graduation. Mentoring in all of these areas helps students feel like they belong at Iowa State.

Recruitment. When I arrived at Iowa State, the initiative to attract underrepresented students had only gained interest from a couple of students. Today, MOCA not only helps recruit Black students to enter the program, but also helps retain them. So far six of our members have either finished or are finishing their PhDs. This success has made it easier to reach out to other Black students around the country and recruit them to the university. As potential graduate students come to check out the program, we invite them to visit with our group for an evening of conversation. They discuss what it is like forming such a small percentage of the population and the unique challenges that they will face, particularly the culture shock.

MOCA has been fortunate in that the people who join the group are incredibly supportive of each other. This is easily seen during the recruitment process. Often, students just need a bit of a push to make that final decision. We want potential students to see the benefits Iowa State has to offer as well as let them know we understand the problems they may face. By visiting with MOCA, prospective students can see that they will have others who understand what it will be like and can offer a protective layer that would otherwise be absent so far from home.

We also provide assistance in understanding the financial aspects of working towards an advanced degree. Many graduate students are new to handling their own finances, and for minority students, discussing personal finances can be viewed as taboo. With many of our members uncertain and uncomfortable about how to obtain necessary financial aid, we know that this is a key component of not only recruiting but also retaining members.

Sometimes what prospective students need is to feel supported while they are preparing for their qualifying exams. There is currently a post-baccalaureate program to help them take a year to fully prepare so that they don't feel rushed through the process. Feeling prepared for the exams increases the likelihood that they will pass, and it proves that they are more than good enough to join the program.

Personal support. Personal support is essential, no matter who the student is. With some students arriving without familial support, the situation can make them feel like they don't belong. MOCA members support each other—if they encounter academic problems, there will be people who can help them through it. Building these relationships as soon as students arrive in the department is important because scheduling meetings becomes more problematic as the semester progresses. Once the friendships have formed, sometimes all it takes is a meeting or two of quietly working side by side to get students to feel like this is where they belong. These relationships now extend beyond graduation as previous members still feel comfortable calling and asking for advice.

Personal support also works as the best way of retaining our students. It is much easier to get discouraged when you feel like you are alone, which increases the odds that a student will drop out of the program. We have been successful in our retention in large part because MOCA members feel they have a strong, reliable group of personal support from people who are like their own personal cheerleaders.

Mentoring. Mentoring is the most enjoyable aspect of working with students. It is also tricky because there are so many components to it. My favorite aspect of mentoring is taking the graduate students to conferences and professional development seminars. These experiences provide encouragement and motivation, particularly when students are struggling because they can see just how much they have to gain by persevering. For those who aren't quite sure what direction to go, I help them understand their options and set goals. I always encourage them to set their own personal goals because, in the long run, it always benefits the student to have a gauge that they set and measure themselves by instead of simply accepting what others say about them.

We also rely heavily on peer and group mentoring. A person who has been through the process can provide a more focused approach for those just starting out. While administrators are there to help, many of them do not have the same personal background and have not faced the same kinds of problems. Our mentors fill the gap, helping those just starting out to better manage their time and letting them know that there is someone else who can understand what they're going through.

Post-graduation. MOCA graduates have been a great source of assistance to our current students, and so far they have not required much assistance themselves. Given how in-demand mathematicians are, our graduates quickly learn that they have a lot of options once they finish their advanced degrees. Finding employment does not appear to be a problem—they have been able to find positions that they enjoy within months of finishing their degree programs. By seeing others succeed, more people of color will begin to recognize that earning an advanced degree in math is a real possibility.

A Personal Look at the Effects of MOCA

While the retention of students has been phenomenal, it is the way our students feel about MOCA that really matters.

One of the founding members and first MOCA graduates. Shanise was one of the first Black students from Iowa State to complete a graduate degree in mathematics. Although she came from Georgia, the culture shock was mitigated by the fact that she knew it wasn't a permanent move. Knowing that her time at Iowa State was finite and that she had a strong support group there made it easier for her to buckle down and graduate.

Figure 1. Michael Dairyko, Shanise Walker, and Chassidy Bozeman at Iowa State University's 2018 graduation.

The biggest problem for Shanise was that she did not feel that her research skills were at a level where she could achieve success. MOCA provided her with the encouragement she needed to not only improve her research skills, but also to foster the confidence to keep going. She explained that in the early days she wasn't sure how to budget her time when she was conducting research because there were so many other things that needed to be done. Learning how to carve out time to conduct meaningful research was difficult, but it has been a skill that she has mastered. Now she is able to put those skills to work in an entirely different environment.

It took her six years to earn her degree, and she remembers the time at Iowa State with a lot of fondness and humor. Looking back, that first year was the hardest because it was so difficult to connect to other students. After the founding of MOCA, things became a lot easier. Today, she occasionally takes calls from students looking for a little insight into the program or who need some advice on things to do in the area. She found a fantastic job at the University of Wisconsin–Eau Claire, about ninety minutes away from Minneapolis, a turn of events she never would have imagined in those early days. When she travels, she actively encourages prospective students to consider Iowa State.

A successful graduate. Mike initially had no intention of going for an advanced degree in mathematics because he didn't think that he was smart enough to do it. Having earned his undergraduate degree at Pomona College in Claremont, California, Mike was ready to start looking for a suitable position. Then he met Michael. Michael convinced Mike to just check things out and see if there was a school where he would be interested in working on a graduate degree. Mike visited two different schools, but after he visited Iowa State, he knew that was where he wanted to go if he was going to pursue another degree.

One of Mike's biggest problems was his self-described Impostor Syndrome. He just never felt like he belonged. Then he talked to the other members of MOCA, and found that two of them felt exactly the same way. Other students began to say that they felt that way too, but that it was a matter of ignoring that doubt and working through those first few weeks to see that they were there because they had earned it.

Some of his fondest memories are of the meetings and potluck meals the group had. While the study sessions were fantastic, it was nice to have the socialization of the meals to really help them talk about things. They could discuss their problems and experiences. Their conversations validated some of the things he felt and helped him be more open. He was an active member of the water polo team during his time at Iowa State, so he felt like he constantly had people cheering him on in many different areas of his life.

Today, he works with the Milwaukee Brewers, a position that he accepted less than two months after earning his doctorate. His time at Iowa State taught him that he is not only good enough and smart enough, but also that he can dream in ways that go much further than he had originally imagined.

The unique experience for a current member. Scarlitte first had a taste of what it would be like to go to Iowa State in 2015 when she attended the Research Experience for Undergraduates (REU) program. She knew that the people in the mathematics department were welcoming, and that she would stand out a bit as Mexican, but this did not seem like a reason to ignore the great experience she had during her REU. It just meant doing a bit of adjusting to the entirely different community dynamic.

When Scarlitte joined MOCA, that made her experience that much more enjoyable. One of the things that she has found to be most beneficial is having the chance to see other people of color in the field. She has been able to attend conferences and lectures where people of color have served as primary speakers, reinforcing the idea that she is in the right place—that she belongs. She wishes MOCA had more regularly scheduled meetings. However, that is always a delicate balancing act because each graduate student has his or her own unique research and schedules that make this a challenging task. Still, she loves spending time with other MOCA members because she loves to see how others in similar positions are doing and know that she is not alone.

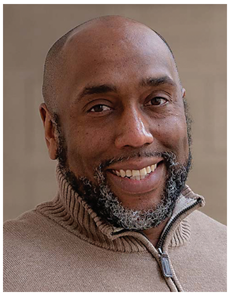
Concluding thoughts. Trying to go at it alone in the field of mathematics is like trying to learn a language without ever speaking to another person in that language. Everything is more challenging. Yet this is the situation that most minorities face, not only when they work towards their bachelor's degree, but also when they strive for a master's degree or a PhD. I think that what makes it all the more difficult is that students may not even realize that it is a problem because they have been desensitized to it. Being a part of a community creates a feeling of belonging. However, it is unreasonable for us to expect graduate students to create this collective when they arrive on campus to study mathematics. The onus is on us, mathematics faculty, directors, program coordinators, and leadership, to do what is required to give each student the opportunity to be successful.

Since the publication of the book in 2021, there have been notable changes in the professional trajectories of the individuals mentioned. All have transitioned to new institutions and assumed different roles that reflect their evolving expertise and aspirations.

Michael Young has joined the Mathematical Sciences faculty of Carnegie Mellon University, where he serves as the inaugural associate dean for diversity, equity and inclusion in the Mellon College of Science. He also leads the ASCEND Mentor Network and is the creator and executive producer of the podcast Mathematically Uncensored.

In 2022, Shanise relocated to her home state of Georgia, where she is an assistant professor of mathematics at Clark Atlanta University. Shanise continues to be active in research, mentoring, and in the mathematics community. She recently coedited the book Aspiring and Inspiring: Tenure and Leadership in Academic Mathematics.

After graduating from Iowa State, Mike started his career in business intelligence for professional sports. He currently is the director of advanced analytics for the Milwaukee Bucks.



Michael Young

Credits

Figure 1 is courtesy of Michael Dairyko.
Photo of Michael Young is courtesy of CMU/Jonah Bayer.

How Can We Know What We Deserve When Toxicity is the Norm?

Marissa Kawehi Loving

The author would like to thank the organizers of the Hidden NORMS (Navigating Obstructive Rules in the Mathematical Sciences)[1] webinar series for inviting her to lead a discussion on "pseudo norms" in the mathematical community out of which this article sprung. You can find the webinar recording here[2] and an excerpt from the webinar below.

> *There's this feeling that we should just be grateful. We should be grateful that we made it. We don't deserve to ask for more. What I want us to walk away from this discussion with is the idea that we should always be willing to ask for more for ourselves and for the people around us until we are all being treated with humaneness [sic], with dignity, and with respect. That is what we deserve and anything less than that is toxic. And that is a hard and very difficult standard to hold for yourself in a space which is always degrading our value and the value of those around us. It is hard not to internalize that we deserve less. So if the only thing you take away from this is to say, "Whatever you say I deserve, I deserve more." That's good. Because it's true.*

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[1] `https://aimath.org/hiddennormsdescription.pdf`
[2] `https://www.youtube.com/watch?v=Eq8CmhxJfnY&ab_channel=HiddenNORMS`

Setting the Scene

When I was an undergraduate student, I was sexually harassed by one of my computer science (CS) professors for well over a year. It started with small things, like him hanging out in the majors-only upper-division CS lab. Then he asked me to add him on Facebook and started messaging me often. Before I knew it, he was finding excuses to touch me. Eventually he told me he loved me (needless to say, I did not feel the same). I was only 19 at the time (I first met him at 16 and was 17 when I took my first class from him). I felt overwhelmed and out of my depth. Meanwhile, the men in my CS cohort started making snide jabs suggesting I was receiving preferential treatment from faculty (it didn't help that I consistently outperformed my all male cohort on both homework and exams). I knew something wasn't right, but I didn't know what to do about it. I didn't even know how to name what was happening. It wasn't until I shared my experiences with several women in my summer REU program that I was given the words to describe my experience: sexual harassment and abuse of power. In retrospect, it all seems so obvious, but abuse thrives in isolation. Without those women in my REU listening to me and affirming my experience I don't know how long it would have taken me to speak the truth of my situation.

Fast forward to my first year of graduate school. Once again I was quite isolated as one of the only women of color in my graduate program. I was consistently treated as *less than* by both my peers and my professors, from being excluded from departmental leadership roles that highlighted mathematical drive (like organizing the graduate student geometry and topology seminar) to never once receiving departmental recognition for any of my accomplishments (grants such as NSF postdocs are usually celebrated with an email to the department, but not mine) to being told to my face time and time again that I didn't belong. Suffice it to say, my graduate school experience was less than ideal (you can read a little bit more about it here[3]), but the experience nonetheless felt very different than the sexual harassment I had suffered in undergrad. My negative experiences in graduate school weren't confined to a damaging relationship with one faculty member or consistently obscured by the power that faculty hold over their students; they happened all the time and right out in the open in front of everyone.[4] At times it felt almost impossible to talk about these negative experiences, not because I felt like I had to keep quiet, but because it felt just too damn normal for anyone to care.

[3] `https://awm-math.org/awards/student-essay-contest/2020-student-essay-contest-results/2020-student-essay-contest-grand-prize-winner/`

[4] *For example, before the end of my first semester multiple white men in my cohort told me that I only received the NSF Graduate Research Fellowship because I was a woman of color and they would have been awarded it if they weren't both white and male. This is only the tip of the iceberg.*

It's been four years since I graduated with my PhD. Three (extremely terrible) years of postdoc and one (pretty good) year of tenure-track later, I've realized more and more how pervasive this culture of isolation, discrimination, and exclusion is in mathematical spaces, especially for Black mathematicians like me.[5] In that time, I've also come to believe more strongly than ever in the power of organizing with the people around us to demand what's ours: respect, dignity, and fair treatment. Unfortunately, what we consider to be fair is so often shaped by how the people around us are treated, and even more unfortunately, academia is a pyramid scheme in which exploitation is normalized especially for people at the base of the pyramid.[6] So how can we talk about what to expect, what we deserve, and what we can demand when toxicity and exploitation is the norm?

I certainly don't have all the answers, but my hope is that this article can help some up-and-coming mathematicians—especially Black mathematicians, Indigenous mathematicians, disabled mathematicians, trans mathematicians, and those living in the intersections of any of these communities or who belong to other communities that continue to be marginalized within academic mathematics—decide to take up more space for themselves and for those around them. I hope that this article will help you (yes, you, dear reader) develop strategies for creating and maintaining high expectations for how you and those around you deserve to be treated even when you are surrounded by norms that tell you that you are lucky to even be in mathematics in the first place and you should just be happy with whatever you get. I want us all to find the strength to say, "This is how I deserve to be treated and I will not accept anything less."

Building Some Shared Language

Not only do I not have all the answers, since my professional training is in research mathematics, I don't even have all the language to describe the problems I want to find answers to (language which I'm sure exists in the wealth of social science literature on workplace culture and harassment). So in order to make this conversation more meaningful, I will do what every mathematician does: I'm going to offer some definitions that will give me enough structure to discuss the problems at hand in a meaningful way. In particular, I want to highlight two types of harmful norms that I have encountered within academic spaces.

Toxic norm: A negative experience that is widely shared within a community regardless of your individual background or identity.

Pseudo norm: A negative experience that you are expected to accept as normal but is not typical for other individuals in your community or those with different marginalized identities than your own.

Remember, I made these terms up to help me articulate my experiences and to give us a framework for discussion. These are not distinct categories by any means and how you interpret your experiences as either stemming from toxic norms or pseudo norms will vary significantly especially depending on your own identities and on your local community!

For me the stories I described in the introduction very broadly illustrate the distinction between these two types of norms: the sexual harassment I experienced in college was a pseudo norm, the horrible experience I had in graduate school (especially during the first couple of years) was much closer to a toxic norm although many of the individual negative incidents of racism or sexism that I experienced were manifestations of pseudo norms. That is, having a bad experience during your first year of graduate school is very common and the community expectation that we should just accept that graduate school will be terrible at times (to the point of destroying our mental health and wellness) lends itself as cover to many specific instances of racism and discrimination. After all, everyone is having a hard time so why should your experience be any different? But having a hard time because of too high teaching loads, uncaring faculty, and a lack of healthcare or a living wage and having a hard time because you are experiencing racist microaggressions from your classmates and professors *on top* of all of those other problems are just *not* the same experiences. This brings me to my final observation about toxic norms vs pseudo norms: toxic norms are often specifically weaponized against marginalized students in a way that creates pseudo norms. Let's consider the following case study to drive this point home.

Case study: Imposter syndrome. Wikipedia defines imposter syndrome as "a psychological occurrence in which people doubt their skills, talents, or accomplishments and have a persistent internalized fear of being exposed as frauds." Academics love throwing around the term imposter syndrome, especially in the face of every marginalized scholar who gives voice to the fear that there isn't a place for them in the academy. It's the perfect way to make structural problems into personal failings. Because while imposter syndrome is a real experience for some people, for many marginalized scholars the fear of being an "imposter" isn't just a "persistent internalized fear." The reality is that many of us are told day in and day out that we do not belong, our accomplishments are undermined and diminished, and we can find ourselves on the receiving

[5]*And, let's be real, this toxic culture and straight-up racism in mathematics is far harsher for Black mathematicians whose features (skin tone, hair texture, and facial features), unlike mine, read as unambiguously Black and often make them subject to even further discrimination rooted in colorism, texturism, and featurism.*

[6]*The examples of this are too numerous for me to enumerate, but one very concrete instance is the poor pay, benefits, and working conditions that so many graduate students, postdocs, and other non-tenure-track faculty experience. Yes, even in mathematics and other STEM fields!*

end of months of racist harassment for simply announcing that we accepted a job which other people don't think we deserve.[7] So while imposter syndrome is aptly classified as a toxic norm to be expected in any pyramid scheme like academia, it often morphs into a pseudo norm used to silence marginalized people who dare to express their experiences of exclusion.

Strategies for Success

Now that we have a way to articulate what we're up against, it's time to talk about how to challenge these harmful norms as you encounter them. If you find yourself having a negative experience in your department, research group, or any other mathematical space, here is a checklist that I hope you will find useful in deciding what to do next.

Step 1. Are you experiencing a toxic norm or a pseudo norm? If you're not sure, think about the experiences of the people around you. Is what's happening to you a common experience for people with a variety of different identities or does it seem to only be happening to you either because of your identity or because of the power that someone holds over you (like your advisor, department chair, or the chair of your tenure committee)? Remember, these norms aren't completely distinct, but usually experiences tend to lean one way or the other.

Step 2. Regardless of your answer in Step 1, the way forward is never alone. Community is the antidote for any kind of harmful norm.

1. Toxic norms crumble in the face of supportive and care-based communities.
    a. Because toxic norms are often widely experienced regardless of individual identity, they can be effectively tackled through collective action!
    b. Organize with your peers![8] Brainstorm solutions! Push for change!
2. Pseudo norms thrive in silence and isolation.
    a. Find trusted friends and accomplices to share your experiences with. This can be scary and overwhelming. Take things at your own pace.
    b. Pseudo norms are not as commonly experienced as toxic norms (although they are still pervasive!), which makes collective organizing more difficult. They are often imposed on you by people with direct power over you.
    c. Each situation is different, but shining a light on the problem is crucial. Nevertheless, you don't owe anyone your story.

Step 3. Document everything as a defense against gaslighting.[9]

1. One of the ways that toxic and pseudo norms persist is by convincing us that our experiences are not as terrible as we've experienced them to be or that they are a result of personal failings rather than external environmental causes.
2. An effective way to combat gaslighting is by documentation! This is especially important when dealing with pseudo norms that may not be widely experienced by your peers.

Step 4. Don't become a toxic norm enforcer!

1. The "if I had to suffer, so should you" mentality is VERY easy to fall into! And it's a common way that toxic norms are perpetuated.
2. The "we've always done it this way" mentality is probably the PRIMARY way that toxic norms survive because it gives us cover to not do the hard work of figuring out what needs to change and how. An unwillingness to disrupt the status quo keeps us perpetuating the same cycles of harm without even needing to actively recognize the damage we are doing.
3. Of course, norms can also be good! But they need to be set intentionally and revisited frequently. Good norms also need to be enforced (just like the toxic ones have been for so long)! Accountability is key.

Know Your Worth

At the end of the day, living in a world where people enrich themselves on the backs of our dehumanization is a constant assault on our sense of self. It is a truly revolutionary act to stay grounded in a deep understanding of our innate worth and the innate worth of the people around us. So I want to end with some affirmations. Read them. Say them. Share them. Believe them. And then go out and help build a mathematical community that realizes them.

- I deserve to be treated with the utmost dignity and respect.
- I deserve to have my mathematical talent acknowledged and supported.[10]
- I deserve to be treated as an expert on my own experience.
- I deserve to be recognized as a complete and complex human being with identities and experiences that materially impact the way I am treated in mathematical spaces.

[7] *Yes, this is a real thing that has happened to me and other marginalized scholars.*

[8] *I'm talking about collective action here, not organizing a research seminar, just so we're on the same page. This might look like getting involved with your local union, like forming a group to lobby the department chair/college dean to fulfill specific asks, or even forming an affinity group where people can share their experiences and realize they are not alone.*

[9] *Explanatory comma—Gaslighting refers to the act of undermining another person's sense of knowing their own reality by denying facts, the environment around them, or their feelings.*

[10] *Sometimes this affirmation needs to be directed inward before we can direct it outward. We deserve to recognize our own mathematical talent and potential! Especially when those around us are working overtime to convince us we don't have any.*

- I deserve to voice my experiences and concerns without suffering retaliation or professional harm.
- I deserve justice and restorative care.

Marissa Kawehi Loving

Credits

Photo of Marissa Kawehi Loving is courtesy of Marissa Kawehi Loving.

Three Steps for Achieving Equity and Access in the Math Classroom

India White

Introduction

In recent years, there has been a growing recognition of the importance of equity and equality in math education. This holds true particularly in the math classroom, where historically marginalized students, i.e., students of African American and Hispanic descent, have faced significant barriers to success. Achieving equity in the math classroom is crucial for ensuring that all students have equal opportunities to excel. According to the 2022 National Assessment for Educational Progressing (NAEP) data, there has been a decrease in academic achievement among underserved diverse learners. Further, there has been a decrease in access to advanced courses, noting that only 24% of all 13-year-olds took algebra in 2022 compared to 34% of 13-year-olds a decade ago (NAEP, 2022).

NAEP data reported the following regarding all fourth- and eighth-grade math students, stating that:

> In 2022, the average fourth-grade mathematics score decreased by 5 points and was lower than all previous assessment years going back to 2005; the average score was one point higher compared to 2003. The average eighth-grade mathematics score decreased by 8 points compared to 2019 and was lower than all previous assessment years going back to 2003. In 2022, fourth- and eighth-grade mathematics scores declined for most states/jurisdictions as well as for most participating urban districts compared to 2019. Average scores are reported on NAEP mathematics scales at grades 4 and 8 that range from 0 to 500.

In this article, we will explore three essential steps that educators can take to promote equity in their math classrooms, which are to **1) Cultivate an inclusive classroom environment; 2) Differentiate Instruction in the math classroom; 3) Address Implicit Biases and Stereotypes**. As we further understand these strategies, educators will be able to effectively have an equitable experience for all learners in the classroom.

Step 1 (Cultivate an Inclusive Classroom Environment). In their article, "Learning Mathematics in an Inclusive and Open Environment: An Interdisciplinary Approach" Demo et al. (2021) mention how researcher Mel Ainscow defines inclusion as, "aiming at the presence of all in school... the meaning participation and learning for all (p.3)." Creating an all-inclusive classroom environment is a foundational component for teachers striving to achieve equity in the math classroom. It involves fostering a sense of belonging and respect for all students, regardless of their background or abilities. In his book *College Students Sense of Belonging: A Key to Academic Success*, Strayhorn (2012) found that when a student's sense of belonging increases, it increases their academic achievement (p.59). However, sometimes teachers lack the ideal strategies needed to create an environment where all students feel like their presence and contributions matter in the math space. This can create social awkwardness, causing students to hold back from contributing to their math lesson.

Some strategies to consider for implementing inclusion in a math classroom are to: **1) Include Diverse Representation; 2) Explore Collaborative Learning; and 3) Create a Safe Place for Mistakes.** We will look deeper into each strategy to better comprehend how to implement each one so that results of proficiency and success for all students may follow.

Strategy 1. Include Diverse Representation: Teachers should strive to incorporate diverse mathematical examples, problems, and real-world applications that reflect the experiences and cultures of all students. This helps students see themselves in the math curriculum and promotes engagement. Sometimes, teachers can open up the learning space for students to partake in how to include diversity in the math content and in math discussion during learning. For instance, teachers can find out who their

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students' favorite athletes are and give a real-world math example for students to work out during the lesson. Further, teachers can be intentional at highlighting mathematicians who contributed to the field of math from diverse backgrounds. Teachers can also highlight various facets of culture to celebrate the culture of learners from various regions. When implementing diverse representation, it should always feel like it is done with an intention to honor and celebrate various diverse groups, so that students from these ethnic groups can become motivated that they can achieve as much as leaders from their cultural group, causing them to give a greater effort toward learning and proficiency in the math classroom.

Strategy 2. Explore Collaborative Learning: Encourage collaborative learning activities that promote teamwork and cooperation. Group work allows students to learn from one another, share different perspectives, and build empathy. In her article, "Cooperative Learning Structures Can Increase Student Achievement," Dotson (2001) speaks about collaboration for students by using Kagan structures , which are collaborative structures founded by Dr. Linda Kagan that have been proven to increase student engagement (Wetering, 2009), and that these structures improve student achievement in the classroom. When students collaborate with their peers, they benefit from peer-to-peer interaction, teaching each other during the metacognitive process. Further, when students are working on math problems, allowing them to work together in pairs or in groups will help them conceptually grasp the math while using grit to motivate each other to thrive (White, 2020; Duckworth, 2016). Hence, collaborative learning can contribute to a sense of belonging and improve access and equitable practices for all learners.

Strategy 3. Create a Safe Space for Mistakes: In the classroom, a teacher must be intentional about creating a safe place for learning. Seeing some students who struggle from math anxiety, a low sense of self, and are afraid to take risks, educators must be intentional and proactive at helping students to grow beyond the fears and barriers they face have when they approach the process of learning math. In their article, "Effect of Classroom Learning Environment on Students' Academic Achievement in Mathematics at Secondary Level" Riaz Hussain Malik and Asad Abbas Rizvi (2018) mention that, "In a classroom, the teacher's role is much important to enhance the morale, self-concept, self-confidence of a learner because he/she may be tool of inspiration or torture (p. 212)." They elaborate, stating that teachers can "humiliate or humor, hurt or heal a student in a class (p. 1)." Hence, educators must be intentional at creating a safe space where students of all backgrounds feel comfortable making mistakes and taking risks.

Students must be convinced that their teacher is an advocate and is there to support them when they are learning a new concept or struggling to solve a rigorous math problem. Teachers should encourage students to learn from their errors and emphasize that mistakes are an essential part of the learning process. As they do this, learners will become convinced that they can understand math and become proficient in it.

Step 2 (Differentiate Instruction). Differentiating instruction is a process in which teachers are able to diversify instruction in a way that meets students where they are in their learning and simplifies concepts so that all students can understand what they are learning. Differentiation is crucial for meeting the diverse needs of students in the math classroom. Since students are learning at various paces, and enter their math classrooms on different levels, choosing to differentiate instruction can help teachers check students' understanding as they help support students who are in various levels on their path toward proficiency in the math classroom. Further, by tailoring instruction to individual students, educators can ensure that all students have access to high-quality math education. In their research article titled "Using Assessment to Individualize Early Mathematics Instruction," Connor et al. (2017) found that, "teachers' instructional practices do affect students' mathematics achievement gains and are more effective when individual student differences in mathematics skills are considered." This takes place when educators tailor instruction to individual students. **Hence, to effectively differentiate instruction, educators can start with three strategies: 1. Give Pre-Assessments; 2. Allow Flexible Grouping, and 3. Have Varied Instructional Materials.**

Strategy 1. Pre-Assessment: When starting with a new classroom of learners, it is in the best interest of educators to conduct pre-assessments to identify students' prior knowledge and skills. As teachers do this, it can help them plan and determine what skills need to be revisited before introducing new content. In their research article titled "Using Assessment to Individualize Early Mathematics Instruction," Connor et al. (2017) found in their research that, "Accumulating evidence suggests that assessment-informed personalized instruction, tailored to students' individual skills and abilities is more effective than more one-size-fits-all approaches." Further, it will help teachers identify strong math identities that exist among learners or see students who are really exposed and vulnerable during learning. In addition, having pre-assessments for students allows teachers to differentiate instruction by providing appropriate challenges and support to each student.

Strategy 2. Flexible Grouping: When grouping students based on their needs and abilities teachers can provide targeted instruction and support to different groups, ensuring that all students receive the attention they need; flexible grouping can set students up for academic success.

Further, it can help build student-to-student relationships in the classroom, fostering a stronger sense of community in the learning environment. Flexible grouping can help support natural leaders in the classroom as they contribute to the environment by sharing their strengths with their peers through scaffolding and modeling how they understood the content. It also motivates lower-performing learners in a way that helps them see that not all hope is gone, and that with grit and persistence they will eventually comprehend the subject at hand (White, 2020; Duckworth, 2016). Through using flexibility grouping, math students are opened to a world of opportunity and access during their learning experience in the classroom.

Strategy 3. Varied Instructional Materials: Utilizing a variety of instructional materials, such as manipulatives, visual aids, technology, and real-world examples is a great way to create access to math for all learners. These methods help to support various learning styles and ensure that all students can access the content. For instance, educators can expose students of diverse backgrounds to understanding math by allowing them to explore various parts of the world through technology and real-world examples, (i.e., pyramids, building structures, etc.) Sometimes, having an expert speak on their field of interest, i.e., National Geographic Explorers, Black Women in Math, NASA engineers, etc., can help make sense of the math and create a sense of reality for learners that might struggle with abstract concepts within a lesson.

When these tools are utilized, learners can pursue learning with an increased amount of engagement, which affects their learning progression in a positive manner. Some colleges have begun adopting these ideas via internships for diverse learners with "institutional diversity partnerships." In their article entitled "Cultivating Diversity and Competency in STEM: Challenges and Remedies for Removing Virtual Barriers to Constructing Diverse Higher Education Communities of Success," Whitaker et al. (2015) concluded that "institutional diversity partnerships are generally established on the basis of leveraging the strengths of HBCUs' success with producing a large proportion of students of color who complete degrees in STEM majors with the fiscal and infrastructure resource bases of large majority institutions," suggesting that various internships at universities for diverse learners can help create a more equitable, and inclusive learning environment that will also retain students of ethnic backgrounds.

Step 3 (Address Implicit Bias and Stereotypes). There are three types of biases that are commonly discussed among educators: Implicit biases, explicit biases, and unconscious bias. In their article titled "Teachers' Bias Against the Mathematical Ability of Female, Black, and Hispanic Students," Copur-Gencturk et al. (2020) define explicit biases as "discriminatory attitudes and stereotyping behaviors that individuals are consciously aware of, are intentional, and are under the control of the individual." They define implicit biases as "biases that individuals are unaware of, operate below the surface of consciousness, are out of the control of the individual." Further, Copur-Gencturk et al. (2020) define unconscious biases as, "overestimation or underestimation of students' ability, as measured by the variance in teachers' assessments of their students' ability not explained by the students' performance on a direct assessment." These biases and stereotypes can significantly impact students' experiences and achievement in the math classroom.

Biases and stereotypes can cause students of diverse and ethnic backgrounds to have a low sense of math identity, creating a mindset of hopelessness. Some learners might experience little to no motivation because they feel like their math teacher does not believe they can excel in their math class because of their skin color, language, or cultural background. It is crucial for educators to be aware of their biases and actively work to counteract them so that invisible social barriers that exist may be removed, and students can develop trust in their teachers and thrive in their learning. **Three strategies to consider when addressing implicit biases and stereotypes include 1) Reflect on personal bias; 2) Promote positive attributes of learners; 3) Encourage a Growth Mindset.**

Strategy 1. Reflect on Personal Biases: Sometimes, people develop biases due to daily interactions with society. However, these biases can trickle into the learning environment in math class, affecting the perspective of teachers toward students of various backgrounds. As a result, grading can be subtly detrimental to learners, interactions can be rigid, cold, and uninviting for diverse learners. Copur-Gencturk (2020) mentioned how biases affected teacher grading for students of color, finding that "for partially correct responses, teachers' ratings of White-sounding names were rated significantly higher than those of Black- and Hispanic-sounding names." Hence, in efforts to eliminate biases that might exist in various tasks such as grading, educators must be honest by engaging in self-reflection to identify and challenge personal biases. This can be done through professional development, workshops, or discussions with colleagues. Other ways is by documenting interactions between teachers and students with diverse backgrounds to see progress that has been made in social interactions, establishing trust and strengthening of teacher-student relationships.

Strategy 2. Promote Positive Attributes of Learners: Highlight positive role models and success stories from diverse backgrounds in math. This helps challenge negative stereotypes and encourages all students to believe in their own abilities. In their article, "*Equity in Mathematics Education*," Vital et al. (2023) emphasize scholarly research that points to various frameworks to promote social justice for learners of diverse backgrounds and caring for diverse learners in the math classroom. As this is accomplished inequities are exposed, and teachers are better equipped to help level the learning experience. Vital et al. (2023) also summarize Anne Watson's work on caring of learners in mathematics education, stating that "care for mathematics, care for learners and care for learning mathematics are developed through collaborative inquiry and power-sharing among teachers, learners and communities" and that "listening and noticing are important for both the emotional and cognitive work involved in learning mathematics." In her book, *Care in mathematics education: Alternative educational spaces and Practices*, Watson (2021) emphasizes the possibility of transformative math spaces that can become a part of a learner's math identity when there is care as a part of the learning environment. Hence, through caring for learners and pointing out their strengths during learning, students can improve their ability through these equitable math practices.

Strategy 3. Encourage a Growth Mindset: In the article, "Carol Dweck Revisits the 'Growth Mindset,'" Dr. Carol Dweck (2015) elaborates on what a growth mindset is stating that it is when "people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point. This view creates a love of learning and a resilience that is essential for great accomplishment." Hence, teachers can foster a growth mindset in diverse students by emphasizing that intelligence and math abilities can be developed through effort and perseverance. As teachers cultivate a growth mindset for learners, it triumphs over a fixed mindset that would convince them that they can't do math, or that the math is just too hard for them (Dweck, 2006). A growth mindset coupled with grit, which Duckworth (2016) defines as "passion and perseverance to obtain long-term goals," can help students feel like they are invincible mathematicians in the classroom, and will cultivate an intrinsic motivation toward learning math.

Conclusion

Achieving equity in the math classroom is a continuous process that requires intentional efforts from educators. By cultivating an inclusive classroom environment, differentiating instruction, and addressing implicit bias and stereotypes, educators can create a space where all students have equal opportunities to succeed in math. Further, when educators promote equity in the math classroom, it benefits marginalized students and enhances the learning experience for all students. When students from diverse backgrounds and abilities are given the support and resources they need, they can contribute unique perspectives, ideas, and problem-solving skills to the math classroom. As educators, it is our responsibility to ensure that every student feels valued, respected, and empowered in the math classroom. By implementing these three steps, we can create a more equitable and inclusive learning environment where all students can thrive and reach their full potential in mathematics. Let's work together to build a math classroom that celebrates diversity, promotes equity, and fosters a love for learning.

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India White

Credits

Photo of India White is courtesy of Big Ideas Learning and Larson Texts Incorporated.

Dear Early Career

I am applying for jobs this year. What should I put on my CV?

—*Job Applicant*

Dear Job Applicant,

Let's start with the essentials that your CV should include. Your CV should include your education, appointments and positions, research interests, and your publications. Your education includes a list of degrees along with the institution and date received, as well as your advisor's name. Your appointments and positions, also known as employment, should include your job titles, the organization names and locations, as well as your start and end dates. Your research interests include primary and secondary interests, and you may choose to include MSC numbers. Your publication list consists of published papers; It may also include work on the arXiv and work in progress (more on this momentarily), just indicate this by including an arXiv identifier or the words "work in progress."

It is also helpful to employers if you include a list of courses that you were involved with teaching. Early in your career, you may have limited teaching experience, and it can be helpful to use descriptive language to specify your roles. For instance, "lead recitations and graded for Calculus II," "mentor for Business Calculus" and"fully responsible for teaching Linear Algebra."

Your CV should be easy to navigate, and one should be able to locate each of the basic items listed above within a few seconds. You can use formatting, bold text, and bulleted lists. Many mathematicians include a direct link to their CV on their professional websites, and so there are many samples out there that you can use to gain inspiration for formatting, organization, and content. Put yourself in the shoes of a member of the hiring committee; many of these folks look through hundreds of files. Their job is to figure out who you are as a mathematician, researcher, teacher, and member of the mathematical community, and your CV needs to make their job easier.

With the basics out of the way, there are other items that can boost your CV, including mentoring experience, conference organization and participation, upcoming and past talks, grants and awards received.

In fact, you may choose to list work-in-progress in your publication list to spotlight that you have work in the pipeline. I especially encourage this if your forthcoming work is in a new area. A postdoc that I know, for instance, is doing some fantastic research in a new area that is outside his immediate area of expertise. Working in this area required him to first learn the relevant background and took up a significant amount of time. Given that he had already given some talks on his results, I encouraged him to list the project in his publication list as forthcoming work so that it would be more visible to potential employers. In general, listing work-in-progress in a new area is a good way to communicate forthcoming work and may add depth to your research expertise.

Finally, put some positivity into your CV while creating it. You have made it this far, and your CV is a celebration of all your accomplishments.

—*Early Career editors*

Have a question that you think would fit into our Dear Early Career column? Submit it to `Taylor.2952@osu.edu` or `bjaye3@gatech.edu` with the subject Early Career.

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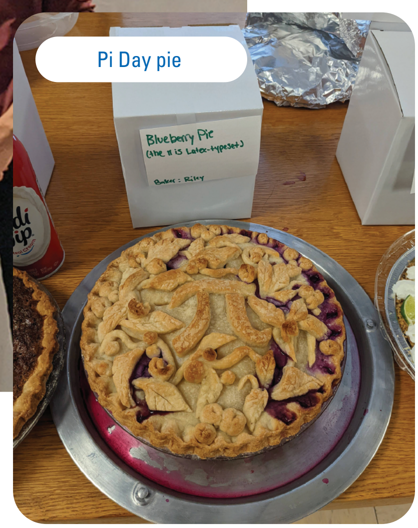
How do AMS Graduate Student Chapters support the mathematical community and beyond?

University of Minnesota chapter members



Pi Day pie

University of Minnesota

Chapter members run a variety of events throughout the academic year, including weekly teas, periodic social events, and a seminar series. Past events have included Pi Day pie and trivia competitions and the Unary Day, featuring Guest Speaker Max Engelstein, Assistant Professor.

Photos courtesy of their respective chapters.

AMS AMERICAN MATHEMATICAL SOCIETY
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Utah State University

The Utah State graduate students invite guest speakers throughout the academic year. One lecture, by Dr. Charles Torre, Professor and Associate Department Head of the Physics Department at Utah State, focused on general relativity and solutions to Einstein's equations.

Utah State University Chapter

Presentation at the Duke University Chapter

Duke University

This Chapter builds community among students by promoting conversation, collaboration, and friendship across research and institutional milieus. They will host the 9th annual Triangle Area Graduate Mathematics Conference (TAGMaC) in 2023, at which graduate students from Duke, University of North Carolina at Chapel Hill, and North Carolina State University give talks about research-level mathematics.

The Chapter Program makes meaningful contributions to the professional development of graduate students in the mathematical sciences and connects graduate students with AMS offerings.

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MEMORIAL TRIBUTE

What I Know for Sure

Peter Eley

"*You will not graduate until you learn to write better.*" These were Dr. Lee Stiff's words to me as I sat in his office reviewing my heavily edited dissertation manuscript. He went on to explain that a well-written dissertation could serve as the catalyst for my first publication. A decade after they were spoken, these words still ring in my memory, representing the turning point that led to two publications based on my dissertation. Dr. Stiff's challenge, issued to me and to many other students, produced excellence.

The late Dr. Lee V. Stiff was many things to many people in academia: dissertation advisor, academic father, hero, pioneer, precursor, humanitarian, and my personal favorite, teacher. He was a quarterback for mathematics justice and a coach for the soul. It was not uncommon to attend a capacity-filled presentation in which Dr. Stiff, with the passion of a Baptist preacher, would strike the iron of your soul while at the same time moving you to tears. It was part of his majesty and Southern charm that he spoke simultaneously to your head and your heart. His voice lives on through the students he advised, papers he authored, books he published, policies he challenged and changed, and lives that were enriched by his work.

The body of work Dr. Stiff created during his lifetime supported change and challenged inequities in mathematics and mathematics education. An example of his work is demonstrated in his 2016 National Council of Teachers of Mathematics (NCTM) Ignite talk entitled, "What You Know for Sure!" In this talk he outlines the barriers preventing access to rigorous mathematics that we are aware of yet fail to address and explains how students labeled "at-risk" are "at-risk" due to teachers' belief systems. The impact of his lifelong work on the field is evident in the current positions of the American Mathematical Society (AMS), Mathematical Association of America (MAA), and Conference Board of the Mathematical Sciences (CBMS) on diversity, equity, and inclusion in the mathematical sciences.

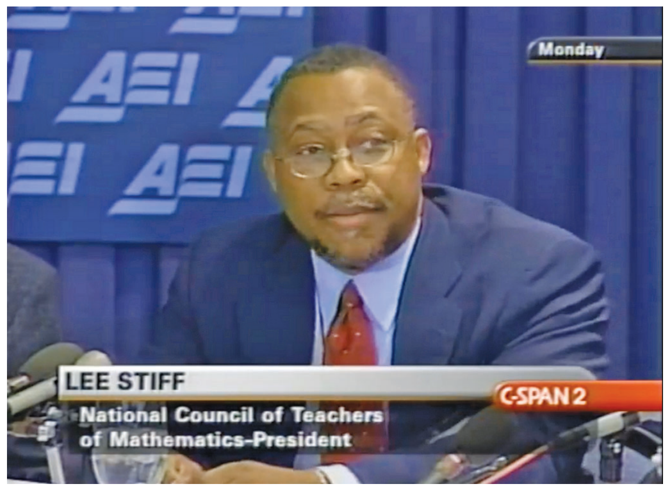

Figure 1. Dr. Stiff defending the NCTM and their recommendations for school mathematics.

Lee Vernon, as he was affectionately known to his family and friends, grew up in Murfreesboro, NC, where he attended Riverview Elementary School and graduated from C.S. Brown High School in 1967. During his high school years, Lee had a significant mentor in his life, Mr. Dudley Flood. Lee spent a lot of time with Mr. Flood, who in addition to serving as his mentor was his Boy Scout leader, mathematics teacher, and football coach. It was Mr. Flood who convinced Lee that excelling in mathematics would be his ticket to college.

During Lee's senior year of high school, the North Carolina public schools began desegregation. When students were given the option of moving to the white school, Lee chose to stay where he was because he was the star quarterback and, in his words, "a big man on campus." He stated that he was already getting a good education at his current school. While in high school, Lee met his future wife,

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Renee Flood; they were married for 39 years and had two daughters, Dr. Adrienne Stiff-Roberts (professor of electrical engineering, Duke University) and Morgan Stiff (movie producer).

After high school, Lee attended the University of North Carolina at Chapel Hill and earned a degree in mathematics in 1971. He continued his education at The Pennsylvania State University (1972) and Duke University (1974), earning Master of Arts degrees in mathematics from both institutions. In 1978, he completed his PhD in mathematics education at North Carolina State University (NCSU).

Dr. Stiff began his teaching career at Murfreesboro Senior High School in Hertford County, NC, and later taught at Daniels Junior High School in Raleigh, NC. In 1978 he accepted a position as assistant professor of mathematics education at the University of North Carolina at Charlotte, where he remained until 1983. In 1983, Dr. Stiff returned to NCSU as an assistant professor of mathematics education; he was promoted to professor in 1998.

In 2000, at a critical moment in mathematics education, Dr. Stiff was elected president of NCTM, a position he held from 2000 to 2002. After completing his tenure as president, he returned to his position as professor of mathematics education at NCSU. He concluded his 42-year career in higher education as Associate Dean for Faculty and Academic Affairs in the College of Education at NCSU. While at NCSU, he chaired 15 dissertations and served on numerous dissertation committees. He authored countless journal articles and over 40 textbooks. He was a sought-after keynote speaker and delivered numerous conference presentations.

Dr. Stiff was a dynamic speaker and influencer in the mathematics education community and his talks would fill rooms to capacity. He spoke with a quiet force that often moved his audience to tears. He was masterful in using voice inflection to emphasize essential elements of a speech and employing body language to command a crowd's attention.

Most importantly, Dr. Stiff was an advocate for students and teachers alike. His talks made you pause to wonder about the possibilities for more equitable mathematics education, in which all students could access the resources they needed to succeed. He understood the pain, the frustrations, and the disappointments of students and teachers, and he enabled his audiences to imagine how it felt to be in their place. He was keenly aware of how decisions made at the highest levels affected parents, students, and other key stakeholders.

In his "Spread the Word" President's Message published in the July/August 2000 issue of the *NCTM New Bulletin*, he wrote,

> Parents, school board members, community leaders, and others must understand that NCTM's vision of school mathematics recognizes that basic computational skills are important but are not enough. Students need to understand the underlying concepts of the mathematics they are taught and have the opportunity to go beyond rote learning in mathematics.

Dr. Stiff's advocacy work was situated in the context of the "math wars" that ensued from the proposed updating of mathematics education standards. In the 1989 document "Curriculum and Evaluation Standards for School Mathematics," NCTM created national standards to support the quality of mathematics education. This presentation of standards sparked the math wars, which pitted traditional mathematics education, with its emphasis on teaching algorithms and rote procedures, against reform mathematics, which advocated inquiry-based approaches to teaching.

In the wake of NCTM's 1989 standards, organizations outside mathematics also began issuing standards for their various disciplines and states. This led to the publication of the "Professional Standards for Teaching Mathematics" (1991) and "Assessment Standards for School Mathematics" (1995). Under Dr. Stiff's leadership, NCTM later published the "Principles and Standards for School Mathematics" (*PSSM*, 2000). The *PSSM* was the first set of standards to include equity as a guiding principle for all mathematics classrooms.

Dr. Stiff played a critical role in presenting these standards. He traveled the country educating mathematicians, mathematics educators, teachers, parents, and the public on the definition of a high-quality mathematics curriculum, instruction, and assessment and the importance of adopting these standards. In his "Multiple Paths to Success" President's Message in the January 2001 *NCTM New Bulletin*, he stated:

> *Principles and Standards* is a flexible and adaptable vision that works with and within local and state or provincial frameworks. This vision enables us to make wise choices—and one of the most important of them is the choice of curriculum materials.

The publication of the *PSSM* (2000) led to a debate on Capitol Hill in which Dr. Stiff defended NCTM's position, using the platform to convey his message of hope for the future and reinforce the importance of access to rigorous mathematics for all. In classic Lee Stiff fashion, he argued before a national television audience on behalf of mathematics education for all students, advocating new methods of teaching mathematics and improving math and science education (C-SPAN clip: starts at 43:00).

Dr. Stiff's leadership and influence are evident in his televised C-SPAN debate with Lynne Cheney, David Klein, Tom Loveless, Mike McKeown, and Gail Burrill in 2002. In this debate, Dr. Stiff argued that the NCTM standards supported greater math access for teachers and students and that rigorous mathematics should be accessible to all, not only a select few. He claimed that for students who were not "hardwired" for mathematics, skilled teachers could "put the wiring in." In other words, with correct implementation and good teaching, anyone can learn mathematics. I strongly encourage you to watch the C-SPAN debate.

Figure 2. Dr. Stiff and Dr. Peter Eley at North Carolina State University graduation ceremony.

For many years Dr. Stiff displayed his genius by writing textbooks. He wrote over 40 classroom textbooks in addition to over 100 other publications. Dr. Stiff received numerous awards, including the NCTM Lifetime Achievement Award (2019), the TODOS Iris M. Carl Equity and Leadership Award, and the Benjamin Banneker Association's Lifetime Achievement Award (2015).

During the 2000s, Dr. Stiff increasingly focused on advancing equity and access in mathematics, as illustrated by his work with Edstar Analytics, an organization that helps educational entities develop data-driven models for decision-making. Dr. Janet Johnson, Dr. Stiff 's first PhD student, cofounded the company, and Dr. Stiff served as president and cofounder. Along with Dr. Johnson and Dr. Patrick Akos, professor of counselor education at University of North Carolina at Chapel Hill, Dr. Stiff uncovered issues of tracking that prevented eligible students from accessing rigorous mathematics courses in Wake County, NC schools. These findings led to more questions and a critical review of data related to course-taking patterns in the school system.

As a result of their inquiry, the researchers discovered that the school district was not following its own long-standing policy on course enrollment. Many minority students could not access rigorous mathematics courses for which they were qualified and eligible to enroll. Dr. Stiff was a staunch advocate for giving students access to the highest and most rigorous curriculum for which they were prepared, subscribing to the theory that students meet the expectations adults have for them. Dr. Stiff tested this notion often and was repeatedly proven correct by numerous minority students.

For example, through Edstar Analytics he had the opportunity to examine students' course-taking patterns; armed with that data, he convinced local education agencies to enforce the policies they already had on the books. As a result, many academically qualified students gained access to rigorous mathematics courses for the first time, helping to narrow the achievement gap. Interestingly, one student for whom Dr. Stiff championed access to rigorous mathematics courses later graduated high school as the class valedictorian. These kinds of accomplishments made Dr. Stiff special as an educator, advocate, and mentor.

In his role with Edstar Analytics, Dr. Stiff was once speaking to a group of high school math teachers about the overlooked high-achieving minority students who were not being recommended for advanced math courses. Edstar Analytics had helped the teachers review the data to identify these students. When the math teachers argued that although high achieving, these students could not succeed in advanced math due to a lack of resources and support, Dr. Stiff offered to teach the course himself if they would enroll the students. The school accepted his offer and enrolled the students, and he taught the course. The students were successful. On another occasion, when Dr. Stiff was demonstrating to elementary school teachers how to teach mathematical concepts, the teachers complained that their students could never understand the concepts. Dr. Stiff volunteered to teach their classes and model good mathematics teaching; once he did this, the teachers were convinced.

Working with a program that promoted equity in education, Edstar Analytics helped a school district identify sixth-grade, low-income minority students who were predicted to be successful in advanced mathematics, so they could be placed on that path. These students were enrolled in an elective course designed to support them in switching to the rigorous math track. When school administrators saw how many Black males were in the course, they assigned a special education teacher who had no math training to teach it, and could not be convinced to do otherwise. Drs. Stiff and Johnson went to the school for a year and taught this course. The following year, these students were identified as academically gifted and an appropriate teacher was assigned to the program. These are just a few examples of how Dr. Stiff went far beyond simply preaching the word.

When Dr. Stiff was not championing equity and access, during his personal time at home he had an affinity for watching British television mysteries. He had considerable creative talent as a songwriter and garnered movie credits for small-budget film soundtracks. He was the executive producer of the movie *Mississippi Damned* and wrote the lyrics and sang the history in *History of Mathematics, The Musical*. His very entertaining renditions illustrate the

Figure 3. Dr. Stiff and Dr. Janet Johnson. Janet Johnson was Dr. Stiff's first doctoral student at North Carolina State University.

cultural diversity of those who have contributed to mathematics.

As a seasoned faculty member in the NCSU College of Education, Dr. Stiff was frequently sought out by school districts for his advice and tremendous insight. He often served as a voice of reason, imparting wisdom to younger faculty of color and women faculty that prevented them from doing or saying things they might later regret. He avidly championed and promoted the work of women in math. He was highly respected by his colleagues, peers, and students throughout the mathematics and mathematics education communities. In his last years, Dr. Stiff accepted the challenging administrative role of Associate Dean for Faculty and Academic Affairs in the College of Education at NCSU. In this role, he actively participated in dissertation committees and taught courses while working on faculty issues for the college.

The passion and enthusiasm Dr. Stiff brought to education was charismatic and unique. He embodied a strength that endured in tough times and an integrity that allowed him to forge new paths rather than following the crowd. His infectious smile and passion for providing opportunities for others, no matter who they were, left a lasting impression on the mathematics and mathematics education communities. His life and work remind us how critical it is for those who are outstanding among us to serve others. In his words: "Those who can, do; those who care, teach!"

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Peter Eley

Credits

Figure 1, Figure 2, and author photo are courtesy of Peter Eley. Figure 3 is courtesy of NCTM.

The Legacy of Evelyn Boyd Granville (1924–2023)

Johnny L. Houston

Figure 1.

An internationally recognized mathematician, computer scientist, scientist / engineer, scholar, educator, mentor, and pioneer / barrier breaker has departed. Dr. Evelyn Boyd Granville is the *second* known Black woman[1] to earn a PhD in mathematics. She died peacefully on June 27, 2023 in her Silver Spring, Maryland, home at the age of 99. She performed pioneering work in **academia** [Smith College, Yale University, New York University, Fisk University, California State University at Los Angeles, University of Texas at Tyler, and Texas College], in **government** [National Bureau of Standards (NBS)/Diamond Ordnance Fuze Laboratories/National Aeronautics and Space Administration (NASA)], and in **industry** [International Business Machines (IBM)/US Space Technology Laboratory (USSTL)/North America Aviation (NAA)].

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[1]*Euphemia Lofton Haynes, Catholic University, 1943; Evelyn Boyd Granville, Yale University, 1949; Marjorie Lee Browne, University of Michigan, 1950; are the three first known Black women to earn a PhD in mathematics.*

Early Years

Figure 2.

Evelyn Boyd was born on May 1, 1924 in Washington DC. She was the second daughter of William and Julia Walker Boyd. After her parents separated when she was young, she and her sister Doris were raised by her mother and their mother's twin sister Louise. Both her mother and Aunt Louise worked as examiners for the US Bureau of Engraving and Printing. Evelyn and Doris often spent portions of the summer at the farm of a family friend in Linden, Virginia. Their mother and Aunt Louise came from a large family in Orange County, Virginia. However, the family was small in DC. Doris had one son, Kurt Barnes, and Evelyn had no children. During most of Evelyn's childhood, the family lived on 2336 Ontario Rd, the homestead from which Evelyn recalls walking to elementary school, junior high and high school. Because of segregation, there were only three high schools for Blacks in DC at the time (Dunbar—academic, Armstrong—tech, and Cardoza—business). Evelyn attended Dunbar and was one of five valedictorians for the class of 1941.

After Dunbar, Attending College

Seeking an opportunity to go to college, Evelyn applied to the women's colleges of Smith and Mount Holyoke, both in Massachusetts. She was accepted at both colleges, but she did not receive a scholarship from either one. Evelyn preferred Smith based on her having met a Black professional lady in DC that she admired who attended Smith. Since her family wanted her to attend a reputable college in the north, they decided to cover Smith's annual cost of $1100 for tuition, room and board for year one. Her mother gave Evelyn $500 and Aunt Louise also gave her $500. Moreover, a professional organization of Black educators, Phi Delta Kappa, gave her a $400 scholarship, which was $100 each year for four years. Beginning with her second year, Smith gave Evelyn a partial scholarship annually and she worked part-time at Smith. With support from her mother and Aunt Louise, and her summer work at the National Bureau of Standards, Evelyn had a comfortable and successful college career. Evelyn entered Smith in the fall of 1941 with the intention of becoming a French teacher, but mathematics, physics, and astronomy drew her away from "uninteresting" French literature. She was elected to Phi Beta Kappa and graduated summa cum laude in 1945 with a bachelor's degree in mathematics and physics.

After Smith, Earning a PhD in Mathematics from Yale/Teaching at NYU and Fisk

Figure 3. L–R: Dr. Vivienne Malone Mayes, Dr. Boyd, and Dr. Etta Falconer.

Evelyn won awards for graduate study (she applied to and was accepted at Yale University and the University of Michigan). These included a Smith Fellowship, a Julius Rosenwald Fellowship, and an Atomic Energy Fellowship. She entered Yale in the fall of 1945, earning a Master's of Arts degree in 1946 (math) and a PhD degree (math) in 1949 under the renowned Functional Analyst, Einar Hille. She was elected to the scientific honorary society, Sigma Xi, and her dissertation was entitled: "On Laguerre Series in the Complex Domain." After earning her PhD, Dr. Boyd did a one-year post-doctorate at New York University Courant Institute for Mathematics, performing research and teaching. From 1950 to 1952, Dr. Boyd was an associate professor at Fisk University, a historically Black college (HBCU) in Nashville, Tennessee. At Fisk, Dr. Lee Lorch was chair of mathematics and Vivienne Malone Mayes and Etta Zuber Falconer were two of Dr. Boyd's students who later earned a PhD degree in mathematics.

Careers Outside of Academia in the 1950s

Figure 4. Dr. Boyd showing a young man, how they programmed the orbits of space vehicles.

In 1952 Dr. Boyd was recruited to become an applied mathematician at the *National Bureau of Standards* (NBS) in Washington, DC, where she worked on missile fuses. Her division of NBS was later absorbed by the United States Army and became the Diamond Ordnance Fuze Laboratories. There she became interested in the new field of computer programming. Dr. Boyd Wrote: "This work entailed consulting with Ordnance engineers and scientists on the mathematical analysis of problems related to the development of missile fuses. . . I met several mathematicians who were employed. . . as computer programmers."

At that time the development of electronic computers was in its infancy. Dr. Boyd recalled:

> "The application of computers to scientific studies interested me very much, which led to my giving serious consideration to an offer of employment from International Business Machines Corporation."

In 1956, another door opened for Dr. Boyd. She was recruited by the corporation, *International Business Machines* (IBM), where she was immediately seated before a 650 Magnetic Drum Data-Processing Machine and asked to do computer programming. At first, she worked in Washington, writing programs in the assembly language SOAP and later in the programing language FORTRAN for the IBM 650, which was the first computer intended for use in businesses, and for the IBM 704. In 1957 she moved to New

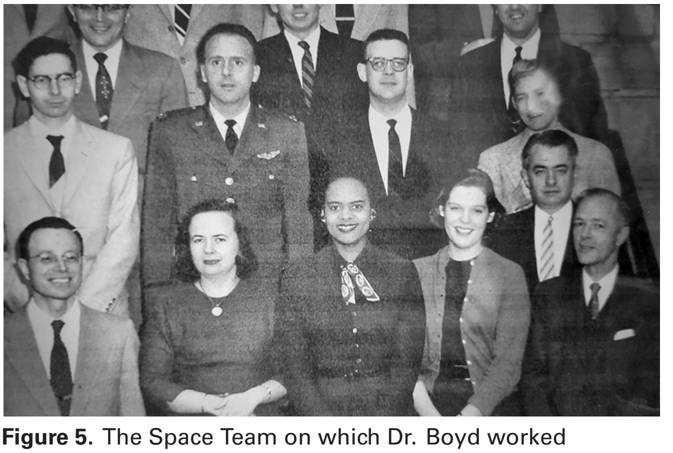

Figure 5. The Space Team on which Dr. Boyd worked consisted mostly of white men; she was always the only Black.

Figure 6. The *Vanguard* satellite and the manned *Mercury* spaceship, the first two vehicles for which Dr. Boyd helped to write computer programs to put in orbit around the Earth.

York City to take up a post as a consultant on numerical analysis at the New York City *Data Processing Center of the Service Bureau Corporation*, which was part of IBM. When the United States space program began to move rapidly forward, the *National Aeronautics and Space Administration* (NASA) contracted IBM to write software for them. She was then assigned to join IBM's Vanguard Computing Center in Washington, DC, where she wrote computer programs that tracked orbits for the unmanned Vanguard satellite and the manned Mercury spacecraft. Dr. Boyd was happy to return to Washington DC to be on a team of IBM mathematicians for this assignment. Dr. Boyd stated:

> **"I can say without a doubt that this was the most interesting job of my lifetime—to be a member of a group responsible for writing computer programs to track the paths of vehicles in space."**

It is noteworthy that Dr. Boyd studied astronomy at Smith, earned degrees in mathematics and physics there, and became an expert in computer science before it became a recognized discipline.

Figure 7. Dr. Evelyn Boyd Granville at her computer/printer.

Careers Outside of Academia in the 1960s

Dr. Boyd left IBM in 1960 to move to Los Angeles, where she participated in research studies on methods of orbit computation with the *Computation and Data Reduction Center* in the aerospace firm: *United States Space Technology Laboratories*. She did further work there on satellite orbits. In 1962, she joined the aerospace firm in Los Angeles, California named, *North American Aviation* (NAA). With NAA, she became a research specialist in celestial mechanics, trajectory, and orbit computation, and developed digital computer techniques for engineers working on the Apollo project. Dr. Boyd returned to IBM in 1963 and joined its *Federal Systems Division* in Los Angeles as a senior mathematician. In 1967, IBM was reorganizing and asked her to relocate to their division in Northern California; however, she declined. She decided that she wanted to remain in Southern California.

In 1960, Dr. Boyd married her first husband, Rev. Gamaliel Mansfield Collins. One of her great memories from that marriage was her husband's close affiliation with the Rev. Dr. Martin Luther King, Jr. This affiliation caused her and her nephew Kurt Barnes to have premier seats on the steps of the Lincoln Monument when Dr. King gave his "I Have a Dream Speech" during the March on Washington (August 28, 1963). In several of her presentations/speeches, she mentioned the impact of this experience on her life. This marriage ended in 1967. Before we chronologically follow her latter years in teaching, education, and mentoring, the author wishes to share a second human interest story that occurred during her outside of academic careers in the 1950s–1960s.

John Glenn, Dr. Katherine G. Johnson, and Dr. Evelyn Boyd Granville All Agreed

Figure 8. Dr. Katherine G. Johnson with her calculator and globe.

On October 4, 1957, a Soviet R-7 intercontinental ballistic missile launched *Sputnik* (Russian for "traveler"). This was the world's first artificial satellite and first manmade object to be placed into the Earth's orbit. *Sputnik*'s launch came as a surprise. In April 1961, the Soviet cosmonaut Yuri Gagarin became the first person to orbit earth, traveling in the capsule-like spacecraft Vostok 1. For the US effort to send a man into space, Project Mercury was established.

Created in 1958, NASA engineers designed a smaller, cone-shaped capsule far lighter than Vostok. They tested the craft with chimpanzees and held a final test flight in March 1961. This was before the Soviets were able to pull ahead with Gagarin's launch. On May 5, 1961, astronaut Alan Shepard became the first American in space, though not in orbit. Later that month on May 25, 1961, **President John F. Kennedy made the bold, public prediction, before a Special Session of Congress: "the US will land a man on the moon before the end of the decade." In February 1962, John Glenn became the first American to orbit the Earth.** By the end of that year, the foundations of NASA's lunar landing program had been established with Project Apollo. Before agreeing to be the first American astronaut to orbit the earth, John Glenn insisted on reviewing the calculations for orbit trajectories calculated by computers on Dr. Boyd's team and other teams. He then insisted that the "Human Computer," Katherine G. Johnson (as recounted in the book and movie: *Hidden Figures*), validate by hand the orbit trajectory results that had been calculated by the computer teams. Johnson validated the results that the computer calculations had produced. John Glenn then agreed to fly around the earth. These two African American mathematicians played fundamental roles in the early successes for both the development of the American Space Age and in NASA honoring the bold prediction of placing a man on the moon in the 1960s, as President Kennedy stated in 1961. On July 20, 1969 American astronauts Neil Armstrong (1930–2012) and Edwin "Buzz" Aldrin (1930–) became the first humans on the moon.

These Two Pioneers Never Met or Communicated, but the World Knows Both

Figure 9. Mr. Edward V. Granville and Dr. Boyd.

Although the pioneering contributions of these two Black mathematicians were not recognized by the public and by historians in earlier times, they have become clearly recognized in recent years. It is interesting to note that these two mathematicians never met or communicated; however, both lived beyond 99 years, and both recently, passed (Johnson: August 26, 1918–February 24, 2020; Granville: May 1, 1924–June 27, 2023). When the book and movie: *Hidden Figures* were shared with the world in 2017, they were no longer hidden from the public. Both of these ladies lived long enough to be able to go to the theater to see the movie and to recognize some of the notice that they were receiving. At its 50th Anniversary Celebration in Baltimore (January 18, 2019), the National Association of Mathematicians (NAM), recognized both mathematicians: Dr. Johnson with a Centenarian Award and Dr. Granville with a 50th Anniversary Legacy Award. The author last saw and spoke with Dr. Granville in person at this event when both of these pioneering mathematicians were still alive.

Dr. Granville Was the Exception, But She Knew About Segregation/Racism/Sexism First Hand

At North American Aviation and IBM, she was part of divisions aiding the Apollo missions and also providing technical support to engineers working on moon landing calculations years ahead of the first steps on the lunar surface in 1969. In the year 2000, in Texas she told the *Tyler Morning Telegraph*: "There was such a need for talent, that companies stopped looking at race and gender."

That was true, up to a point. She just happened to be part of a small cadre of women involved in the space program. Moreover, Dr. Granville had exceptional skills in mathematics, physics, and computer programming. Often times she was the only woman *and* the only Black person on her team. But Dr. Granville also knew segregation and sexism firsthand. She often lamented that women and minorities remained significantly underrepresented in mathematics and the sciences. She spent her childhood in Washington, DC, and did all her education before college in the segregated education system of Washington, DC. As she wrote in 1989 for the scholarly journal *Sage*: "We accepted education as the means to rise above the limitations that a prejudiced society endeavored to place upon us."

Return to Academia

Figure 10.

After nearly two decades in the private sector and government, Dr. Boyd decided to return to academia. Although she thoroughly enjoyed her two years at Fisk, it should be noted that at the end of her post doc year, Dr. Boyd applied to white institutions with PhD programs in mathematics who had advertised entry faculty openings in their mathematics department. However, none of them offered her an interview, even though she had a PhD in mathematics from Yale, impeccable credentials, and her advisor, Prof. Einar

Hille, had been a former president of the American Mathematical Society (AMS). Dr. Boyd once wrote:

"Early in life, I saw black women, well-dressed women—teaching school, and I wanted to be a teacher. . . "

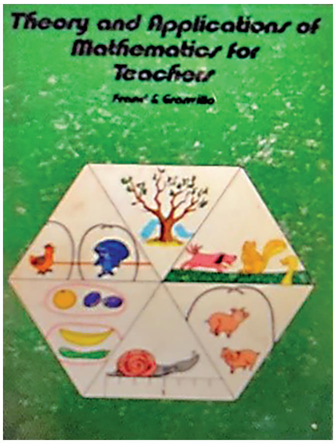

Figure 11. First Edition of Dr. Granville's book.

In 1967, she applied for a faculty position in the mathematics department at California State University, LA. She was hired and rose to the rank of a full professor before retiring in 1984. Her pay in academia was at least a 50 percent cut from what her pay had been in the private sector. In 1970, she married her second husband, Edward V. Granville, a realtor in LA. After this marriage, she used the name, *Evelyn Boyd Granville*.

At California State University in Los Angeles, she taught several courses in mathematics and computer science, including numerical analysis and a course required for all persons aspiring to become elementary school teachers. The latter course caused her to develop a real interest in mathematics education for the training of elementary school teachers. This interest in mathematical education led to her involvement with the *Miller Mathematics Improvement Program* and as part of this program she taught mathematics for two hours each day at an elementary school in Los Angeles during the 1968–1969 academic year. Out of this experience came her joint publication with a colleague at CSU, LA, Jason Frand. The title of the book is *Theory and Applications of Mathematics for Teachers* (1975). The book was well received and adopted by 50 or more institutions of higher learning. Three years later a second edition was published but fashions changed in the teaching of mathematics and soon after the second edition ceased to be as relevant as the first edition. She wrote:

> *My husband was born and raised in East Texas and had planned to return to the area when he retired from his business. I often accompanied him on visits to Texas and, after making several trips, I was convinced that a move to a rural setting in East Texas would be a welcome change from the Los Angeles metropolis. We found an ideal setting near Tyler, Texas.*

Figure 12.

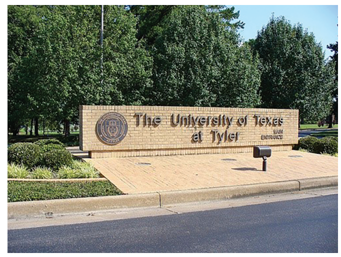

Figure 13.

In 1984, she and Edward V. Granville left Los Angeles to live on this rural 16-acre plot, and a four-acre lake where they shared a lovely, quiet home. However, Dr. Granville began teaching again, first at the local public schools in Tyler, and then at Texas College, a HBCU in Tyler, TX, beginning in 1985. She retired in 1988. At Texas College, she taught a newly instigated computer science course. Two years later (1990), she was recommended and invited to teach at the University of Texas in Tyler, where she was the Sam A. Lindsey Professor of mathematics, developed elementary school math enrichment programs, and retired in 1997. In addition to Dr. Granville's teaching, while on their 16 acres oasis in Texas for over 20 years, the Granville's rode their horses, raised chickens, sold eggs, and even sold the catfish they caught from the lake. After her husband Edward died in 2008, she returned to Washington, DC, in 2010, which was to be her final retirement.

Who Was the First Black Woman to Earn a PhD in Mathematics?

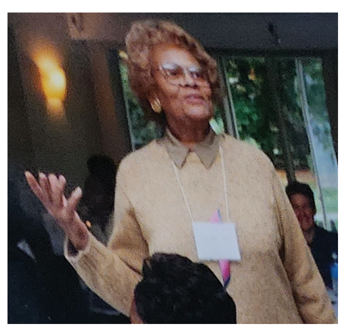

Figure 14. Dr. Granville speaking in Texas.

The author went to Texas to visit Dr. Granville to discuss with her a 50-year-old error that had been printed in the literature that Dr. Evelyn Boyd Granville was the first Black woman who earned a PhD degree in mathematics; receiving her degree from Yale University in 1949 and that Dr. Marjorie Lee Browne was the second Black woman to receive a PhD in mathematics from the University of Michigan in 1950. In 1999 the author and others learned that the first Black woman to receive a PhD in mathematics was Euphemia Lofton Haynes who received a PhD in mathematics from Catholic University in 1943. The author and others were about to put this correction in the literature. However, the author and NAM decided that this situation needed to be discussed with Dr.

Granville in person before anything was printed. It was a difficult and embarrassing conversation that had to occur in person. Even today, I do not know how so many of us articulated and printed that error for 50 years (1949–1999). In examining the official records at Catholic University and in discussing them with persons who could verify what the records stated, we had to accept as being accurate that Euphemia Lofton Haynes was the first known Black woman to receive a PhD in mathematics. The literature has now stated, for the past two decades, that the first three known Black women to earn a PhD in mathematics are

1. Euphemia Lofton Haynes, Catholic University, 1943;
2. Dr. Evelyn Boyd Granville, Yale University, 1949;
3. Marjorie Lee Browne, University of Michigan, 1950.

Figure 15. The Granville's in rural Texas.

Figure 16. Johnny Houston visits the Granville's in Texas.

Both Haynes and Granville were born and grew up in Washington, DC; both Haynes and Granville attended Smith College; both earned baccalaureate degrees in mathematics, from Smith, Haynes (1914) and Granville (1945). Haynes earned a MA degree from the University of Chicago. Neither of the first three Black women mathematicians had children. All three taught at an HBCU for some period of time. The work and archives of Dr. Haynes are housed at Catholic University. The work and archives of Dr. Granville are housed with Smith College and NAM. And the work and archives of Dr. Browne are housed at the University of Michigan and North Carolina Central University. This 50-year error or disconnect between when Haynes received her PhD degree in mathematics and when Granville received her PhD in mathematics caused the author to do several years of in-depth research to learn, as accurately as feasible, who were the first 100 Black women known to earn a PhD degree in mathematics by year. Houston is now finishing an article (with verification) of the first 100 known Black women (in less than 20 pages) who earned a PhD in mathematics. Thus, the article being written is titled "**The First 100 Black Women Known to Have Earned a PhD Degree in Mathematics, J. L. Houston List.**"

Figure 17.

Dr. Evelyn Boyd Granville has served many distinguished boards, panels, public service and professional organizations, and advisory committees before / during / after her retirements, including the Mathematical Association of America (MAA), Association of Women in Mathematics (AWM), the National Association of Mathematicians (NAM), the National Council of Teachers of Mathematics (NCTM), the American Association of University Women (ASUW), and even on the Psychology Examining Committee of the Board of Medical Examiners of the State of CA (appointed by Gov. Brown). Dr. Granville has received numerous awards and recognitions. ***The Ada Project***, originally developed at Yale University, is designed to serve as a clearing house for information and resources related to women and computing. Given its aim and authority, it is noteworthy that **the project lists twelve women as "*pioneering women of computing.*"** They are listed in this order:

1. Ada Byron King, Countess of Lovelace (1815–1852)
2. Edith Clarke (1883–1959)
3. Rosa Peter (1905–1977)
4. Grace Murray Hopper (1906–1992)
5. Alexandra Illmer Forsythe (1918–1980)
6. ***Evelyn Boyd Granville* (1924–2023)**
7. Margaret R. Fox
8. Erna Schneider Hoover
9. *Kay McNulty Mauchly Antonelli*
10. Alice Burks
11. Adele Goldstine
12. Joan Margaret Winters.

Thus, she has the distinct honor of being recognized as the first African American computer scientist, and the first African American woman computer scientist.

Dr. Granville received honorary doctorates degrees from:

1. Smith College (1989), the first African American to receive such a degree from Smith
2. Lincoln University (1999)
3. Yale University (2001)
4. Spelman College (2006)

Figure 18.

Awards she received include:

1. Wilbur Lucius Cross Medal, Yale Graduate School Alumni Association's highest honor (2000);
2. Featured in a *Yale Alumni Magazine* cover story about 150 Years of Women at Yale;
3. National Academy of Engineering honoree;
4. Inducted into the National Academy of Sciences Portrait Collection of African Americans in Science;
5. New York Outstanding Scientist Award at the same occasion where Jackie Robinson received the Man of the Year Award;
6. Dow Chemical selected her to be a Legendary Ambassador of Math for several states (2000);
7. NAM Lifetime Achievement Award (1996); and
8. NAM Golden Anniversary Legacy Award (2019).

NAM has a Lecture Series named in her honor at NAM's Annual Meeting at the JMM**: the Haynes-Granville-Browne Presentations by Recent Doctoral Recipients.** She received her 2019 award at NAM's 50th Anniversary Celebration. And she was a frequently invited honoree and/or speaker for many different occasions.

The author considered listing many of the different persons who came to know her personally and professionally and were highly impacted by her. However, many of these have passed. He does know that there are scores of persons alive today that she impressed, inspired and who view her as one of their greatest role models. Many of these are mathematicians; especially women and American minority mathematicians. In the references listed at the end of this article, one can find many more details about this amazing pioneer.

Dr. Granville shared many interesting ideas, some are listed below:

"... growing up in the thirties in Washington, DC, I was aware that segregation placed many limitations on Negroes,...However, daily one came in contact with Negroes who had made a place for themselves in society; we heard about and read about individuals whose achievements were contributing to the good of all people. These individuals, men and women, served as our role models; we looked up to them and we set our goals to be like them."

"Two of my favorite teachers at Dunbar taught me math: Ms. Mary Crumwell and Mr. Bassett."

"Among my teachers at Smith College, Mr. Neal McCoy was particularly supportive of women mathematicians, perhaps in part because his own sister was a mathematician."

"I was fascinated by the study of astronomy and at one point I toyed with the idea of switching my major to this subject. If I had known then that in the not-to-distant future the United States would launch its space program, and astronomers would be in great demand in the planning of space missions, I might have become an astronomer instead of a mathematician."

Dr. Granville also gave her views on the current problems of teaching mathematics in American schools in a lecture at Yale University:

"I believe that math is in grave danger of joining Latin, Greek, and other subjects, which were once deemed essential but are now, at least in America, regarded as relics of an obsolete, intellectual tradition."

"... (Math) must not be taught as a series of disconnected, meaningless technical procedures from dull and empty textbooks."

"We teach as if there is only one way to solve a problem, but we should let children explore various techniques... But we're not training teachers to provide this new approach."

"... children end up crippled in mathematics at an early age. Then, when they get to the college level, they are unable to handle college classes. It's tragic because almost every academic area requires some exposure to mathematics."

"Make children learn how to add, subtract, multiply, and divide, and they won't need calculators. How do you teach the beauty of mathematics, how do teach them to ... solve problems, to acquaint them with various strategies of problem solving so that they can take these skills into any level of mathematics? That's the dilemma we face."

Throughout her life, she often smiled when she heard anyone say "women can't do math." In an interview with Lorentz Hall in 1994, **Dr. Granville was asked to summarize her major accomplishments. She said: "First of all, showing that women can do mathematics." She then said, "Being an African American woman, letting people know that we have brains too."**

We have lost a ***Giant*** in our communities of mathematical scientists, scholars, teachers, mentors, and advocates. She was a ***True Pioneer*** in many unchartered terrains and

Figure 19.

she inspired many to reach their highest heights. Those who knew of her, or who knew her, will always remember the footsteps that she left in the sands of time and her impact on helping America to build the most sophisticated and successful space program on this earth, and even helping America to get to the moon. She was a scholar, great teacher, great mathematician, great computer scientist, and great scientist, as well as a great inspiration and mentor for so many. The author feels very honored to have known her personally and professionally.

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Johnny L. Houston

Credits

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Queer of Color Justice in Undergraduate Mathematics Education

Luis A. Leyva

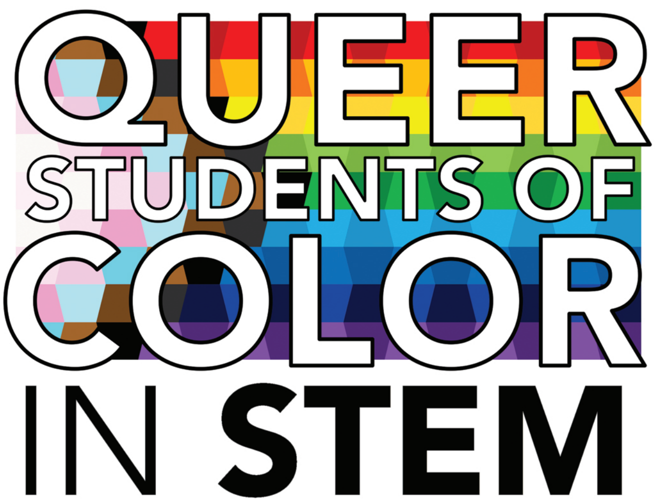

Figure 1. Project logo for the "Queer Students of Color in STEM" study.

Black and Latin*[1] students report racial oppression in undergraduate mathematics classrooms, including isolation, limited opportunities to participate, and underestimated ability due to stereotypes [1, 2, 3]. Intersectional analyses have shown how cisgender women of color navigate racialized and masculinized constructions of mathematical ability, such as Latina women pushing themselves to earn high grades for disproving male classmates' views of them becoming young mothers rather than future scientists [4]. Queer and trans* (QT) students also experience undergraduate mathematics classrooms as exclusionary because their gender and sexual identities are deemed irrelevant or unwelcome in these 'neutral' environments [5, 6].

In response to oppressive realities in mathematics classrooms, students with minoritized racial, gender, and sexual identities demonstrate agency and empowerment, such as building peer networks of support and expressing their identities in performative ways (e.g., avoiding use of slang, passing as straight) [2, 4, 5]. However, research has yet to center intersectionality[2] of mathematics classroom experiences among QT students of color [8, 9]. Filling this research gap is important to generate nuanced insights on features of undergraduate classrooms that promote racial and QT justice for multiply-marginalized populations. My research about the experiences of undergraduate QT students of color in STEM majors adds to this area of needed research.

The present article proposes a guiding vision for undergraduate mathematics pedagogy to promote identity-affirming and equitable learning opportunities among QT students of color. Such a vision, therefore, contributes to developing mathematics classrooms as spaces that advance *queer of color justice* (justice for QT people of color).

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[1] *The asterisk in Latin* considers fluidity in gender identities across the Latin American diaspora. The term, Latin*, responds to (mis)use of Latinx as a gender-neutral term originally intended for explicit inclusion among gender nonconforming peoples of Latin American origin and descent.*

[2] *Intersectionality refers to unique forms of agency and oppression that historically marginalized people experience at the juncture of multiple systems of oppression, including racism and cisheteropatriarchy (a system of oppression that marginalizes QT people and cisgender women by upholding misogyny as well as reinforcing heterosexual and cisgender identities as normative)* [7].

In what follows, I first lay out a conceptual foundation for my vision by synthesizing prior theorizing of inclusive pedagogies for racially minoritized and QT populations. My pedagogical vision encompasses practices of curricular design, instruction, and student support. Next, I engage findings from my research about the intersectionality of Latin* QT students' experiences as STEM majors, which were presented during my Spectra Lavender Lecture at the 2023 Joint Mathematics Meetings, to illustrate how mathematics pedagogy limited and expanded opportunities for queer of color justice. I argue that such justice is contingent on pedagogy that affirms students' identities as QT people of color and disrupts white cisheteropatriarchy (systemic oppression at the juncture of racism, white supremacy, and cisheteropatriarchy). I conclude the article with implications for practice in undergraduate mathematics.

A Pedagogical Vision for QT Students of Color in Undergraduate Mathematics

Two concepts of inclusive pedagogies in and out of mathematics education provide a foundation for my proposed vision. The first concept is the *liberated mathematics classroom* for justice among Black and Latin* learners [10], and the second concept is *anti-oppressive education* for promoting justice among QT students of color in education [11]. In what follows, I first present each of these concepts. I then apply the anti-oppressive education perspective as an intersectional lens to extend the concept of a liberated mathematics classroom for queer of color justice. In doing so, I propose a set of pedagogical approaches to meet the unique needs among QT students of color as mathematics learners.

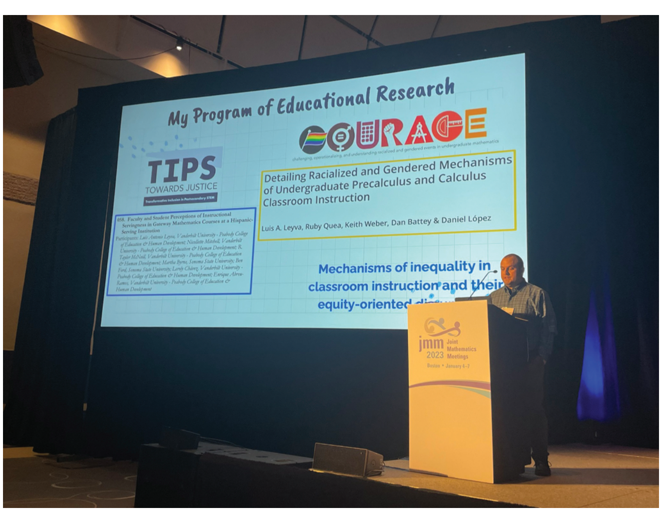

Figure 2. Photo from the 2023 Spectra Lavender Lecture.

The liberated mathematics classroom and disrupting whiteness. Brittany Mosby outlined a vision of liberatory mathematics education for Black and Latin* learners [10, 12]. This vision posits that mathematics learning as a liberatory experience is disruptive of the status quo, collaborative, discovery-oriented, and humanizing for racially minoritized populations. There are six elements of a liberated mathematics classroom: (i) affirming students' cultural knowledge and mathematical intuition; (ii) highlighting the utility of mathematics as a language to understand problems across multiple fields; (iii) encouraging metacognition and agency in the learning process; (iv) engaging students as active cocreators of knowledge and not relying solely on lecture-based instruction; (v) balancing rote, single-skill practice with student discovery and complex, contextualized multistep problems, and (vi) decolonizing content by decentering whiteness, maleness, and Europeanness as well as exposing learners to non-western foundations of the discipline. As summarized in Figure 3, liberatory mathematics classrooms allow students to experience the process of learning mathematics as active, culturally-affirming, freeing of oppressive realities, interdisciplinary, and relevant to social justice efforts.

Mosby's vision interrogates and disrupts dominant forces of whiteness in mathematics education that I have identified elsewhere [13]. The classroom element of affirming students' cultural backgrounds resist ideologies of whiteness that frame mathematics as a socially neutral and cultureless space. Black and Latin* students are relieved of the undue labor of aligning their participation (e.g., behaviors, language use) with white ideals. As a result, their contributions are more readily acknowledged as mathematical and thus have increased opportunities to build positive identities with the discipline. The classroom element of agency in students' learning, including active roles in coconstructing knowledge, disrupts racialized distribution of mathematical authority between educators and students as well as among students in classrooms. Such agency, coupled with another classroom element of decolonizing curricula, empowers Black and Latin* students to critique whose knowledge is valued in mathematics and the racially oppressive purposes that the discipline has served (e.g., wealth disparities). Thus, students are equipped with tools to interrogate whiteness in society, which contributes to solving social problems explored in other fields (e.g., ethnic studies, sociology) that are traditionally left unexamined in mathematics education.

Anti-oppressive education and advancing justice for QT students of color. In the book *Troubling Intersections of Race and Sexuality*, Kevin Kumashiro introduced the perspective of anti-oppressive education that challenges racism and cisheteropatriarchy at the intersections, thus providing

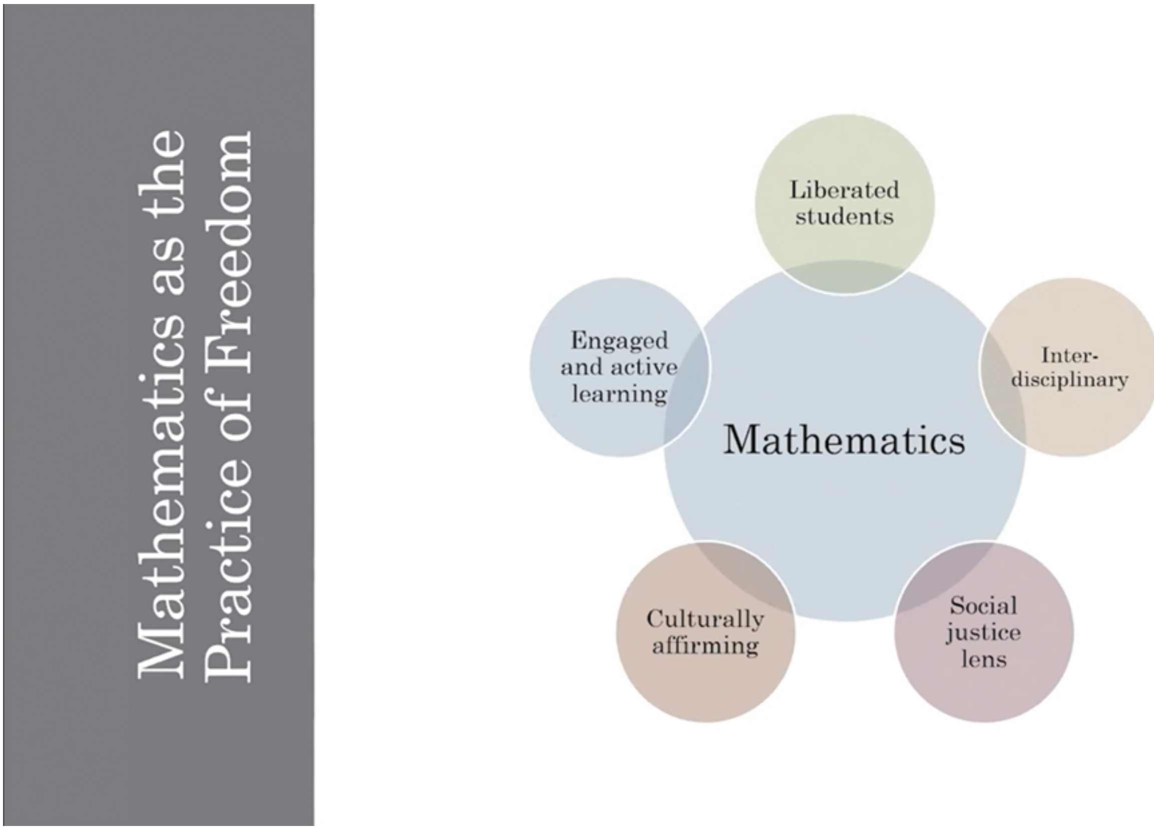

Figure 3. A slide from Brittany Mosby's plenary at the MSRI Workshop on Math & Racial Justice (2021).

identity-affirming opportunities to learn for QT students of color [11]. This perspective redresses cisheteronormativity in antiracist pedagogies that overlook QT identities, in addition to whiteness embedded in anti-heterosexist pedagogies that decenter QT students of color. Therefore, anti-oppressive education (also referred to as antiracist, anti-heterosexist education) is intersectional by design. There are four pedagogical approaches to anti-oppressive education. The "education for the Other" approach develops spaces of learning where QT students of color experience affirmation and safety for their full identities. Another approach, "education about the Other," resists partial understandings and stereotypical views about QT people of color, including use of curricular content by and about QT people of color that captures variation in issues of intersectionality.

For the "education that is critical of privileging and Othering" approach, educators increase learners' awareness of social issues that oppress QT people of color to generate critiques of systems and structures entrenched in white, cisheteropatriarchal ideologies. This approach also creates space for learners to interrogate how they are complicit in reinforcing systemic oppression and empowers them to transform structures for queer of color justice. The "education that changes students and society" approach encompasses educators entering into self-reflective, repetitive practices that disrupt resistance to traditional ways of knowing and being in the world, thus challenging complicity with queer of color oppression. Kumashiro argues that this approach invites educators into a "crisis" that troubles assumptions about identities, ways of engaging with oppression, and efforts for justice-oriented change. Anti-oppressive educators support students to work through such a "crisis" for themselves via pedagogy that advances justice for QT people of color.

A vision for queer of color justice in undergraduate mathematics classrooms. With the anti-oppressive education perspective extending the race-specific concept of liberated mathematics classrooms to consider intersectionality, I propose three approaches for my vision in advancing queer of color justice through undergraduate mathematics pedagogy:

I. Empowering educators and learners to interrogate their complicity in systemic oppression as well as use mathematics as a tool for intersectional justice;
II. Creating an interactional space that QT students of color experience as affirming of their full identities and contributions to mathematical knowledge; and

III. Decolonizing curricula by infusing contributions from QT mathematicians of color and problem-solving opportunities relevant to students' lives.

Approach (I) focuses on instruction, (II) on support, and (III) on curricular design. Figure 2 maps elements of liberated mathematics classrooms (in yellow) and anti-oppressive education (in green) onto my proposed vision.

Later in this article, I share findings from my research about Latin* QT students' experiences that I presented during the Lavender Lecture. These findings depict how mathematics pedagogy expanded or limited opportunities for Latin* QT students to experience support for their intersectional identities. The analysis, thus, serves an illustrative purpose of capturing how pedagogy in undergraduate mathematics can advance or inhibit queer of color justice as outlined in my proposed vision. Such insights inform pedagogy that accounts for issues of intersectionality among Latin* QT students and QT students of color more broadly. In addition, the analysis fills a research gap on inclusive STEM pedagogies for QT students with an explicitly intersectional focus [8, 14].

Queer Students of Color in STEM Study

This section provides an overview of the larger study, titled "Queer Students of Color in STEM" (QSOC-STEM), from which the analysis of Latin* QT students' mathematics experiences is derived. QSOC-STEM explores the intersectionality of experiences among Black, Latin*, and Asian QT students in STEM majors. More details about the study can be found elsewhere [8, 15].

Study context and data collection. The QSOC-STEM team recruited participants across five US universities, including four large, research-intensive, and historically white institutions (HWIs) and a historically Black university. Three HWIs are public and one is private.

One public HWI site received the federal designation as a Hispanic-Serving Institution, and another has a strong record of STEM support for underrepresented groups. The historically Black university is a small liberal arts school with a strong record of QT student support. A total of 60 QT students of color participated, including 25 Latin* participants whose data was the focus for the analysis presented here and in the Lavender Lecture. This study's Latin* sample is diverse in terms of ethnoracial identity (e.g., Costa Rican/Puerto Rican, Mexican, mixed Colombian-white), queer sexuality (e.g., bisexual, gay, pansexual) and STEM major (e.g., computer science, mechanical engineering). Ten of the 25 Latin* participants hold gender-expansive identities, such as female/questioning, nonbinary, and transmasculine.

Participants completed a demographic survey to report their race, gender, sexuality, year of study, STEM major, coursework, and campus involvement. They also submitted a written autobiography about being QT students of color in STEM, including memorably positive and negative experiences, identity-affirming academic and cocurricular spaces, influential people, and the role of mathematics. Throughout the study, participants kept a journal of events in STEM spaces (e.g., classrooms, study groups) that they experienced as supportive and affirming or unsupportive and disaffirming of their identities. Participants completed an individual interview (Interview 1), group interview (Interview 2), and member check interview (discussed later). Eight group interviews with Latin* QT students were included in the present analysis.

Data analysis. QSOC-STEM explores oppression and agency among undergraduate QT students of color in STEM, in addition to structures and practices that disrupt oppressive experiences. My framework, *STEM Education as a White, Cisheteropatriarchal Space* (WCHPS), presented in Figure 5 guided data analysis. The framework provides researchers with a model for guiding analyses of interplay between systemic forces of racism, whiteness, and cisheteropatriarchy in STEM education, which shape intersectional oppression and agency for QT people of color and other multiply-marginalized groups [8].

Each WCHPS dimension attends to a level at which white cisheteropatriarchy operates and can be disrupted in STEM education. The ideological dimension addresses beliefs, norms, and values that organize educational practices. The institutional dimension explores structural inequities of achievement and participation. The relational dimension addresses interactional forms of oppression, agency, and resistance. WCHPS dimensions are interconnected, allowing the framework to uncover complex ways that white cisheteropatriarchy operates.

Our team completed four stages of data analysis. First, we inductively coded data to flag instances corresponding to each WCHPS dimension in terms of oppression, agency, and supportive disruptions. Next, we synthesized codes in each participant's data set and followed the critical race methodology of counter-storytelling [16] to construct an analytical narrative for each participant. Counter-storytelling challenges or "counters" deficit portrayals of racially minoritized communities in educational research. It centers racially minoritized people's lived experiences as sources of knowledge for theorizing resistance to racism and other interlocking systems of power. Participants' counter-stories capture interplay between ideological, institutional, and relational forces in their experiences.

The third stage of analysis was a cross-case analysis of counter-stories to identify themes of oppression, agency, and supportive disruptions across the intersectionality of STEM experiences. The final stage of analysis was

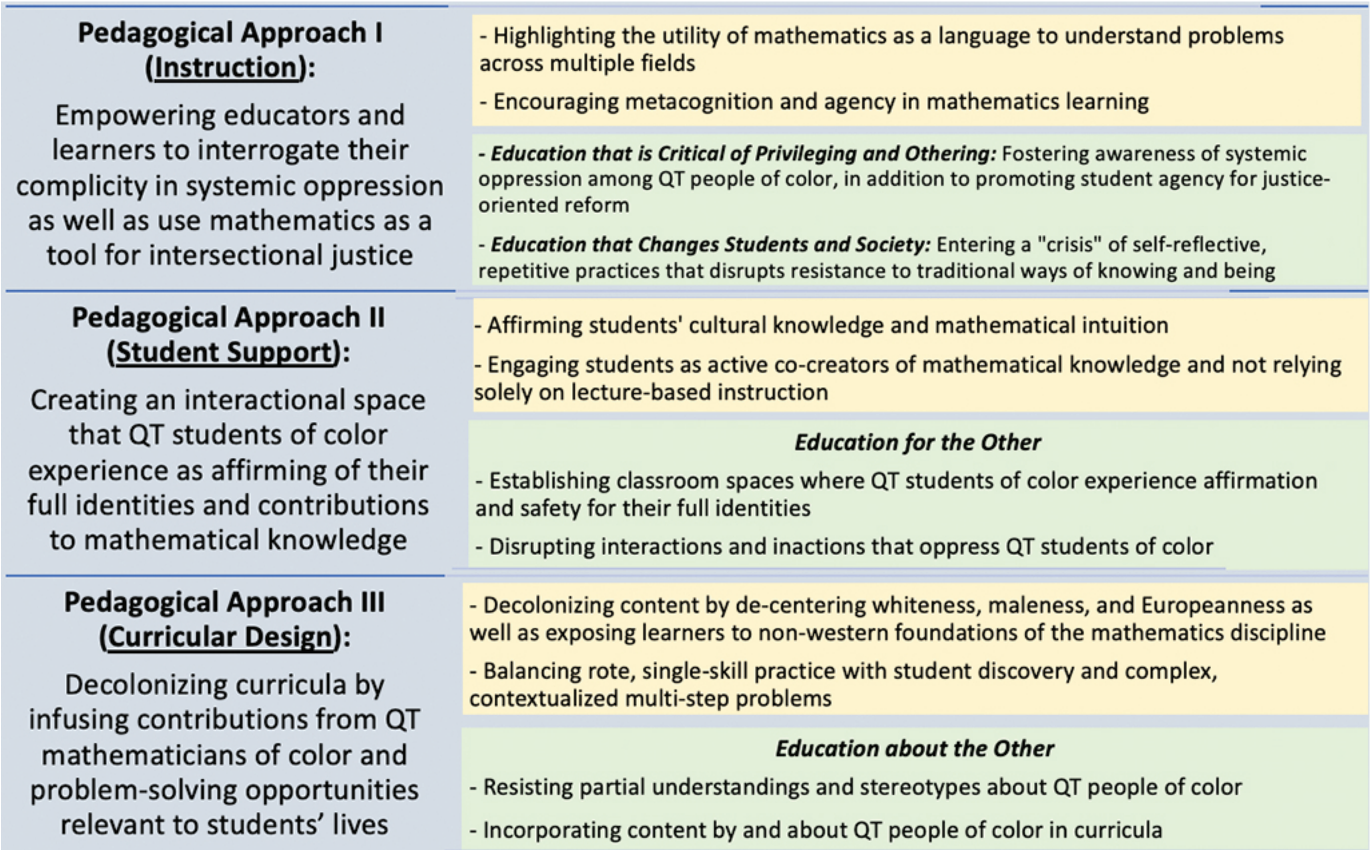

Figure 4. Pedagogical vision for queer of color justice in mathematics.

conducting a member-check interview (Interview 3). Member checks sharpened claims from data analysis. We presented participants with written sections of their counter-stories corresponding to each WCHPS dimension and asked them to suggest edits that reflect their experiences more accurately. The team also asked student participants questions to clarify and further elaborate on important ideas in their counter-stories. Thirteen of the 25 Latin* QT students completed member check interviews.

In preparing for the Lavender Lecture and writing this article, I independently completed a fifth stage of data analysis. This stage involved identifying themes from our team's analysis about the influence of mathematics. I identified two contexts (pedagogy and peer relationships) where participants experienced dissonance and resonance with their intersectional identities in mathematics. In this article, I present findings about pedagogy. (I report findings about peer relationships elsewhere [15].) The WCHPS framework shed light on how ideological, institutional, and relational aspects of how pedagogy influenced experiences of undergraduate mathematics as a white, cisheteropatriarchal space. When reporting findings later in the article, I apply my vision for queer of color justice in mathematics to depict how pedagogical practices in Latin* QT students' counter-stories limited or expanded support for their intersectional identities.

Positionality. Our research team (one faculty member, four doctoral students, ten master's students, and two undergraduate students) are members of the Power, Resistance and Identity in STEM Education (PRISM) Lab at Vanderbilt University. The team has robust social diversity across intersections of race (African American, Black, Latin*, biracial, white), gender (cisgender, nonbinary, transmasculine), and sexuality (demipansexual, gay, lesbian, pansexual, queer, heterosexual, unsure). Most of the team collected and analyzed data from Latin* QT participants.

Rich Milner's framework on positionality in educational research [17] provided a set of guiding perspectives for self-reflection to avoid dangers of approaching our study without being conscious of the influence of our identities and experiences. These reflections avoided the seen danger of not interrogating our respective areas of privilege and oppression. The team adopted an asset-based research approach by making space in the analysis for agency and resistance, thus avoiding the seen danger of positioning participants as powerless victims of

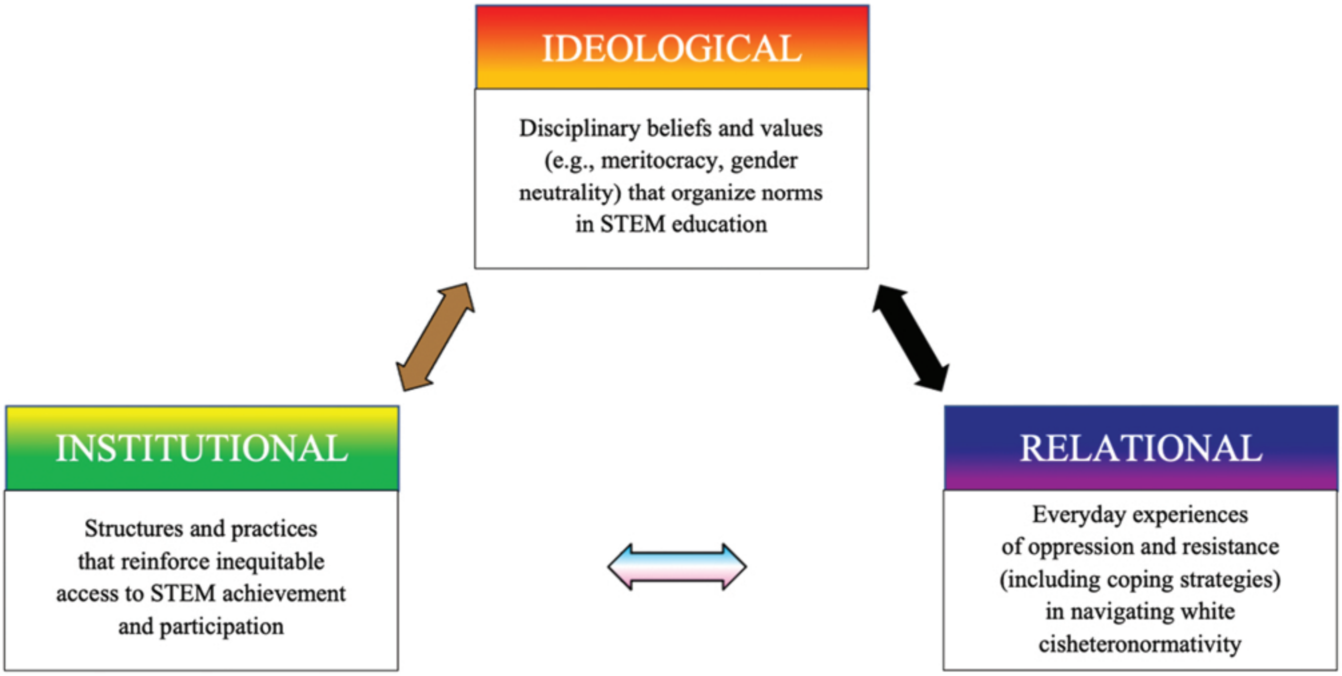

Figure 5. *STEM Education as a White, Cisheteropatriarchal Space* [8].

oppressive systems. To mitigate the unseen danger of misinterpreting participants' sensemaking, we completed interviews and coding in pairs to have multiple perspectives present when collecting and analyzing data. We also completed member checks to strengthen the trustworthiness of our findings. Team members bracketed their lived experiences from those of participants to avoid the unforeseen danger of distorting participants' realities, all while remaining critical of oppressive STEM educational structures and practices through counter-storytelling methodology. I avoided the unforeseen danger of flattening variation in experiences of oppression and agency when reporting findings by looking across two counter-stories. Although this is a solo-authored article, I solicited feedback on early drafts from team members who analyzed Latin* QT participants' data. This feedback ensured data analysis reflects our collective work.

Counter-storytelling Queer of Color Justice in Undergraduate Mathematics Education

This section presents two counter-story cases of how pedagogy in undergraduate mathematics shaped Latin* QT students' intersectionality of experiences as STEM majors. One counter-story case features Jay (he/him), a Colombian-American gay male in his sophomore year as a biology major and on the pre-medicine track. The second counter-story highlights Teresa (she/her), a Costa Rican-Puerto Rican cis bisexual woman in her junior year pursuing majors in mathematics and economics with a minor in gender studies. Jay and Teresa attended two different HWIs that were both large, public, and research-intensive.

I feature Jay's and Teresa's counter-stories here because, of the 25 Latin* QT participants' mathematics experiences, these two cases were most illustrative of themes uncovered through data analysis. These themes reflect how pedagogical practices (curricular design, instruction, and student support) produced dissonance and resonance with students' intersectional identities. The themes include: (i) the culture of individualism and competition in mathematics; (ii) the influence of racialized and cisheteropatriarchal constructions of mathematical ability; and (iii) mathematics curricula as sites of asociality (i.e., the depersonalized nature of STEM pedagogical and interactional contexts that render identities and social issues as inappropriate and irrelevant [14]). Each counter-story offers a unique account of how a Latin* queer student's experiences of oppression, agency, and support reflect these themes. After presenting the two counter-stories, I conclude with a cross-case analysis using my proposed vision as well as implications for educational practice.

It is important to note that Jay's and Teresa's cases are two examples from a spectrum of ways that mathematics pedagogy shapes intersectionality of experiences for Latin* QT students. These counter-stories, for instance, do not account for unique struggles and agency among Latin* gender nonconforming students in mathematics, which I explored elsewhere [15]. Jay's and Teresa's cases also depict racialized realities specific to their ethnoracial identities, families' immigration history, and language backgrounds that vary in the Latin* diaspora. Their counter-stories, thus, are intended to serve as illustrative accounts and should not be interpreted as erasing or reducing social complexities in Latin* queer experiences.

Jay's counter-story. Jay's case depicts how pressures of assimilation through instruction, the STEM culture of competition, and stereotypes of mathematical ability presented barriers to support. His counter-story shows his agency to overcome such struggles by way of "play[ing] catch-up" to his peers, forming peer networks of safety, and performative displays of confidence.

Struggles with assimilation. Jay shared dedicating more time to his academics than socializing due to struggles of assimilating to US mathematics education. He reflected on the influence of mathematics on his STEM trajectory, "The idiom that says math here is the same as everywhere else is only partially accurate... People from Latinx and other underrepresented communities are often assumed to have completely normal backgrounds... without taking into account that sometimes we might need extra assistance" (Autobiography). The idea of "normal backgrounds" in Jay's reflection refers to having early experiences of learning mathematics in US contexts. Jay described how, in addition to the progression of mathematical content being more advanced in the US than in his home country Colombia, the written algorithms for different operations (e.g., fraction multiplication) differed. These disparities across national contexts, on top of navigating instruction in English as an emergent bilingual, led Jay to feel pressure to catch up to classmates when learning content.

> *When you come to US, the levels are just through the roof and it was complicated for me to grasp it. It takes me three times as much to do this stuff... You come here and you have to play catchup. But it's just difficult... The concepts are to an extent different... They're explained differently and they're performed differently to an extent... Having to adapt here, it's difficult.* (Interview 1)

Despite the challenges of assimilation, Jay made several efforts to ensure his mathematical success, such as seeking various forms of help (e.g., tutoring), always attending class, and translating class notes to Spanish.

Jay found it challenging to connect with peers on a personal level in mathematics and other STEM courses. The "air of competitiveness" (Interview 1) in the pre-medicine track contributed to this challenge. As a college student older than the majority of his peers and a first-generation immigrant adjusting to the US context, Jay felt his time for socializing was limited in order to succeed in a highly competitive STEM program.

> *When you have everything cut out for you, you get to do that... But when there has been some experiences in your life when you're just trying to play catchup to a lot of things, it sucks... Here, I am a little older and I'm still trying to get a social aspect and still have the content of stuff [from STEM courses]. But it seems like in STEM, it's very cutthroat... It's just difficult to create those connections.* (Interview 1)

The competitive culture in pre-medicine collided with Jay's struggles as a nontraditionally aged, first-generation student to make peer networks less accessible.

Peer interactions that focused on course content and were less personal, such as groupwork, made Jay anxious. His anxiety stemmed from classmates' perceptions of his academic ability as a student developing fluency with English and who speaks with a Spanish accent. His most memorably negative STEM experience was encountering linguistic racism in groupwork, "I felt excluded or undervalued by some of my peers due to my accent or language insecurity... It was clear that some students did not want to collaborate or work with me, and I felt that my knowledge was being questioned" (Autobiography). This reflection captures how Jay felt vulnerable as an emergent bilingual to being academically underestimated when collaborating in STEM classrooms. Jay shared how, on top of feeling "a little anxious... when [his] English gets a little too fast" (Interview 1) or when he struggles with language translations, the inability to select groupwork partners added to his lack of calm. He further reflected on the negative groupwork instance, "Maybe it's also my anxiety of everything and I had to be [one of] the very, very last ones that was paired up... You got to make it work... [and] do what you can with what you have" (Interview 1). Although groupwork is intended to foster positive peer connections in mathematics and other STEM courses, Jay's insecurity with speaking in English and being undermined made collaboration anxiety-inducing and exclusionary.

Jay also viewed his accent and struggles with verbal communication in English as adding to difficulties in connecting personally with classmates. He described a shift in how peers interacted with him upon hearing his accent, which he interpreted as them being less inclined to engage in conversations unrelated to the class with him.

> *It's notorious sometimes because when you say something... I don't know if people are trying to be respectful or what, but there's a slight change in attitude of like 'Okay, so we're just here to talk to a specific thing and then that's it.' There's no... 'Oh... where were*

you born?' There's not that special connection... 'Oh yeah, I have a friend that was born there.'... But then you see other people, they're just talking about where they were raised... They went to this high school, it's the same place. (Interview 1)

Classmates' evasion of off-task topics in response to Jay's accent left him with limited opportunities to connect on a personal level. Jay perceived this disconnect as putting him at an academic disadvantage in mathematics and other STEM courses with strong representation of fellow pre-medicine students. He described how the competitive pre-medicine culture resulted in the formation of "cliques" that were exclusive in extending support and sharing resources, "The competitiveness of the... pre-med and all these pre-health tracks... We don't collaborate if we don't have to collaborate. And if I'm gonna share my notes, it's going to be with this group specifically and that's it" (Interview 3). Linguistic racism decreased Jay's access to strong peer connections as an emergent bilingual, thus leaving him at the margins of "cliques" in class.

Agency through classroom behavior. To manage the pressures of "trying to fit in a clique" (Interview 3) and secure peers' academic support, Jay was strategic about his behavior in precalculus and other STEM classrooms to avoid deficit views of his ability. Jay was conscious that, in addition to being academically underestimated for his accent, his effeminate self-expression as a gay male could subject him to being stereotyped as lacking mathematical ability, "Gay people, very feminine gay people, we don't tend to be good at math. It's just the stereotypical situations" (Interview 2). He described "not let[ting] people make an opinion" (Interview 2) about his ability based on these racialized and gendered views by withholding participation, "Talk as little as possible in public and sometimes don't ask the question that you think you need to ask. Wait 'til the very end [of class] and hopefully you remember your question for your professor" (Interview 3). Another strategy in Jay's classroom behavior, particularly to evade homophobia, was forming and exclusively interacting with a "clique" of peers with whom he felt safe collaborating. He shared navigating the precalculus classroom in this way.

If somebody is nice to me... For example, my pre-calc class, there's this girl, she's from India, sweetest girl ever. And that's my clique... If I found my safe person, I stay with her and that's it... I had that person that we can rely on each other to explain each other stuff... I found someone that I related to... We had a good dynamic and I just stick to it. (Interview 3)

In addition to the comfort and relatability that Jay felt with an Indian female peer, he perceived precalculus classmates as safe for collaboration if they seemed to either identify as queer or signal QT inclusion. Jay described indicators of safety that he sought, "The way they're dressed, the way that they are social... You just feel it... Just little marks... like a gay flag or something on the backpack" (Interview 3). The inability to readily build peer connections, coupled with navigating stereotypes and the threat of homophobia, made Jay grateful that he found a precalculus peer for safe and productive collaboration, "It was great to have it [his clique] in the class because... I suck sometimes working in teams. But that collaboration part was fantastic for both of us" (Interview 3). Despite this positive relationship, Jay was taxed with undue labor of limiting his participation and finding peer safety to protect himself as a Latin* queer mathematics learner.

Jay also managed deficit views of ability by adopting masculine-presenting behaviors (e.g., speaking in a deeper voice, keeping an upright posture), which he saw as exuding confidence as a student.

Como esa confidence... Uno trata pero a veces no sale. A veces el inglés no sale... En la parte como hombre gay, hasta cierto punto es eso... Uno sabe cuando la gente tiene algún prejuicio contra un hombre que es muy femenino... Si uno nota mucha 'pluma' entre comillas... También lo veo en ese sentido... Uno se presenta con alguien y tiene de que 'Tengo que poner con la voz más ronquita, Tengo que pararme más derecho... Presentarme.' [Like that confidence... One can try, but sometimes that confidence does not come through. Sometimes, the English language does not come through... On the part about being a gay man, this holds true to a certain extent... One knows when people have some prejudice against a very feminine man... If a man is perceived as too feminine.... I also see it from that view. One presents an image of like 'I need to make my voice a little deeper. I need to stand more upright. Introduce myself]... I want to present that image of my confidence and hopefully that confidence translating [to] being more capable of doing stuff... to avoid them [classmates] to think, 'He might not be the smartest of them all.' (Interview 3)

Jay viewed his performative display of confidence coded in masculinity as shielding him from being academically undermined as an emergent bilingual and a gay male with feminine gender expression. He described how growing up in a Latin* household with narratives of machismo shaped his association of masculinity with performances of confidence. When Jay reflected on gendered beliefs in his family, he shared, "El machismo... En la familia latina, el hombre siempre va ser el hombre de la casa y si no, no eres un hombre [Machismo. In the Latin* family, the man will always be the man of the house and if not, you're not a man.]" (Interview 3). Jay recalled early advice from his parents grounded in machismo that later influenced his embodiment of confidence in STEM classrooms, "Like derechito [stand with upright posture], have a deeper voice y trate de mostrar su confianza… Creo que eso es… lo que más instilaron desde pequeñito [and try to show your confidence. I believe that is what they most instilled in me since I was a young child]" (Interview 3). Jay's reflections depict how cisheteronormativity from his upbringing both served and censored him as a Latin* gay male in the mathematics classroom. His parents' advice served to guide strategic displays of confidence to deflect linguistic racism and homophobia in peer perceptions of ability. At the same time, such gendered performances of confidence censored authentic expressions of Jay's identity to appease a white, cisheteropatriarchal gaze.

Teresa's counter-story. Teresa's case shows how asociality in mathematics curricula and instruction separated knowledge production from social realities. There were limited coursework opportunities for Teresa, as a mathematics major, to experience affirmation as a Latin* queer learner and support for her justice-oriented career goals. Teresa's case highlights disruptions of such asociality through pedagogical practices that rehumanized the mathematics discipline [18] and fostered resonance with her identity. Her case also depicts agency as a queer Latina navigating exclusionary and unsafe contexts of participation in 'neutral' mathematics classrooms.

Erasure of identity and social issues. Teresa shared how "most of [her] math classes do not involve discussion of people" (Interview 2) and that "many professors… don't talk about social issues" (Autobiography). She felt dissonance between such asociality in mathematics and her justice-oriented career goals. Claiming that "math is super important for econ" (Interview 1), Teresa picked up mathematics as a major to advance her career goals of addressing socioeconomic disparities in her community and fostering equitable P–12 educational opportunities. She also viewed majoring in mathematics as allowing her to get involved with data analytics and motivate younger generations with shared identities to pursue a similar path, "With math, I don't wanna tokenize myself, but sometimes I'm like, 'Hey, it would be cool to be involved in data analytics and even potentially inspire someone else to pursue data analytics that's younger than me'" (Interview 1). With social issues unexplored in mathematics, Teresa did not have coursework opportunities to nurture her social justice commitments.

Asociality in mathematics made it difficult for Teresa to ascertain faculty views on social issues. She was able to do that more in economics and gender studies courses, where topics of identity and social justice were common.

> *Math… is just like, 'Oh, well, we don't deal with people.' That's the overarching feeling I get from professors… I think about this a lot compared to my econ and gender studies classes, where we do have to talk about people a lot, and so it's very easy to find what my professors' political leanings are.* (Interview 2)

Along with the presence or absence of identity discussions in class, Teresa learned about professors' ideological orientations on social matters through their research, which was more often the case in economics than mathematics, "I can look up my [economics] professors, see what kind of research they do and, usually from that, I can tell where they lie on a political spectrum… But with math, there's no way of knowing" (Interview 1). Teresa's justice-oriented career goals made such knowledge about faculty important.

Teresa saw a contrast between the erasure of social issues in her mathematics major and socially-relevant content in an immersive summer research program at another university (Silver). The program supported students from underrepresented backgrounds to develop research skills and learn about graduate study in mathematics. Teresa's goals of using mathematics for social justice and her minoritized identity were recognized in the Silver program.

> *We listened to a lot of projects and presentations. Some of them on imposter syndrome… One of them was on gerrymandering… That had an impact on me, because I was like, 'Oh wait, that could be a route I could take if I wanted to go to grad school in math, that I don't have to abandon the social aspect of a lot of what I want to do… Since it's [the Silver program] geared towards minorities, we're all kind of in the same boat… The imposter syndrome thing definitely resonated with us, so we felt really seen there, and just having mentors who also were very understanding of our situations of not always having the same resources that other people, other demographics in our field have.* (Interview 2)

Teresa reflected on learning opportunities in mathematics that would have more readily nurtured her equity-oriented goals, "The [Silver] experience is probably the most salient… We don't have an applied math program [at her university]. We only have pure math. So a lot of the upper levels I took were like topology and it was just basics of topology. We couldn't really apply it" (Interview 3). The focus on pure mathematics in the major precluded opportunities to apply concepts in exploring social issues, which Teresa found inspiring in the Silver program.

Gender studies coursework was a space where Teresa felt more seen in her intersectional identity than in mathematics, "As a gender studies minor, most of the... classes I've taken have an intersectional lens and I feel very seen in those spaces" (Autobiography). She contrasted the extent to which her experience as a bisexual Latina was seen as relevant and a source of knowledge between her gender studies and STEM courses.

> *It's* [Teresa's identity] *relevant to the conversation. A lot of my gender studies courses, we have a lot of discussion, a lot of opportunities to even cite anecdotes, which is not going to be super common in STEM classes. So in group discussion or class discussion, you'd be like, 'Oh yeah, I know what this author is talking about because I'm bisexual. I've experienced that kind of conversation.* (Interview 1)

Teresa similarly viewed engagement of her lived reality as a bisexual Latina more readily valued in her campus job at the center for social justice than in STEM contexts, "Our programs are backed by research and theory, but we still recognize and see anecdotes as valid. Like we can say, 'That resonates with me because as a Latina, I had this and this experience,' and anecdotes aren't super welcome in STEM environments" (Interview 3). Dominant beliefs of what 'counts' as knowledge in STEM, including valorized notions of abstraction and objectivity in mathematics [13, 19], shaped asociality in coursework that made Teresa feel her queer of color identity was irrelevant and unproductive for learning. Such disciplinary values, which are anchored in ideologies of white patriarchy, orient pedagogical practices that often leave mathematics as a product of western colonization uninterrogated in higher education.

Teresa argued that adding more historical perspectives in mathematics instruction can normalize discourse on social issues and allow QT students of color to be seen in their identities. As an example, she called for discussion of how statistics was used to justify racism in the eugenics movement—a topic in her gender studies courses where she felt her identity as a bisexual Latina was affirmed.

> *The history of eugenics and how it was based on a lot of statistics and how people thought it was scientific. I learned that in my gender studies class instead of my math class, which I thought was weird because I think that should be something that all math majors should learn about... Computer science majors are required to take an ethics class, but something like that for math, I would have appreciated.* (Interview 2)

Teresa saw topics of history and ethics in mathematics as socially-relevant content crucial for students in the major.

Although Teresa yearned for engagement of social issues in mathematics, exceptional instances of faculty infusing topics of social power and individual histories felt disconcerting. For example, Teresa described how it "felt really out of place" (Interview 3) when a professor for an upper-level course raised mathematicians' realities of religious persecution and overcoming misogyny.

> *I also had another professor... who did make an effort to spotlight women mathematicians that made an impact on the field... He would also talk about if a theorem was named after a person... There was one mathematician... that was persecuted for being Jewish... I appreciate that he even brings stuff like that up because... these names of theorems are real people. I'm seeing a bit more humanness in math, but you can tell it's rare because the few times I've seen it, it felt out of place.* (Interview 3)

Discussion of mathematicians' lived experiences as a pedagogical practice added a humanizing element missing in mathematics courses. Teresa feeling disconcerted with such practices shows how such pedagogical disruptions of asociality, which she valued, were not the norm.

Resistance through femininity. Despite pedagogy reinforcing views of mathematics as a 'neutral' space, Teresa viewed classrooms as sites of gendered struggle. Gender played a more central role than her race and sexuality in navigating STEM contexts, including mathematics classrooms, "I'm white-passing. My bisexuality isn't super obvious in the way I present physically... A lot of my negative experiences when it comes to being in STEM, it's just me being a woman" (Interview 1). In Teresa's first year of college, participation in mathematics classrooms, including groupwork, was uncomfortable. She was concerned about "feel[ing] like [her] ability to contribute was automatically underestimated" (Interview 1) along with not "want[ing] to seem too confident" (Interview 1) when presenting ideas. With Teresa's older brothers encouraging her to participate more actively in class and her ex-boyfriend suggesting that she major in mathematics, she viewed her mathematics trajectory as involving "straight men who seemingly wanted to share their confidence" (Autobiography). Her gendered struggles with participation decreased in upper-level classes where she connected with other women.

Teresa viewed her feminine gender expression contributing to being academically undermined in first-year mathematics courses, "I used to dress very feminine. I would put on makeup and paint my nails and I knew I wouldn't get taken as seriously" (Interview 2). At the same time, such displays of femininity concealed her bisexuality and protected her from being fetishized as a queer Latina.

> *I was okay with my bisexual identity being covered by me being feminine because I didn't want anyone to know that I was both Latina and bisexual because that is just a recipe for fetishizing that I didn't want to welcome... I wouldn't say I was in the closet. I was very active in queer spaces, but in terms of like in the classroom, I wasn't letting that be known because I didn't want to invite any type of comments or sexualization* (Interview 2)

Teresa reflected on how knowledge of her Latina identity based on her Spanish last name already made her vulnerable to being sexualized. She recalled cisgender men in a first-year statistics course making comments like, "Oooh, can you say your last name again? It's so sexy" (Interview 3). Such racialized-gendered sexualization brought Teresa to feel her "comfort comes down to how many cishet, white men are in the room" (Interview 2). To navigate this white cisheteropatriarchal gaze in mathematics, she concealed her queerness to prevent being fetishized as a queer woman of color and avoided interactions with cisgender men, "I also don't talk to cis men in any of my classes... I don't want them to know any more about me because that's enough [being Latina]... Therefore, [I] hide my bisexuality" (Interview 3). Being mathematically successful as a feminine-presenting Latina was also a form of resistance, "It was almost like a 'F you' because the men that wouldn't take me as seriously, I would do better than them in school and then they would end up having to ask me for help" (Interview 2). Teresa's femininity, thus, was a source of both oppression and agency for navigating first-year college mathematics classrooms as a bisexual Latina.

Rehumanizing disruptions. Teresa accounted for two pedagogical disruptions of asociality that she experienced as rehumanizing mathematics and affirming of her Latin* queer identity. The first disruption, which she described as the "most human interaction [she's] had with a STEM professor" (Autobiography), took place in a calculus class. The professor, a white heterosexual cisgender man, dedicated half of a lecture to recognize his privilege as well as advocate for students who share his identities to do the same and ensure equitable opportunities for classroom participation. In describing a vision for STEM classrooms that affirm QT students of color, Teresa alluded to this moment from the start of the semester.

> *The one instance I can think of... was a cishet male professor who... spent the entire half of the lecture just telling all the other cishet white males like, 'Hey, you hold a space of privilege in STEM. Recognize that. Give space to other people who don't hold those identities... I make time to have these conversations. It is part of my curriculum that I spend half of this lecture talking about it because it's so important to me.' Even something like that, I wouldn't mind if it was in every class, every semester... I think it's important and needs to be drilled.* (Interview 2)

Teresa valued her professor's interrogation of privilege because it expands opportunities for QT students of color to experience classrooms as "more affirming [and] more safe" (Interview 1). She expressed how this moment personally impacted her as a bisexual Latina, "It made me feel like he understood who I was and what I go through as a minority woman in STEM" (Autobiography). Having a mathematics professor who asked white male students to "step back when marginalized people are talking" (Interview 1) potentially allowed Teresa to have her gendered struggles with classroom participation feel seen.

Teresa perceived her calculus professor's instruction overall as rehumanizing mathematics for QT learners of color like her. She recalled the professor taking time during the first day of class to build community and learn students' pronouns, "First day of class... He was just like, 'I want you guys to get to know me. I want to get to know you... If you have pronouns that you want me to refer to you by, please let me know. I don't want to misgender you"' (Interview 1). When advocating for students to recognize and interrogate their privilege, the professor stated, "This isn't a political statement. This is a human statement" (Interview 1). Such centering of identity notably disrupted asociality in Teresa's mathematics experiences, "None of my math professors, besides him, have ever just stated that during class time—the importance of identity... Some professors don't bring it up because they might think gender and race [are] political when it is a human issue" (Interview 1). Teresa saw it valuable for all students continuing with mathematics, including QT students of color, to experience such humanizing instruction in introductory courses, "Especially because he's teaching lower-level math. So if you're gonna go into higher levels, this is how you should start your journey, knowing where you stand and how you can help marginalized communities" (Interview 3). Thus, identity-affirming calculus instruction elevated consciousness for Teresa and classmates about white cisheteropatriarchy in mathematics as well as how educators and students can work to disrupt it.

The second pedagogical disruption of asociality in Teresa's mathematics experience took place in an upper-level course with a queer professor of color. At the start of the semester, the professor shared two video clips he recorded—one with a course overview and another with his self-introduction that was optional to watch. The optional video highlighted the professor's work in mathematics and outside of the field, such as his involvement with a QT initiative on campus and his written work about QT people of color. When asked about instances of feeling seen as a Latin* queer student, Teresa raised this video.

> *He had a little 'About the Professor' thing... and he talked about how he worked on [QT campus initiative]... He writes [about queer people of color] and he's my math professor, so I was like, 'Oh, this is amazing. You do things that aren't just math, and you do human things. Not like math isn't human, but you know what I mean—more things that relate to people... He's a very well-respected professor, so it's like, 'Oh, that's a queer person of color doing the damn thing at my school.'* (Interview 2)

Learning about the professor's work in the QT community at the university made Teresa "feel safer since the math field feel[s] so dominantly straight and white" (Autobiography). It also made her feel comfortable with talking to the professor about academics and events at the campus center for social justice, where she worked on QT programming. Since Teresa completed this course at the start of the COVID-19 pandemic when classes were moved online, she found it challenging to connect and discuss queer-related topics, "I want to talk to him more about that [his writing about QT people of color], but it just being an online setting makes it really hard... A lot of classes have [online platform], that's very public and I want to have a private conversation... when it comes to stuff like that" (Interview 1). Despite such barriers to out-of-class connections, this course was a uniquely positive mathematics experience for Teresa. The professor noting his involvement in queer activism via the optional video disrupted traditional pedagogies in mathematics that reinforced a separation of the discipline and humanity, "The fact that he mentioned [the QT initiative on campus] still felt good because it felt like he found that an important thing to mention about his work" (Interview 1). Teresa perceived such recognition of queer realities as a way the professor worked to "demonstrate that math is a very human issue" (Interview 3). This instructional practice, thus, rehumanized mathematics and allow Teresa to feel affirmed in her Latin* queer identity.

Concluding Perspectives: A Cross-case Analysis and Implications for Practice

This concluding section applies my pedagogical vision for queer of color justice in undergraduate mathematics (Figure 4) to guide a cross-case analysis of Jay's and Teresa's counter-stories. I structured this analysis in three parts that address each pedagogical approach in my vision—instruction, student support, and curricular design. The cross-case analysis elucidates how pedagogy across the two Latin* QT participants' mathematics experiences limited or expanded opportunities for their queer of color identities to be supported. I use the WCHPS framework (Figure 5) to elucidate how white cisheteropatriarchy was reinforced or disrupted at multiple levels (ideological, institutional, and relational), which impacted opportunities for queer of color justice through pedagogy. Adding to recent *Notices* discussions of supportive practices for the QT community in mathematics [20], I raise implications for pedagogical practice from my analysis and findings.

Instruction. Jay's and Teresa's cases differed in terms of the first approach in my pedagogical vision—*empowerment of educators and learners to interrogate complicity in systemic oppression as well as use mathematics as a tool for intersectional justice*. Pressures of assimilation that Jay faced as a first-generation, emergent bilingual student through instruction limited access to content. His case reflects educators potentially falling short in interrogating how instruction that assumes English language fluency and prior mathematical knowledge can lead to differential opportunities for success. Applying the WCHPS framework, Jay's counter-story depicts how dominant ideologies that recenter US-based ideas of mathematical success (e.g., communicating in English) framed instruction as an institutional practice tied to inequitable learning for QT students of color.

Teresa's counter-story presents a pedagogical disruption of complicity with systemic oppression going unchallenged, thus advancing the first approach in my vision. This disruption is reflected in Teresa's calculus professor naming his privilege and calling upon white male students to monitor how much space they occupy in class. He modeled critical self-reflection of his social position and empowered students to take action in challenging white cisheteropatriarchy. This instructional instance made Teresa feel seen as a bisexual Latina navigating gendered struggles of participating in class and being sexualized. At the ideological level of the WCHPS framework, Teresa's case exemplifies a pedagogical disruption of asociality in mathematics anchored in ideologies of whiteness and cisheteropatriarchy. The professor's instruction resisted the ideological framing of mathematics as 'neutral' to hold himself and his students accountable in pushing back on queer of color oppression. Such disruption at an ideological level is also evident in Teresa's summer research program experience, where she explored use of mathematics to address issues of social justice (another aspect of my pedagogical vision's first approach). Teresa's mathematics coursework fell short in this pedagogical approach to advance queer of color justice due to absent discussions of identity and social power. She was left relying on outside opportunities like the summer program to receive support for her justice-oriented motivations as a mathematics major. Teresa's experience of instruction in the calculus course as rehumanizing also disrupted undergraduate mathematics as a white, cisheteropatriarchal space on a relational level. The professor's pedagogy carved opportunities for building community and honoring identities, which Teresa had not experienced in mathematics where her identity as a bisexual Latina went unrecognized.

Jay's and Teresa's counter-stories raise implications for instructional practice about the importance of critical self-reflection among mathematics educators. Faculty must interrogate how their areas of social privilege shape instruction that reinforces inequities of access and disallows QT students of color to feel seen as mathematics learners. Teresa's case depicts the value of infusing social relevance in instruction to empower students' engagement with mathematics for queer of color justice.

Student support. Jay's counter-story illustrates pedagogical shortcomings for the second approach in my vision for queer of color justice—*creating an interactional space that QT students of color experience as affirming of their full identities and contributions to mathematical knowledge*. The unstructured nature of groupwork in mathematics left Jay's limited access to peer support unchallenged. Linguistic racism presented barriers in forming personal connections with classmates, placing him at the margins of "cliques" in class. The pre-medicine track's competitive culture contributed to the hoarding of resources and support in "cliques." As a feminine-presenting Hispanic gay man, Jay navigated racialized-gendered views of his mathematical ability that pressured him to prove his worth as a peer collaborator.

Through the lens of the WCHPS framework, Jay's counter-story shows how mathematics pedagogy as an institutional practice failed to intercept individualism as a dominant ideology in STEM. Competition among students on the pre-medicine track led to the formation of exclusionary "cliques" whose support was inaccessible to Jay due to linguistic racism and homophobia in peer interactions. At the same time, Jay's case depicts agency in managing such individualism by forming his own "clique" in the precalculus classroom to ensure access to peer support. He actively sought indicators of classmates being inclusive of QT people through their dress, behavior, and appearance when determining who might be someone safe with whom to collaborate. Although Jay's strategy of forming his own "clique" provided a safe environment to express himself during groupwork, the silos of peer support and culture of individualism in the mathematics classroom persisted.

Jay's case also captures white cisheteropatriarchy operating at both ideological and relational levels. Intentional choices of behavior, such as withheld participation and displays of confidence coded in masculinity, served to protect Jay from stereotypes of mathematical ability in the relational space of the classroom as a feminine-presenting gay Latin* man. Ideologies of machismo in Latin* culture, despite being anchored in cisheteronormativity, guided Jay's performances of masculinity to exude confidence and avoid peers' gendered perceptions of his ability.

Insights from Jay's counter-story raise implications for mathematics pedagogy that ensure equitable distribution of student support. Faculty, for example, can adopt norms for forming groups and guiding collaborative interactions that explicitly challenge racialized, cisheteronormative beliefs of mathematical ability. Such norms establish an interactional space that disrupts the dominant culture of individualism in STEM and expands opportunities for competence among QT students of color to be embraced.

Curricular design. Teresa's counter-story illustrates advancing the third pedagogical approach in my vision—*decolonizing curricula by infusing contributions from QT mathematicians of color and problem-solving opportunities relevant to students' lives*. A self-introduction video from a queer professor of color that acknowledged his campus work in the QT community and writing about QT people of color disrupted the silence of queer humanities that Teresa experienced in mathematics. These highlighted achievements also provided temporary reprieve from mathematics as a racialized, cisheteropatriarchal space often unsafe for Teresa as a queer Latina. Through the lens of the WCHPS framework, the video is an institutional practice that disrupted traditional forms of curricular design, which subscribe to dominant ideologies of mathematics as a 'neutral' space.

Learning about faculty involvement in social activism, mathematicians' social realities, and possibilities of using the discipline to solve social justice problems rehumanized mathematics for Teresa as a bisexual Latina. However, these curricular opportunities were so uncommon to the point that Teresa felt disconcerted when she experienced them in class. Even the self-introduction video was optional to watch. Teresa's counter-story raises implications for faculty to provide and normalize learning opportunities where students readily consider connections between mathematics and broader sociopolitical issues. Such curricula allow QT students of color to experience mathematics with relevance to their identities and lives.

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Luis A. Leyva

Credits

Figure 1 is courtesy of Lee Druce.
Figure 2 is courtesy of Nicole Joseph.
Figure 3 is courtesy of Brittany L. Mosby.
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THE NEXT GENERATION

Early-career AMS members take a moment

Favorite memory from an AMS event: The excitement of seeing the big mathematicians whose papers I've read.

Were you inspired by a mathematician?: I was inspired to study math just in order to make it simpler for others. This is because I was the only one that passed math in my graduating high school class and I thought this could be simpler because I believed my classmates were smart too.

Favorite color: Blue

Field of research: Differential Equations

Maria Amarakristi Onyido

Assistant Professor, Northern Illinois University

What does the AMS mean to you?: The AMS is a good resource for staying up-to-date on the latest research and networking.

Hobby: Playing Soccer

Favorite color: Green

Favorite food: Biryani

Field of research: Matroid Theory

Describe a situation when you first fell for math: When I won the math competition at my school during the 7th grade.

What is the best piece of advice you received that helped you get through graduate school?: Think about the end product and stay positive.

Were you inspired by a mathematician?: Yes, Srinivasa Ramanujan

Dony Varghese

Visiting Professor, University of North Alabama

Favorite memory from an AMS event: Meeting my friends and seeing old colleagues.

Field of research: Statistics and Applied Mathematics

Describe a situation when you first fell for math: I was at middle school when I first took a geometry class. I was totally fascinated with the elegance and the beauty of mathematics. Geometry helps discover patterns and understand the world around us.

Mariam Khachatryan

Assistant Professor, Augustana University

OF MATHEMATICS

to share a little about themselves:

#AMSMember

Hobby: Harpist (I've played for a former prime minister of England)

Field of research: Probability and Statistics

What does the AMS mean to you?: Math has always been a way to connect with a community of people who love what I love. The AMS has enabled that connection at both my institution and more broadly at conferences and online.

PhD Candidate, University of Louisville

Vicki Modisette

Hobby: Soccer

What does the AMS mean to you?: Opportunity, Growth, Success

What do you think is the most important service the AMS offers?: Conferences, Opportunities

Field of research: Applied Mathematics

Assistant Professor, University of Nevada, Las Vegas

Rachidi B. Salako

Favorite number: Mertens Constant

Field of research: Analytic Number Theory

How would you describe math to a non-math person?: The rhythm of a metered rhyme:

The fractal shape of falling snow
A contour changing over time,
both swift and slow
The logos of the wise and great
The mind unlocked, alert, and free
The dance of chaos, facing fate.

PhD Candidate, Dartmouth College

Grant Molnar

www.ams.org/membership

Nobody Majors in STEM to Fail

Daniel Zaharopol

"How could he miss question two? That was a gimme."

"There was a question just like that on the homework, and she never came to office hours."

"He's not taking the work seriously. He needs more maturity to succeed in this class."

Faced with struggling students in their classes, many instructors search for explanations. It can be easy to find ways that students seemed to fall short of expectations. However, that perspective leaves out where students are coming from: what experiences they might have had before college, how those experiences are impacting their trajectories now, and how adjustments to college could better improve their outcomes.

In short, without considering the context, it's easy to dehumanize students, and to think more about their failures than the ways things could have been designed to allow their success.

The goal of this article is to demonstrate that, indeed, "nobody majors in STEM to fail." We will draw a line from middle school, through high school, to college; consider how those experiences might influence students' college academic work; and provide concrete ideas for how to create better environments that support students' successes and see them as humans. The aim is to see the highly varied and individual journeys that students take, and then draw those threads together with a special focus on the experiences of those who are underrepresented, such as Black and Latino students.

This article will closely parallel the author's recent invited address at the Mathematical Association of America's MathFest conference. It is, however, meant to be the start of a conversation, not the end; it will raise ideas and hopefully provoke more reflection, but it nevertheless represents the observations of just one person who continues to seek the best ways to support young people in mathematics.

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Taking a Longitudinal View

Discussions at the college level often focus on where college students are at *now* without necessarily considering the contexts they came from. Hence, perhaps our most important contribution is illustrating how students' backgrounds are crucial to designing effective interventions.

The observations here are drawn in part from the author's work with Bridge to Enter Advanced Mathematics (BEAM[1]), a program that supports students in New York City and Los Angeles. BEAM works to create pathways for students from low-income and historically marginalized communities to become scientists, mathematicians, engineers, and computer scientists.

BEAM's longitudinal program provides support from sixth grade through college graduation, including intensive enrichment classes and individual student advising and mentoring. This support gives its staff a holistic viewpoint on students' journeys. For more about BEAM's program, see [Zah20].

The Middle and High School Experience

It's common to break students down into groups and discuss them as monoliths: the need to improve, say, the experiences of Black students in STEM or of gender minorities. While it's important to call out adverse impacts on large groups, this practice can also obscure diversity and individualism within each group. So here, we will take an opposite tack: while still focusing on the experiences of underrepresented minorities, we will explore their individual uniqueness and the wide variety of experiences they have growing up—experiences which, in turn, have a dramatic impact on their adjustment to and success in college.

Nonacademic experiences. Consider, for example, students' families, home communities, and schools. We will

[1] `www.beammath.org`

paint a picture of different environments, and consider later how some of these contribute to college experiences.

The BEAM program has worked with highly engaged families pushing for every opportunity; families that prioritize education but see it as driven by school, focusing students on their schoolwork but not extracurriculars; families that don't have time to support students and leave their child to their own initiative; immigrant families with a strong push to succeed but a lack of knowledge about the system; and more. Families might be close knit, or more individualistic; students may have responsibilities to their family while growing up (especially care of younger siblings), or be more independent. Students may have role models in the family who work in math and science, or they might not. Students might be the first in their family to go to college. All of that, of course, is just scratching the surface.

Similarly, students' home communities might be large or small; close-knit or diffuse; more communal, or more individualistic; primarily immigrant or primarily multigenerational American; highly educated or not.

Schools exhibit many differences as well. Of course schools might be small, where each student gets a great deal of individual attention (often in contrast to college!) or large, where students tend to be more independent. Some schools are highly structured, where student expectations are clearly laid out for every minute of the day, which is especially common in some charter schools that serve Black and Latino students. Schools may have differing expectations for student achievement or college attendance and varying levels of support. (As just one example, there are an average of 408 students to one guidance counselor nationwide [Ame], with huge variation by school. Of course, this ratio will have a much greater impact on a student whose family and community have less knowledge about college.)

Schools themselves serve very different populations of students, in part due to the segregation of housing and the school system overall. In the US, over two thirds of Black and Latino students attend schools that are more than 60% Black or Latino, while over two thirds of White and Asian students attend schools that are more than 60% White or Asian.[2] These schools are still diverse along many metrics: a school that serves predominantly Black students might be serving multigenerational Americans, recent African or Caribbean immigrants, Black Latinos, as well as students with a variety of different religions and family backgrounds. In any case, school demographics have clear consequences for students' feeling at home at different universities.

[2]*This statistic was calculated based on data from* [Nata].

Academic experiences. There are well-known disparities in academic performance across students from different backgrounds. For example, the National Assessment of Educational Progress (NAEP) reports that by 12th grade, only 24% of students score at the "proficient" level in mathematics. However, when we break it down by income, 33% of more affluent students (those not eligible for free- or reduced-price lunch) are rated proficient, while just 11% of low-income students (those who are eligible) score at proficient. The gap becomes even more stark at the "advanced" level of achievement: 4.79% compared to 0.69%. [Natb]

Most analyses stop with data like that—a simplistic look at "achievement gaps." However, we should dig deeper to understand STEM preparation. Consider a typical student who might go on to a college STEM major. A common expectation might be that they did well in high school science and math classes, including (most likely) a calculus course, along with some high school exposure to physics, chemistry, and biology. Some communities also engage heavily with extracurricular work: camps, independent reading, competitions, computer programming, or independent learning (via books or even YouTube).[3]

Each of these expectations can be better understood with more digging. For example, consider the pathway to calculus. There are four years of standard high school mathematics, typically divided up as Algebra 1, Geometry, Algebra 2/Trigonometry, and Precalculus. To fit Calculus into a high school timeline, students must take Algebra 1 in eighth grade, or, less commonly, double up in math at some point during high school. Nationwide, 19% of students pass Algebra 1 in eighth grade, but not surprisingly, there are stark disparities when one disaggregates by race. While 29% of Asian students and 24% of White students pass Algebra 1 in eighth grade, only 14% of Latino students and 10% of Black students do so (see Table 1).

It is also worth noting that this data may understate the issues: Some studies have indicated that Algebra 1 courses at high-minority schools generally *cover less content* than the corresponding courses at schools that are not predominantly minority students; in other words, "passing Algebra 1" may have a different meaning at different schools. [MRC20]

There is no single cause for this opportunity gap, and again it helps to dig deeper. First of all, only 59% of middle schools offer Algebra 1,[4] and high-minority middle schools are precisely those that will have fewer or no seats

[3]*Although it does not address STEM, the enrichment spending gap is documented. See, for example,* [KMW11].

[4]*Although this statistic is slightly misleading: smaller middle schools are less likely to offer the course, and so overall 80% of students have access to Algebra 1.*

Demographic Group	Percent enrolled in Algebra 1	Pass rate	Overall percentage passing Algebra 1
All students	24%	78%	19%
Asian students	40%	74%	29%
Black students	16%	65%	10%
Hispanic and Latino students	20%	72%	14%
White students	29%	85%	24%

Table 1. Breakdown of the Algebra 1 pipeline in eighth grade, adapted from [USDa].

in the class. [USDb] [PRSM] However, other causes for the gap may include bias (students may be coached out of advanced courses that they are ready for, or they may enroll but then have negative experiences); families who may not know the importance of taking Algebra 1; or students who may wish to stay with their friends or others who share their backgrounds, who may be disproportionately placed into eighth-grade math.

According to studies, prior academic preparation explains about 50% of the gap in Algebra 1 coursetaking. [PRSM] However, it is "turtles all the way down:" some of that lack of preparation is surely due to the same factors playing out in earlier grades.

These factors are present in later grades as well. Black and Latino students are less likely to attend a high school that offers calculus, and rates of passing the AP Calculus exam are correspondingly lower. Overall, 50% of high schools do not offer calculus, already a very high number that might be shocking to those in affluent communities. However, among those with high Black and Latino enrollment, that number goes up to 62% that do not offer calculus. In fact, 35% of high schools do not even offer precalculus [USDa], making it even harder for a potential STEM major to succeed if their peers start college taking calculus or later courses. For a more detailed breakdown of calculus achievement, see [Zah19].

In summary, by the time students arrive at college, they may have had different academic experiences, including starkly different levels of access to courses such as precalculus and calculus (to say nothing of enrichment experiences that also play a huge role in building problem solving skills). Although each student's experience is different, on a societal level this lack of access disproportionately affects Black and Latino students.

Other experiences. While we have been focusing on more systemic factors, students of color are likely to encounter explicit racism as well as implicit bias. Depending on their experiences, they may have felt marginalized, out of place at certain institutions, or been led to question their belonging. A full discussion of these factors would exceed the space available here as well as the expertise of the author, but it is important not to minimize these factors as students navigate college.

Similarly, many students may have experienced housing insecurity, food insecurity, medical insecurity, violence, or other traumatic challenges. Again, a full discussion exceeds the author's expertise, but these challenges are disproportionately faced by groups that are also underrepresented in math.

The College Experience

What actionable insights can we draw from students' varied backgrounds? It can feel overwhelming to try to take those experiences into account once students arrive in a college classroom. In this section, we translate that challenge into some actionable insights.

However, it's important to emphasize that this section provides examples, not a recipe. Our goal is to develop skills to understand students on an individual level, and thus to better support their success, but how exactly to do so depends on the institution, the students being served, and many other factors.

Sometimes, people cite the uniformity of experience as a sign of equality: A STEM major in college may be hard, but "everybody goes through it." This article is to some extent a refutation of that idea: the *ways* in which STEM majors are hard have disproportionate impact on some people based on their past experiences. Changes to the STEM major experience, such as those in this article, can mitigate the disproportionate impact.

We explore this disproportionate impact through a series of questions that show how past experiences impact future ones.

Why do students drop STEM majors? This is clearly a complex question, and not one we will come close to answering in full. Instead, we'll consider some explanations to provide a framework for additional reflection.

Many of the reasons may seem familiar: low grades; less preparation before college; inadequate guidance on how to navigate college/STEM majors; and the overall environment within a STEM major. Any of these can contribute to students' choice to leave before finishing their degree.

However, each of these can also have disproportionate impact on students depending on their backgrounds. Consider:

- Low grades will have a psychological impact on every student. However, some students may have family members or older siblings who completed STEM majors, and can put those grades in context, while other students will compare college grades to high school grades and conclude that they're out of their depth. Additionally, students who already feel different or isolated may find that lower grades further depress their senses of self worth and belonging.
- Becoming aware of gaps in their academic preparation will affect all students. However, some students will have more access to resources such as private tutors, peers, or friends who recommend additional resources. Conversely, students with fewer financial means will have work obligations to pay for college that prevent dedicating time to learning prerequisite material.
- All students find navigating STEM majors difficult, and the guidance provided by colleges is often inadequate. However, while some students again have family members or peers (even older graduates of their high school) to go to for advice, others may be largely on their own, especially if their social group does not have many STEM majors.
- All students experience a change of culture when they go to college. However, some students may find that culture change to be more difficult: there may be fewer peers who share their background, or there may be more of a difference between the expectations and norms in college as compared to their home environment. This is to say nothing of racism or bias that students of color may face.

Some of these challenges are systemic: the way that courses and prerequisites work or the composition of a college class are all about the systems that students find themselves in. However, that is not a reason to give up on making change; any individual can implement changes locally that will impact the student experience, the culture, the advising that students get, or even how grades are communicated to students. (How much can a discussion at office hours change a student's perception of their midterm grade?)

However, it is important to remember that each student experience is different. BEAM has had a number of students of color go on to STEM majors at a huge variety of colleges, from community colleges to the Ivy League. Some of these students attended high school with strong preparation; some with weaker preparation. Some students had been in high schools or summer programs where their peers were predominantly White or Asian; others had not. Some had strong networks of college advising, or family members who were engineers. Each student's story is unique, and it is critical not to make assumptions.

For example, one BEAM student was a young Latino man, who had gone to a strong high school, taken Calculus there, and gotten an A in the course. He then attended a selective public college. However, Calculus 2 proved to be difficult. His past school had been a highly structured charter school; college offered much more freedom, and relied much more on self-study. Although he worked hard, he did not have study skills, he could not afford the textbook (and instead went to the library), and he often worked from home, which was crowded and loud. He felt like his own work was key to his success, and so he did not seek outside help. When COVID hit, he could no longer access the library and he failed his Calculus 2 course. While some obstacles listed above played a role (such as the impact of low grades, or cultural expectations about the college environment and seeking help), others, such as preparation gaps, did not.

The next question asks about a remedy that might have helped this young man: what could have inspired him to seek help?

Why don't students come to office hours? Office hours can be a critical tool for student success. They provide a link between student and instructor; allow the student to get customized help; and help create connections that can lead to letters of recommendation or information about future opportunities. However, attending them can be difficult for students. Going can be intimidating, especially if the student doesn't already know the instructor; the timing may not work in a student's schedule; a student may have poor time management and so may not do problem sets before office hours; and students may be worried that by going, they'll reveal that they "don't really understand the class."

As before, though, each of these concerns may have a disproportionate impact on certain students. For example:

- Office hours can be intimidating for every student, but for students who never met a professor before college (because there were none in their family or network), or whose parents never graduated college, a professor can seem much more intimidating.
- Making time for office hours, similarly, can be hard; any student may have a conflict with another class or obligation. However, some students may have additional family obligations such as caring for younger siblings, may work to pay their way through college, or may live farther away from campus to save money.

- Time management is similarly a challenge for everyone, and especially people in their late teens or early 20s who are often in college. However, the challenge can be much greater if someone's high school was not very challenging (and hence did not require learning good time management skills), or if it was highly structured, where students had very little flexibility on how they spent their time. The adjustment to the freedom of college is not the same for everyone.
- Office hours can feel risky, because no student wants to demonstrate that they "don't understand" the topics in class. However, that worry can be heightened for someone who already "sticks out" (perhaps because they're one of few students of color), or who is the first in their family to go to college, or who is more intimidated by professors than their peers.

It is worth lingering on that last point. Even at BEAM, which is a highly supportive program that works with students beginning when they are 11 or 12 years old, students fear showcasing a lack of knowledge. The implication of the BEAM program (that it is "advanced" mathematics) leads students to worry that they don't belong, and to hide places where they struggle in school math classes. We see this especially with male-identifying students; in any case, it may be a significant struggle for students to admit to having difficulty.

Fortunately, professors can take action to mitigate some of this disproportionate impact. For example, opening up to students and sharing their own stories can help to demystify "the professor" and put students at ease. Similarly, professors can communicate that their goal is student growth, and publicly share their own stories of struggle (or those of students from past years) to normalize the challenges students face.

A second approach is to ensure that all students come to office hours at least once. Professors can make repeated invitations in class, demystifying office hours by explaining what happens and why it is beneficial. Students can also be required to come to office hours at least once in the first month. In this way, students will have broken the ice and may be more likely to come in the future.

Finally, individual follow up can make a positive impact on students. A professor might personally email students who struggled on the first quiz or exam, sharing positive expectations and inviting them to come into office hours to discuss some of the problems. While it does not take much time to write such a message, it can spur a student to take the step to come in!

Why do students struggle in intro courses? Introductory courses are often one of a students' biggest barriers. In fact, about half of students who drop out of college do so in their first year.

Examples of challenges that students might face in their introductory classes include their overall adjustment to college; lacking access to adequate academic support; and a significant jump in the sophistication of the work students are now expected to do.

Also as before, there will be a disproportionate impact on some students. Adjusting to college is more difficult for many students for reasons discussed above, such as less family experience with college, coming from different communities, or feeling different from other students. Financial circumstances and social networks can impact access to academic support. Rather than rehashing these topics, we will consider the jump in the sophistication of work that often comes with going to college.

The "Algebra 2/Trigonometry" Regents Examination is the most sophisticated math exam required to graduate in New York State. In a review of the August 2016 exam [New], the author picked out the following question as the "hardest:"

"Using the identity $\sin^2\theta + \cos^2\theta = 1$, find the value of $\tan\theta$, to the *nearest hundredth*, if $\cos\theta$ is -0.7 and θ is in Quadrant II."

Of course, even the typical college calculus exam question is generally much more difficult than this. For students who are used to the *most difficult questions* both telling you how to do the problem ("Using the identity $\sin^2\theta + \cos^2\theta = 1$") and being relatively simple, it can be a big jump to the *typical* exam question requiring justification or developing a solution method without guidance. This emphasizes again how a good high school calculus course (to which there is uneven access) can help students succeed even without skipping college calculus.

Where, then, do students get access to more sophisticated problem solving that prepares them for college? Sometimes a good high school will offer it as part of their classes, although more often, students may do other enrichment programs or independent study (such as online). Of course, access to those resources is uneven. Many students are told to simply do well in their school classes, and that will be enough—unfortunately, those students are often the most likely to attend high schools that offer less of an education in problem solving.

There are no easy answers to these challenges. However, professors can be explicit about the kind of problem solving necessary, normalize struggle, and prepare students psychologically for a jump from what they are used to. It is also possible to provide more scaffolding, such as earlier problems that guide students through the process of problem solving. Regardless, course design and good teaching is especially important in introductory classes. Finally, if resources are available, advocacy within colleges or universities to offer additional preparation, perhaps in parallel with existing courses, can make a difference for student success.

How does the culture of STEM affect students? The people who do math are a diverse group, and there is no one culture to mathematics. However, there are cultural trends that are common among STEM communities. For example, a focus on inquiry, problem solving, and understanding, or an enjoyment of puzzles. Mathematical communities also often (but not always!) show support to those who are less social or may not be neurotypical.

However, it's also the case that the dominant culture in mathematics may be foreign to students. Moreover, the culture of mathematics is often individualistic and competitive,[5] which may be very different from what they're used to at home.

While there is not space to fully address this claim here, it's not hard to see competitive trends in the classroom. The student in class who raises their hand primarily to show off their knowledge, or the student who scoffs at another for not having learned something yet, are common tropes that can imply a competitive mindset.

As before, individualism and competitiveness also disproportionately impact some students. For example, those who have less preparation, less of a sense of belonging, or who feel less of a connection to others who do STEM may be more negatively impacted by a sense of competition. So, then, how do we address individualism and competitiveness?

Of course, it is best to address these broadly, but even within a classroom there are changes that can make a significant difference. For example, taking the time to explicitly welcome students from varied communities to the class, or setting a mindset based around growth in the class (rather than absolute achievement) can make a positive impact for students. Setting wait times before students can answer questions (to allow more students to develop their answers) will similarly help.

It may not be intuitive, but one common classroom technique that encourages a competitive mindset is the way that grades are often curved.[6] Test grades are often curved by ordering them, looking for bunches, and setting the boundaries between As, Bs, and Cs depending on where there are gaps. While this is easy, it sets students against each other and does not make sense to them: their grades are not based on what they know, but how they compare to others in the class. A grading system where there are absolute measures corresponding to each letter grade that are based on understanding of the course material (rather than how peers do) emphasizes that students must achieve at a certain level, rather than in competition with others. An 'A' will really mean "I fully understood the material" and not "I did better than most other people."

Naturally, these solutions are only the beginning of addressing the disproportionate impact of the culture of mathematics and the classroom. Nonetheless, they can begin to make a difference.

Structural racism. Although the focus here is on individuals, it is important to emphasize that many of the issues discussed above are more likely to impact students from certain backgrounds: racial or ethnic minorities, first-generation college students, and those from low-income families. Systemic racism refers to society-wide practices that advantage particular groups based on race (even if they are on their face neutral); the examples here, then, are examples of how systemic racism works at the college level, although the disproportionate impact does not only affect people based on race. On its face, we say "college is hard for everybody." In reality, it is harder for some than for others because of the overall design of the system.

That said, BEAM's students have encountered not just systemic racism but direct, individual racism. For example, a student at a highly selective high school was called the n-word by her peers. While this article has focused on addressable challenges, it does not intend to minimize the adverse experiences students may have with direct racism.

Conclusions and Next Steps

In the work to better support students from marginalized backgrounds to achieve STEM degrees, it is important to balance expectations. Understanding students' backgrounds and designing an approach to mitigate the disproportionate impact of existing structures can make a positive difference in students' experiences—even if doing so will not magically solve all the inequities that persist. Seeing students as people and supporting their aspirations can improve their experience—even if there remain other negative factors in their environment.

And yet, despite the magnitude of the challenges, there are many steps that individuals and departments can take to make a genuine positive impact. It is possible to create a healthier environment that views students as individuals who are each trying their best to succeed. Nobody majors in STEM to fail; many students do find success, and more will do so with targeted support.

Questions of equity remain core to the future of society; work to address those questions will have rich and

[5] *Thanks to Dr. Nicole M. Joseph for introducing me to these terms.*

[6] *This is also cited directly by students as a problem in STEM courses; see, for example,* [SH19].

important benefits. Those of us educating future STEM professionals have an important role to play if we commit ourselves to the goal.

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Daniel Zaharopol

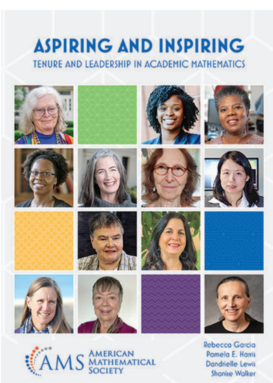
Credits

Photo of Daniel Zaharopol is courtesy of Erin Patrice O'Brien.

BOOK REVIEW

Aspiring and Inspiring: Tenure and Leadership in Academic Mathematics

Reviewed by Deanna Haunsperger



Aspiring and Inspiring
Tenure and Leadership in Academic Mathematics
Edited by Rebecca Garcia, Pamela E. Harris, Dandrielle Lewis, and Shanise Walker.
AMS, 2023, 189 pp., `https://bookstore.ams.org/mbk-150`.

The start of the fall term coincides each year with my return to attempting a regular exercise routine. Aside from walks with friends, I've never enjoyed exercise, and usually my well-laid plans for getting back into the shape I was in twenty years ago peter out after a few weeks. My colleague Katie, thirteen years my junior, talked me into joining a CrossFit-inspired class last spring led by Thad, a challenging instructor we both like. I would watch as Katie lifted barbells that appeared massively heavy and listen to the satisfying clank as they hit the ground after her lift. I, on the other hand, was doing my best to lift the barbell with no additional weight attached, just the bar, and no sound of success. One night after one such training session, I picked up this edited collection of essays and began reading it. From the very first article I saw my struggles—in math, in life, and even in the gym—talked about openly and honestly. Maria Emelianenko said, "I have been hopelessly failing at a task of holding myself to an impossibly high standard..."(p. 20) and "I cannot trust anyone with making decisions on my behalf since no one knows my strength and weaknesses better than me" (p. 21). So true.

How has success been defined in a mathematical academic career? Matriculate to the best undergraduate institution (based on a national ranking) you can, take as many advanced courses as possible, participate in an REU or more than one, publish an undergraduate research paper, next matriculate to the best graduate school that you can (again based on a national ranking) where you work with one of the top researchers in the department and publish several papers. Finish your PhD and start a postdoc at a top university. Accept a position at a top-ranked university and publish, publish, publish to earn tenure; then publish, publish, publish more to earn full professor. It's exhausting just typing it out, let alone living it. Notice what's missing. There's no mention of self-care, a partner or family, a life. For many of us in mathematics, this is not the track we wanted to be on, even from the start. However, it still may loom large in our minds as what we *should* have wanted. What we should be aspiring to. What success looks like.

Jacqueline Dewar implores us to "each come to—and honor—our own definition of success," (p. 3) which she defines for herself as wellness, a balanced life, and a community where she belongs. This sentiment is echoed

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Communicated by Notices *Book Review Editor Emily Olson.*

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throughout the book, as others point out the importance of not letting anyone tell us that there is one best path for our careers. Allison Henrich writes about how she changed her research field to one that has more possibilities for student research. She also amended her scholarship to include writing which "fed [her] soul" (p. 49), like her work on *Living Proof: Stories of Resilience Along the Mathematical Journey. Living Proof* feeds us all as it helps to humanize mathematics by normalizing the struggles that we face. Kathryn Leonard shows us in a poignant description how her career path could be read as either wildly unsuccessful or successful, depending on the lens through which you view it. Cynthia Wyels defines "winning" academia as, "finding ways to contribute to the greater good" (p. 176). This book made me stop and think about how I have defined—and to some extent still may define—success in my own life (including at the gym).

The editors of this volume, themselves leaders in the mathematics community, drew together the experiences of eighteen prominent women and gender minority mathematicians in seventeen different chapters. These people represent academia at liberal arts and research institutions, have served in department and university administration, are leaders in society, have served as journal editors, have directed national programs, and have won numerous prestigious awards. When we learn about the successes of someone in mathematics, it's all too easy to think that it was easy, straightforward, effortless. In this volume we hear the stories of these women and, as if they are sitting next to us at our dining room table, they give us personal advice and tips on how to have success like they have enjoyed.

Erica Graham wrote a compelling chapter about being in the intersection of a number of identities (she self-identifies as a privileged, Black, queer woman) and how that creates an ethos where, it's "very easy to internalize messages of inferiority when we don't see ourselves reflected in the mathematicians standing before us" (p. 32). Returning to her alma mater to teach, she was the only Black science faculty member, and as such, BIPOC students would seek her out to share their stories of racism. "[Racial battle fatigue happened at] that moment when every single microaggression seemed to find all the others, and they collided to paint a huge, ugly… portrait of academia…: the implicit biases; the tired diatribe against weak backgrounds and under-preparedness, which strangely only seemed to apply to BIPOC students; the not-so-subtle discussions about diversity and rigor…; the blatantly racist, classist, sexist, and every other *-ist* hiring practices" (p. 38). Graham calls academics out for the "stock, *Hallmark*-esque promise to do better" (p. 39), and well she should. How can we really do better? Among other things, Graham suggests true allyship by acknowledging our privilege, addressing biases we observe, and practicing microaffirmations.

Some authors provide a particular bit of advice that has worked well for them. Henrich describes a technique she learned from Kerry Ann Rockquemore, the founder of the National Center for Faculty Development and Diversity. Rockquemore suggested setting aside time on Sunday evenings to map your to-do list onto your calendar for the week, including classes, office hours, meetings, and times for scholarship both with collaborators and independently. Henrich has become better over time at estimating the time required for various tasks, and she's more likely to keep time for research when it's on the schedule.

Perla Myers shared the idea of "'multifaceting:' accomplishing more than one objective with a single task" (p. 99). An example was when she was the associate dean of faculty. She would introduce new faculty to the campus with a walking tour. On the tour, she would learn about how they were settling in, explain the resources available to the faculty and where they could find them, introduce the new faculty to others they met along the way, and enjoy some exercise. As another example of multifaceting, she gave the students in her math course for future elementary school teachers the assignment to develop and implement an activity for a Family Math Night at her son's elementary school. This allowed her students the opportunity to learn that they needed to understand the mathematics deeply to be able to explain it. It also provided a resource for the elementary school. And Myers was able to participate in her son's life and meet his classmates and their families. "One activity. Many outcomes" (p. 101).

When Irena Swanson moved from a fifteen-year career as a professor at Reed College to becoming the department head in the Department of Mathematics at Purdue University, she learned a great deal about running a department. For instance, she advocates for learning by asking questions. Also, interrupting a meeting or class with a question "can be a boost to everybody's attention" (p. 136). "Make sure to give credit where it is due, and do so generously" (p. 137), she maintains, along with accepting blame when things go wrong based on your actions. I'll mention one other from her list of things she learned: "Stand up for what is right. You cannot correct all the wrongs in the world, but if they are within your power, do address them" (p. 137).

One could imagine that if each person has their own definition of success, the advice of the authors of this volume to others for climbing the tenure ladder wouldn't share much commonality. However, the similarities in their advice for success are noteworthy.

- Get out of your comfort zone and say "yes" to new opportunities. Jacqueline Jensen-Vallin also points out that you should use times when you

need to say "no" as opportunities to raise up other individuals who may not receive the same offers you do.
- Work on your own self-confidence, deal with your feelings of inadequacy, trust yourself, and encourage, as Emelianenko calls it, your "alpha-female" (p. 26).
- Find advocates and mentors for different aspects of your career and life, and ask for advice when you need it.
- Build in accountability to your schedule by belonging to writing groups or having collaborators.
- Find a community where you can be your whole self.
- Take care of yourself.

Wyels pointed out several things that we can all do for junior faculty, among them: share your teaching resources including syllabi, activities, and assignments; offer to critique job applications, grant proposals, and tenure portfolios; and promote your colleagues to others because "people tend to select those they know for various positions and opportunities" (p. 181). She notes that although her recommendations are written from the perspective of a more senior faculty member, she hopes to inspire more junior faculty members to know they can ask for these things.

The book, organized alphabetically, ends with the chapter by Wyels. It's a fitting end to the book because it is largely looking forward. Wyels reminds us, again, to reach back to people who are climbing the academic ladder after us. Even though, she points out, one is not "responsible for how you were socialized into specific beliefs; you're responsible for what you do going forward" (p. 182). We can, for example, "[i]ntervene on behalf of students and colleagues from marginalized groups..." (p. 182) and even consider taking on administrative roles to change all we can for the better.

All of the authors acknowledged the support and the advice from others that have helped them along their journeys. I was not surprised because I don't think that anyone can thrive without mentorship and a community. However, I was alarmed to read that many attributed some step in their careers to serendipity or just plain good luck. This is not acceptable. The success of women and gender minorities in our profession should not depend on luck. We can do better. We must do better. As Graham suggests, we need to change the academy to make a successful career in mathematics, however one defines it, a possibility for anyone who desires to study math. Emelianenko points to "sowing the seeds of change" by showing our children that women and gender minorities can have successful careers and lives in mathematics. I recommend that you read *Aspiring and Inspiring: Tenure and Leadership in Academic Mathematics* to see how it could affect your future.

I, personally, have a new definition of success. Become an ally. Become a mentor. Sow the seeds of change. Work to reform the academy.

Deanna Haunsperger

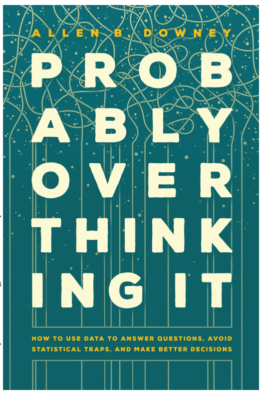
Credits

Photo of Deanna Haunsperger is courtesy of Carleton College.

BOOKSHELF

New and Noteworthy Titles on our Bookshelf

February 2024



Courtesy of Chicago University Press.

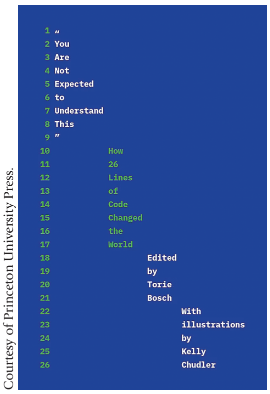
Probably Overthinking It

How to Use Data to Answer Questions, Avoid Statistical Traps, and Make Better Decisions

By Allen B. Downey. University of Chicago Press, 2023, 256 pp.

In the age of Big Data, we might take for granted that we can find the answers to our questions within a data set. While there is some truth to this, we must also acknowledge our limitations in understanding the data. There are paradoxes that we could succumb to without proper investigation. *Probably Overthinking It* started as a blog and is now compiled into a book that discusses many data sets and possible ways to analyze them. Downey explores many data sets: family size, recidivism rates, COVID-19 superspreaders, smoking and birth weights, and ratings of chess players, just to name a few. He discusses distributions and possible reasons why data sets form a certain type of distribution.

Perhaps the most interesting component of the book is the discussion of paradoxes. I learned about a selection bias called length-biased sampling which can result in paradoxical conclusions, such as concluding that there are no cars driving your speed on the interstate, only cars driving much faster or much slower than you. Downey provides many examples of Berkson's paradox and Simpson's paradox. If you cover these in a data science or statistics course, you might be interested in this book. You could also analyze the data sets yourself or have students explore them; Downey invites anyone to use the code he wrote for this book and points readers to relevant and available data sets. The book contains helpful data visualizations without too much technical detail, which should allow mathematicians in many fields to benefit from this book.



Courtesy of Princeton University Press.

"You Are Not Expected to Understand This"

How 26 Lines of Code Changed the World

Edited by Torie Bosch. Princeton University Press, 2022, 216 pp.

Our lives are so integrated with technology that it is hard to imagine life without email or the internet. We long for "likes" and put up with pop-up ads. *You Are Not Expected to Understand This* contains 26 essays about pieces of code that have changed the world in unexpected and extensive ways. The essays are historical as well as anecdotal, and they span from the origins of coding to modern viral images. Many of the essays are authored by those who were on the front lines of coding changes, and their contributions to the book are short, only six or seven pages long.

I was amused when I learned that the popularity of the Roomba (the robot vacuum) is attributed to its slightly chaotic path around a room. I also gained cultural insights when reading essays about the logistical challenges of the gender spectrum when many databases require a binary entry and how coders put Bangla, the language of Bangladesh, online to help more of the world communicate. I appreciated that the book highlights flaws in code. Coders are human, and rather than expecting them to be infallible, we should understand the limitations of their code. I expect that many mathematicians who lived through some of the events discussed, such as the creation of hyperlinks or the JPEG, would appreciate this book. I also think a younger generation that has grown up with the consequences of the choices of programmers would gain insight as well.

This Bookshelf was prepared by Notices *Associate Editor Emily J. Olson.*

Appearance of a book in the Notices *Bookshelf does not represent an endorsement by the* Notices *or by the AMS.*

Suggestions for the Bookshelf can be sent to `notices-booklist@ams.org`.

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AMS BOOKSHELF

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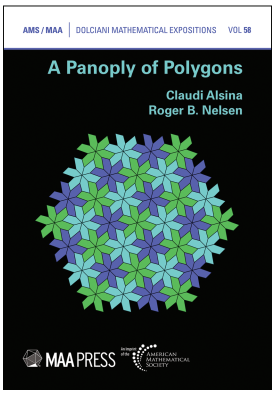
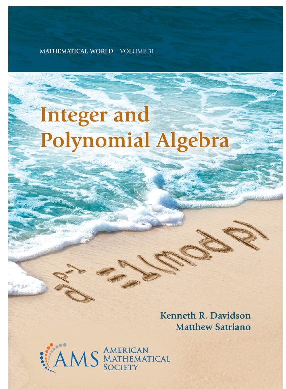


A Panoply of Polygons
By Claudia Alsina and Roger B. Nelsen. DOL/58, 2023, 267 pp.

Are you a polygonophile? If not, *A Panoply of Polygons* will make you one. The book begins with basic facts about polygons and the tools that are helpful for understanding them. Chapter 2 is devoted to pentagons, Chapter 3 to hexagons, Chapter 4 to heptagons. See the pattern? Since it is a basic requirement of any book that it be finite, Chapter 6 treats "many-sided polygons." Miscellaneous classes of polygons are presented in Chapter 7, figures such as centrally symmetric convex polygons called "zonogons" (from an ancient Greek word for belt or girdle). Combinatorics, which makes guest appearances throughout the show, gets a star role in the final chapter, "Polygonal Numbers." The reader will enjoy many theorems about polygons and related figures proved with only a few words (or no words at all). The authors pack an impressive amount of mathematics and history into only 267 rectangular pages.

While many polygons are impossible to construct with straightedge and compass, Major League Baseball's home plate, described in the League's handbook, is impossible to construct with any tools in our world. (The problem is the right angle in front of the batter.) I doubt that many readers already knew this, or that the 1000 kwacha coin from Zambia has heptagonal shape. Factoids such as these appear with illustrations throughout *A Panoply of Polygons*. They are sprinkled spice, but they never overwhelm the theorems and history that are the heart of the book.

A Panoply of Polygons offers a panoply of reasons to be fascinated by its subject. If you love geometry, then this book will delight you. If you have forgotten how beautiful geometry is, then this book will remind you.



Integer and Polynomial Algebra
By Kenneth R. Davidson and Matthew Satriano. MAWRLD/31, 2023, 185 pp.

This small book resulted from a course taught for many years at the University of Waterloo, Canada. The goal of the course — and of the book — is to introduce students to rigorous mathematics using the familiar setup of numbers and polynomials.

The first five chapters are about numbers. Chapter 1 is about integers with the main focus on the most important notions and results: the Euclidean algorithm, unique factorization into primes, and the infinity of the number of primes. Chapter 2, which is about modular arithmetic, explains the Chinese remainder theorem and Fermat's little theorem. Chapter 3 discusses quadratic extensions, explaining, in particular, that integers in some quadratic extensions do not have the unique factorization property. Quadratic reciprocity is also discussed. Chapter 4 is about primality testing and the Rivest–Shamir–Adleman encryption algorithm, while Chapter 5 is a brief introduction to real and complex numbers, which culminates in the proof of the fundamental theorem of algebra.

The last two chapters of the book are about polynomials over the integers (unique factorization and the definition of algebraic numbers) and over the reals (Sturm algorithm of the determination of the number of real roots). Finally, Chapter 7 presents the basics of the theory of finite fields, treating them as quotient rings of the ring of polynomials in one variable over a prime field.

Attractive features of the book are its wealth of exercises (5–10 for each of 50+ sections) and brief notes at the end of each chapter suggesting further reading. With these features, instructors will have many options to tailor a math circle activity, a summer camp program, or an undergraduate course in a way most appropriate for their students.

The first review is by Daniel Silver, emeritus professor in the department of mathematics and statistics at the University of South Alabama. His email address is `silver@southalabama.edu`. *The second review is by Sergei Gelfand, publisher of the AMS. His email address is* `sxg@ams.org`.

DOI: https://doi.org/10.1090/noti2867

COMMUNICATION

Making *Journeys of Black Mathematicians*

George Csicsery

There was zero hesitation on my part when then-director David Eisenbud first approached me on behalf of MSRI, now known as the Simons Laufer Mathematical Sciences Institute (SLMath), in late 2019 to ask if I would be interested in making a film about Black mathematicians. My collaborations with MSRI on producing documentaries about mathematicians stretched back to 2001 and included several biographical works starting with *porridge pulleys and Pi*, about Hendrik Lenstra and Vaughan Jones; *Taking the Long View*, about Shiing-Shen Chern; and *Counting from Infinity*, which documented the remarkable Cinderella-like rise of Yitang Zhang following his breakthrough on the twin prime conjecture. I was on the verge of completing *Secrets of the Surface*, about Maryam Mirzakhani, which premiered at JMM 2020, and I was ready for a new challenge.

As I set out to plan and budget the project, the immense scale of the subject started to reveal itself. This was not going to be a film about a single person, nor was it the story of a program like the one I documented in *Navajo Math Circles*, or a competition like IMO, that I covered in *Hard Problems*. As the critic Darryl Pinckney so aptly put it in a recent *New York Review of Books* piece, "In our present cultural mood of generous reexamination, the hunt is on for the forgotten, the overlooked." Severely underrepresented in mathematics, the stories of the many African-American mathematicians who played important roles as researchers and educators have too often been forgotten or overlooked.

It quickly became obvious that this film about Black American mathematicians would be a panorama, surveying a landscape populated by outstanding characters and dozens of key institutions and programs, while simultaneously highlighting certain individuals and their exemplary stories. It was to span several generations, chronicling pioneers, their travails and accomplishments, and showcasing current programs created to discover and nurture today's students to pursue mathematics in their studies and careers. I hoped that the programs we chose would include several younger people we could follow as they made decisions about their own futures. The comparisons with experiences of the pioneers would lead to new perspectives on the challenges and hopes facing the younger generation.

MSRI's Trustees were enthusiastic and offered seed money. David Eisenbud took the proposal to the Simons Foundation and they too were supportive, providing major funding for the project. Once funding was secured early in 2020, I began to sift through lists of organizations, programs, and individuals to involve. MSRI has a history of supporting programs that expand the inclusion of underrepresented groups in mathematics: it hosted the very first Conference for African-American Researchers in the Mathematical Sciences (CAARMS) meeting in the mid-1990s, and currently hosts summer programs such as MSRI-UP, the African Diaspora Joint Mathematics Workshop (ADJOINT), and the annual Critical Issues in Mathematics Education (CIME) Workshop for K-16 math educators. Mathematicians involved in these programs could tap into the far reaches of the African-American mathematics community, and I began to assemble names of people to interview. Through a Zoom interview hosted by Edray Goins of Pomona College, I watched Johnny Houston present a talk about his top ten list of role models among African-American mathematicians. This convinced me that Houston could help me select the right subjects to interview for the film. When he agreed to do so, I discovered just how lucky a choice I had made. Houston devoted much of

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Figure 1. Johnny Houston and cinematographer Ashley James after a long day of shooting at Howard University.

his career to managing the National Association of Mathematicians (NAM), the first and foremost organization for Black mathematicians in the United States, which has since expanded its mission. He has also collected a massive amount of historical information and is widely known in the community as the *griot* of Black mathematicians. For over three years, he has tirelessly helped me compile lists of interviewees and contacted the subjects, while helping to organize and schedule several of the major shoots we've completed in the Baltimore, Washington, and Atlanta areas, and at Purdue University. The film owes a lot to his skill at arranging access and opening doors that I didn't even know existed.

Early in 2020, Johnny Houston convinced me that there were some pioneers I should interview sooner rather than later because of their advanced age. I made plans, but in March, the Covid-19 pandemic paralyzed the country and much of the world. Access to older subjects for film interviews became too risky without the strictest safety protocols for all involved. Furthermore, it would be months before these protocols were developed and accepted and longer before the first vaccines became available. Travel was curtailed. Programs and meetings we were planning to film were canceled, postponed, or moved to the Zoom online platform—a format by nature inimical to the visual medium that seeks to convey intimacy, nuance, and beauty.

But the urgency of filming with some of the older subjects remained, so we organized the first live filming with Scott Williams, a founder of NAM, who had retired from SUNY Buffalo. Living in California, I was not allowed into New York state in September 2020, and therefore could not conduct the interview near Williams's home near Buffalo. We agreed to meet in Erie, Pennsylvania. Williams and his wife drove there from Buffalo. I flew on an empty plane to Cleveland, where I had hired a local camera crew and drove to Erie. For this first interview I was joined by Albert Lewis of the Educational Advancement Foundation (EAF). The charming bed and breakfast in Erie gave us their entire first floor for a day of shooting. Except for Scott Williams, everyone was masked; equipment and surfaces were frequently wiped down, and distances kept — my team's first foray filming under the latest CDC safety guidance.

Williams told us his life story, explaining how poetry and artistic pursuits provided breaks from his mathematical work. His passion to ensure that the legacies of Black mathematicians were not forgotten led to his creation of the website Mathematicians of the African Diaspora[1] (MAD), the first attempt to collect and preserve the stories of African-American mathematicians in one place. Growing up in Baltimore during the 1950s and 1960s, he had participated in protests organized by Black students at Morgan State University to integrate local movie theaters in 1963. Morgan State, it turns out, was one of the cradles for producing Black American mathematics PhDs, Scott Williams and Earl Barnes being among the first.

The next two interviews were conducted under more stringent conditions. Virginia Newell was 102 years old when we filmed her in December 2020 at her home in North Carolina. A local cameraman set up the camera equipment and a Facetime link at her home, and I asked my questions from my desk in California. Newell had been an educational reformer before the integration of schools in the southern states was mandated by the Supreme Court in 1964. She talked about being the first to bring trigonometry to the schools where she taught, and the prevailing prejudice of the times. Despite the Covid restrictions, her powerful voice and eloquence brought the conditions of those times to life.

Early in 2021 we repeated the successful technical procedure used with Virginia Newell to interview Evelyn Boyd Granville, the second African-American woman to earn her PhD in mathematics. She and a cinematographer were in Washington, DC, while I stayed in California. Masking, sanitizing equipment, and distancing were strictly observed. Granville had enjoyed a stellar career with several NASA projects, including the Apollo and Mercury programs. She went on to consult and design mathematics education programs and textbooks that were widely used in Texas and elsewhere. Earlier this year I was alerted that there would be a 100th birthday conference held in her honor early in 2024. It went on my calendar as an event to cover for the second film in our series. Sadly, this will not happen; Evelyn Granville passed away in June 2023 at age 99. In July we edited a 20-minute memorial piece

[1] http://www.math.buffalo.edu/mad/index.html

Figure 2. Covid-19 was raging in December 2020 and there was still no vaccine when we filmed at Virginia Newell's house in North Carolina. George Csicsery conducted the interview via a remote connection from California, while the cameraman stayed in another room a safe distance from Dr. Newell, who was 102 years old at the time.

compiling highlights from the 2021 interview. It was screened at NAM's Mathfest during a tribute to Granville and her contributions to mathematics.

In 2021, with the pandemic still raging, we filmed with Emille Davie Lawrence, who was at that time chair of the Math Department at the University of San Francisco. Working with cinematographer Ashley James assisted by William McNeill, we worked at Emille Lawrence's San Francisco home, and included scenes with her children. A few months later we were at one of Dr. Lawrence's USF classes, although Dr. Lawrence and her students remained masked throughout. It would be another year before we could shoot in classrooms, at seminars and conferences, with a mostly unmasked group. During her interview, Emille Lawrence, a Spelman College graduate, repeatedly stressed the importance of the role models she had encountered at Spelman in the decisions that led to a career in mathematics, highlighting Etta Falconer and Sylvia Bozeman.

I quickly learned that Historically Black Colleges and Universities (HBCUs) were an integral part of the story for many African-American mathematicians. They soon became central to the film as well. In October 2021 we spent over a week in Atlanta, interviewing a dozen mathematicians with connections to Morehouse and Spelman colleges at the Atlanta University Center's Woodruff Library. With Covid lockdowns still enforced, our access to classrooms at the colleges was delayed until 2023, when we returned for another ten-day shoot centered on the Atlanta schools.

Figure 3. Signs showing directions to the different Historically Black Colleges at the Atlanta University cluster.

Figure 4. The Brothers and Sisters ceremony is a tradition at Spelman and Morehouse colleges. The procession was filmed in 2021 during the height of the Covid pandemic.

During 2022 and 2023, we filmed dozens of interviews in the Baltimore and Washington areas, concentrating on the mathematics faculties at Howard University, Morgan State, the University of Maryland, and the University of Delaware. In September 2022, we were able to cover the first in-person Mathfest put on by NAM since the pandemic. NAM's Mathfest is a unique annual event where professors from several schools bring their personally selected undergraduates to present talks or posters and meet with recruiters from a range of graduate schools. Over the years, Mathfest has been, along with EDGE and MSRI-UP, one of the more effective incubators for increasing the number of Black students in pursuit of PhDs in mathematics. At MathFest 2022, we met and interviewed a number of students from Howard, Morgan State, and Spelman College. We have already followed up with some, and will continue to track them for the second film in the series, scheduled for completion by January 2025. Delivering these stories about why and how students choose their mathematical futures is one of my key goals.

The HBCUs have long understood the importance of mentoring and support networks, especially for students who arrived at their doorsteps unprepared. The importance of providing guidance and career advice echoed in countless interviews, from Sylvia and Robert Bozeman, who had long careers in mathematics at Spelman and Morehouse, to a younger generation of teachers and administrators, such as Naiomi Cameron, Tasha Inniss, and Anisah Nu'Man at Spelman, Duane Cooper and Ulrica Wilson at Morehouse, Dennis Davenport at Howard University, and Asamoah Nkwanta at Morgan State University. Several scholars mentioned that Black students with undergraduate backgrounds at an HBCU earn more PhDs in mathematics than those who completed their undergraduate studies at predominantly white institutions (PWIs). To understand why, we looked at an array of mentoring projects.

An interview conducted in Baltimore with Zerotti Woods led to one of the deeper explorations of this culture of mentoring, which in his case began when he was asked to leave Morehouse College after his freshman year. Woods generously led us on a tour of the Cleveland Avenue neighborhood in Atlanta where he grew up. After a year of misbehaving that included brushes with the law, he returned to Morehouse and was told that if a professor signed off on him, he would be readmitted. Woods credits Duane Cooper with believing in him. Cooper saw it as a chance to nurture raw talent. "The phrase I use a lot when I give talks is to identify the talent, polished or raw. And in this case, this was raw talent," Cooper said.

Zerotti Woods, now a senior level mathematician at the Johns Hopkins Applied Physics Laboratory, is committed to paying it back. At Mathfest, he recruited Spelman senior Janiah Kyle to work on his team during the summer of 2023 as she prepared for graduate school. When we visited the Atlanta neighborhood of his childhood, Woods took us to a recreation center where he mentors children ranging in age from seven to 12, trying to convince them that mathematicians can look like him.

Others have realized that making mathematics even recognizable, if not acceptable, in underrepresented communities must start at an early age. In Baltimore we interviewed Antoynica and Dontae Ryan, a husband and wife team, who run Ryan Academy, a tutoring program for children struggling with math in school. Dontae Ryan and his friend Akil Parker, both Morgan State graduates, like to use words like "ambassador" when they describe mentoring in mathematics as a means of attaining social and economic equality. Their words echo those of Robert "Bob" Moses, who wrote that mathematics is a civil right.

As Covid restrictions eased up, we filmed more interviews and surveyed programs, filming sessions at MSRI workshops including the 2022 MSRI-UP and ADJOINT summer programs and the 2023 CIME Workshop on "Mentoring for Equity." Students and other participants in these programs are among those whose journeys we are tracking through 2024. In July 2023, we visited the home of 2022 MSRI-UP student Elijah Leake in Chicago to learn about his education and career plans as he finishes his undergraduate studies. Another innovative new workshop covered is the MSRI-UP inspired Mathematically Advancing Young Undergraduates Program (MAY-UP) hosted at the Atlanta University Center in May 2023. Coordinated by Duane Cooper from Morehouse College, MAY-UP is aimed at early undergraduates selected from Atlanta-area universities who are on the cusp of deciding whether to pursue advanced study in mathematics and careers in related fields.

Figure 5. William McNeill and Ashley James film with mathematicians Zerotti Woods and Shelby Wilson at NAM's Mathfest in 2022.

In May 2022, we filmed at Horace Mann, a junior high school in Los Angeles, where under the guidance of UCLA mathematician Wilfrid Gangbo, teachers designed a program for students to create and present posters they had created about pioneering African-American mathematicians. One pair of students chose Benjamin Banneker for their poster, another two picked Scott Williams.

Our last shoot in 2023 was at CAARMS, held this year at Purdue University in July, the first live version since the pandemic lockdowns. I first interviewed William Massey in 2001. He had organized the very first CAARMS meeting in 1995, and it was interesting to hear him again as he stressed his unwavering emphasis on maintaining the highest standards in mathematical research. His was an argument for quality over quantity. Among the presenters at CAARMS, we interviewed Abba Gumel, from the University of Maryland. His widely recognized work in applications of statistical models to epidemiology provide palpable evidence that mathematics is essential to solving real world problems. CAARMS also held a surprise for me, as I got to interview Scott Williams again, almost three years after we first met in Erie, Pennsylvania.

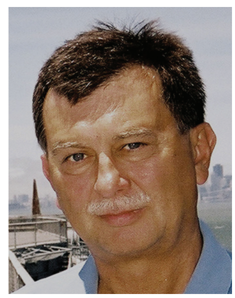
Figure 6. Mel Currie, now retired from the National Security Agency (NSA), recruited numerous African-American mathematicians to work at the agency while he was there. He also had a role in finding government support for several initiatives and scholarship programs that helped advance the careers and visibility of Black and other underrepresented groups in mathematics. He was interviewed at his home in Maryland.

We are now editing over 200 hours of material and plan to screen the first of two films at JMM 2024. *Journeys of Black Mathematicians, Part I* is subtitled *Forging Resilience* because so many of the stories told in it are about persistence in the face of adversity in a society where the legacies of slavery and racial discrimination are too often perceived as baked into the system. When she was a child, the school bus would not stop for Virginia Newell because she was Black. Freeman Hrabowski III, a retired president of the University of Maryland and one of the most influential proponents of mathematics education, was a member of the same church as one of the four little girls killed in an infamous church bombing in the South; he was arrested during the Civil Rights marches that followed. When Donald Cole tried to enroll in mathematics courses at the University of Mississippi as a freshman, he was dismissed. After working his way through an HBCU he was finally readmitted, got his PhD and retired as president of Ole Miss, having produced 15 students who earned PhDs in mathematics. At CAARMS 2023, Johnny Houston took us to the Black Cultural Center at Purdue University where he had earned his own PhD. He had fought for its creation when he was a student, a time when Black students were not allowed to reside in any of Purdue's dormitories and had to find housing on the other side of town.

Among the 51 people interviewed for the film project so far, there are many more equally compelling tales; too many to cram into two one-hour films. Some of the stories described above will be covered in the second film to be completed by late 2024, along with several programs such as EDGE, MAY-UP, ADJOINT, and CAARMS. The second film will continue tracking several of the students introduced in the first. It will no doubt contain surprises, but I expect there to be interesting lessons about the programs we've followed and their impacts on career choices in education and research and developments in mathematics.

It is my hope that once the two films are completed we will be able to showcase most of the individuals interviewed in separate biographical sequences in a series accessible to everyone via the web.

Short clips from several individuals and programs documented in the films can be viewed at: `http://www.zalafilms.com/jbm`.



George Csicsery

Credits

Photo of George Csicsery is courtesy of Nathaniel Dorsky.

AMS Prizes and Awards

NEW! I. Martin Isaacs Prize for Excellence in Mathematical Writing


I. Martin Isaacs

The I. Martin Isaacs Prize is awarded for excellence in writing of a research article published in a primary journal of the AMS in the past two years.

About this Prize

The prize focuses on the attributes of excellent writing, including clarity, grace, and accessibility; the quality of the research is implied by the article's publication in *Communications of the AMS*, *Journal of the AMS*, *Mathematics of Computation*, *Memoirs*, *Proceedings of the AMS*, or *Transactions of the AMS*, and is therefore not a prize selection criterion.

Professor Isaacs is the author of several graduate-level textbooks and of about 200 research papers on finite groups and their characters, with special emphasis on groups—such as solvable groups—that have an abundance of normal subgroups. He is a Fellow of the American Mathematical Society, and received teaching awards from the University of Wisconsin and from the School of Engineering at the University of Wisconsin. He is especially proud of his 29 successful PhD students.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/isaacs-prize`.

NEW! Elias M. Stein Prize for Transformative Exposition


Elias M. Stein

The Elias M. Stein Prize for Transformative Exposition is awarded for a written work, such as a book, survey, or exposition, *in any area of mathematics* that transforms the mathematical community's understanding of the subject or reshapes the way it is taught.

About this Prize

This prize was endowed in 2022 by students, colleagues, and friends of Elias M. Stein to honor his remarkable legacy of writing monographs and textbooks, both singly and with collaborators. Stein's research monographs, such as *Singular Integrals and Differentiability Properties of Functions* and *Harmonic Analysis*, became canonical references for generations of researchers, and textbooks such as the Stein and Shakarchi series *Princeton Lectures in Analysis* became instant classics in undergraduate and graduate classrooms. Stein is remembered for his ability to find a perspective to make a method of proof seem so natural as to be inevitable, and for his strategy of revealing the essential difficulties, and their solutions, in the simplest possible form before elaborating on more general settings. This prize seeks to recognize mathematicians at any career stage who, like Stein, have invested in writing a book or manuscript that transforms how their research community, or the next generation, understands the current state of knowledge in their area.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/stein-exposition`.

Mary P. Dolciani Prize for Excellence in Research

The AMS Mary P. Dolciani Prize for Excellence in Research recognizes a mathematician from a department that does not grant a PhD who has an active research program in mathematics and a distinguished record of scholarship. The primary criterion for the prize is an active research program as evidenced by a strong record of peer-reviewed publications.

Additional selection criteria may include the following:

- Evidence of a robust research program involving undergraduate students in mathematics;
- Demonstrated success in mentoring undergraduates whose work leads to peer-reviewed publication, poster presentations, or conference presentations;
- Membership in the AMS at the time of nomination and receipt of the award is preferred but not required.

About this Prize

This prize is funded by a grant from the Mary P. Dolciani Halloran Foundation. Mary P. Dolciani Halloran (1923–1985) was a gifted mathematician, educator, and author. She devoted her life to developing excellence in mathematics education and was a leading author in the field of mathematical textbooks at the college and secondary school levels.

The prize amount is $5000, awarded every other year for five award cycles.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Nominations should include a letter of nomination, the nominee's CV, and a short citation to be used in the event that the nomination is successful.

Information on how to nominate can be found here: `https://www.ams.org/dolciani-prize`.

Award for an Exemplary Program or Achievement in a Mathematics Department

This award recognizes a department which has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of society. Examples might include a department that runs a notable minority outreach program, a department that has instituted an unusually effective industrial mathematics internship program, a department that has promoted mathematics so successfully that a large fraction of its university's undergraduate population majors in mathematics, or a department that has made some form of innovation in its research support to faculty and/or graduate students, or which has created a special and innovative environment for some aspect of mathematics research.

About this Award

This award was established in 2004. For the first three awards (2006–2008), the prize amount was US$1,200. The prize was endowed by an anonymous donor in 2008, and starting with the 2009 prize, the amount is US$5,000. This US$5,000 prize is awarded annually. Departments of mathematical sciences in North America that offer at least a bachelor's degree in mathematical sciences are eligible.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: A letter of nomination may be submitted by one or more individuals. Nomination of the writer's own institution is permitted. The letter should describe the specific program(s) for which the department in being nominated as well as the achievements which make the program(s) an outstanding success, and may include any ancillary documents which support the success of the program(s). Where possible, the letter and documentation should address how these successes came about by 1) systematic, reproducible changes in programs that might be implemented by others, and/or 2) have value outside the mathematical community. The letter should not exceed two pages, with supporting documentation not to exceed an additional three pages.

Information on how to nominate can be found here: `https://www.ams.org/department-award`.

Award for Impact on the Teaching and Learning of Mathematics

This award is given annually to a mathematician (or group of mathematicians) who has made significant contributions of lasting value to mathematics education.

Priorities of the award include recognition of:

(a) accomplished mathematicians who have worked directly with precollege teachers to enhance teachers' impact on mathematics achievement for all students, or

(b) sustainable and replicable contributions by mathematicians to improving the mathematics education of students in the first two years of college.

About this Award

The Award for Impact on the Teaching and Learning of Mathematics was established by the AMS Committee on Education in 2013. The endowment fund that supports the award was established in 2012 by a contribution from Kenneth I. and Mary Lou Gross in honor of their daughters Laura and Karen.

The US$1,000 award is given annually.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Letters of nomination may be submitted by one or more individuals. The letter of nomination should describe the significant contributions made by the nominee(s) and provide evidence of the impact these contributions have made on the teaching and learning of mathematics. The letter of nomination should not exceed two pages, and may include supporting documentation not to exceed three additional pages. A brief curriculum vitae for each nominee should also be included. The nonwinning nominations will automatically be reconsidered, without further updating, for the awards to be presented over the next two years.

Information on how to nominate can be found here: `https://www.ams.org/impact`.

Ciprian Foias Prize in Operator Theory

The Ciprian Foias Prize in Operator Theory is awarded for notable work in Operator Theory published during the preceding six years. The work must be published in a recognized, peer-reviewed venue.

About this Prize

This prize was established in 2020 in memory of Ciprian Foias (1933–2020) by colleagues and friends. He was an influential scholar in operator theory and fluid mechanics, a generous mentor, and an enthusiastic advocate of the mathematical community.

The current prize amount is US$5,000, and the prize is awarded every three years.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Nominations require CV of the nominee, a letter of nomination, and a citation.

Information on how to nominate can be found here: `https://www.ams.org/foias-prize`.

David P. Robbins Prize

The Robbins Prize is for a paper with the following characteristics: it shall report on novel research in algebra, combinatorics, or discrete mathematics and shall have a significant experimental component; and it shall be on a topic which is broadly accessible and shall provide a simple statement of the problem and clear exposition of the work. Papers published within the six calendar years preceding the year in which the prize is awarded are eligible for consideration.

About this Prize

This prize was established in 2005 in memory of David P. Robbins by members of his family. Robbins, who died in 2003, received his PhD in 1970 from MIT. He was a long-time member of the Institute for Defense Analysis Center for Communications Research and a prolific mathematician whose work (much of it classified) was in discrete mathematics.

The current prize amount is US$5,000 and the prize is awarded every 3 years.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/robbins-prize`.

E. H. Moore Research Article Prize

The Moore Prize is awarded for an outstanding research article to have appeared in one of the AMS primary research journals (namely, the *Journal of the AMS*, *Proceedings of the AMS*, *Transactions of the AMS*, *Memoirs of the AMS*, *Mathematics of Computation*, *Electronic Journal of Conformal Geometry and Dynamics*, and *Electronic Journal of Representation Theory*) during the six calendar years ending a full year before the meeting at which the prize is awarded.

About this Prize

The prize was established in 2002 in honor of E. H. Moore. Among other activities, Moore founded the Chicago branch of the American Mathematical Society, served as the Society's sixth president (1901–1902), delivered the Colloquium Lectures in 1906, and founded and nurtured the *Transactions of the AMS*.

The current prize amount is US$5,000, awarded every three years.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/moore-prize`.

Leroy P. Steele Prize for Lifetime Achievement

The Steele Prize for Lifetime Achievement is awarded for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students.

About this Prize

These prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein, and are endowed under the terms of a bequest from Leroy P. Steele. From 1970 to 1976 one or more prizes were awarded each year for outstanding published mathematical research; most favorable consideration was given to papers distinguished for their exposition and covering broad areas of mathematics. In 1977 the Council of the AMS modified the terms under which the prizes are awarded. In 1993, the Council formalized the three categories of the prize by naming each of them: (1) The Leroy P. Steele Prize for Lifetime Achievement; (2) The Leroy P. Steele Prize for Mathematical Exposition; and (3) The Leroy P. Steele Prize for Seminal Contribution to Research.

The amount of this prize is US$10,000.

Next Prize: January 2025

Nomination Period: February 1 – March 31

Nomination Procedure: Nominations for the Steele Prize for Lifetime Achievement should include a letter of nomination, the nominee's CV, and a short citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

Information on how to nominate can be found here: `https://www.ams.org/steele-lifetime`.

Leroy P. Steele Prize for Mathematical Exposition

The Steele Prize for Mathematical Exposition is awarded for a book or substantial survey or expository research paper.

About this Prize

These prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein, and are endowed under the terms of a bequest from Leroy P. Steele. From 1970 to 1976 one or more prizes were awarded each year for outstanding published mathematical research; most favorable consideration was given to papers distinguished for their exposition and covering broad areas of mathematics. In 1977 the Council of the AMS modified the terms under which the prizes are awarded. In 1993, the Council formalized the three categories of the prize by naming each of them: (1) The Leroy P. Steele Prize for Lifetime Achievement; (2) The Leroy P. Steele Prize for Mathematical Exposition; and (3) The Leroy P. Steele Prize for Seminal Contribution to Research.

The amount of this prize is US$5,000.

Next Prize: January 2025

Nomination Period: February 1 – March 31

Nomination Procedure: Nominations for the Steele Prizes for Mathematical Exposition should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

Information on how to nominate can be found here: `https://www.ams.org/steele-exposition`.

Leroy P. Steele Prize for Seminal Contribution to Research

The Steele Prize for Seminal Contribution to Research is awarded for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research.

Special Note: The Steele Prize for Seminal Contribution to Research is awarded according to the following six-year rotation of subject areas:

1. Analysis/Probability (2020)
2. Algebra/Number Theory (2021)
3. Applied Mathematics (2022)
4. Geometry/Topology (2023)
5. Discrete Mathematics/Logic (2024)
6. **Open (2025)**

About this Prize

These prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein, and are endowed under the terms of a bequest from Leroy P. Steele. From 1970 to 1976 one or more prizes were awarded each year for outstanding published mathematical research; most favorable consideration was given to papers distinguished for their exposition and covering broad areas of mathematics. In 1977 the Council of the AMS modified the terms under which the prizes are awarded. In 1993, the Council formalized the three categories of the prize by naming each of them: (1) The Leroy P. Steele Prize for Lifetime Achievement; (2) The Leroy P. Steele Prize for Mathematical Exposition; and (3) The Leroy P. Steele Prize for Seminal Contribution to Research.

The amount of this prize is US$5,000.

Next Prize: January 2025

Nomination Period: February 1–March 31

Nomination Procedure: Nominations for the Steele Prize for Seminal Contribution to Research should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful.

Information on how to nominate can be found here: `https://www.ams.org/steele-research`.

Levi L. Conant Prize

This prize was established in 2000 in honor of Levi L. Conant to recognize the best expository paper published in either the *Notices of the AMS* or the *Bulletin of the AMS* in the preceding five years.

About this Prize

Levi L. Conant was a mathematician and educator who spent most of his career as a faculty member at Worcester Polytechnic Institute. He was head of the mathematics department from 1908 until his death and served as interim president of WPI from 1911 to 1913. Conant was noted as an outstanding teacher and an active scholar. He published a number of articles in scientific journals and wrote four textbooks. His will provided for funds to be donated to the AMS upon the death of his wife.

Prize winners are invited to present a public lecture at Worcester Polytechnic Institute as part of their Levi L. Conant Lecture Series, which was established in 2006.

The Conant Prize is awarded annually in the amount of US$1,000.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Nominations with supporting information should be submitted online. Nominations should include a letter of nomination, a short description of the work that is the basis of the nomination a complete bibliographic citation for the article being nominated.

Information on how to nominate can be found here: `https://www.ams.org/conant-prize`.

Mathematics Programs that Make a Difference

This Award for Mathematics Programs that Make a Difference was established in 2005 by the AMS's Committee on the Profession to compile and publish a series of profiles of programs that:

1. aim to bring more persons from underrepresented backgrounds into some portion of the pipeline beginning at the undergraduate level and leading to advanced degrees in mathematics and professional success, or retain them once in the pipeline;
2. have achieved documentable success in doing so; and
3. are potentially replicable models.

About this Award

This award brings recognition to outstanding programs that have successfully addressed the issues of underrepresented groups in mathematics. Examples of such groups include racial and ethnic minorities, women, low-income students, and first-generation college students.

One program is selected each year by a selection committee appointed by the AMS president and is awarded US$1,000 provided by the Mark Green and Kathryn Kert Green Fund for Inclusion and Diversity.

Preference is given to programs with significant participation by underrepresented minorities. Note that programs aimed at pre-college students are eligible only if there is a significant component of the program benefiting individuals from underrepresented groups at or beyond the undergraduate level. Nomination of one's own institution or program is permitted and encouraged.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: The letter of nomination should describe the specific program being nominated and the achievements that make the program an outstanding success. It should include clear and current evidence of that success. A strong nomination typically includes a description of the program's activities and goals, a brief history of the program, evidence of its effectiveness, and statements from participants about its impact. The letter of nomination should not exceed two pages, with supporting documentation not to exceed three more pages. Up to three supporting letters may be included in addition to these five pages. Nomination of the writer's own institution or program is permitted. Nonwinning nominations will automatically be reconsidered for the award for the next two years.

Information on how to nominate can be found here: `https://www.ams.org/make-a-diff-award`.

Oswald Veblen Prize in Geometry

The award is made for a notable research work in geometry or topology that has appeared in the last six years. The work must be published in a recognized, peer-reviewed venue.

About this Prize

This prize was established in 1961 in memory of Professor Oswald Veblen through a fund contributed by former students and colleagues. The fund was later doubled by the widow of Professor Veblen. An anonymous donor generously augmented the fund in 2008. In 2013, in honor of her late father, John L. Synge, who knew and admired Oswald Veblen, Cathleen Synge Morawetz and her husband, Herbert, substantially increased the endowment.

The current prize amount of US$5,000 is awarded every three years.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/Veblen-prize`.

Ruth Lyttle Satter Prize in Mathematics

The Satter Prize recognizes an outstanding contribution to mathematics research by a woman in the previous six years.

About this Prize

This prize was established in 1990 using funds donated by Joan S. Birman in memory of her sister, Ruth Lyttle Satter. Professor Birman requested that the prize be established to honor her sister's commitment to research and to encourage women in science. An anonymous benefactor added to the endowment in 2008.

The current prize amount is $5,000 and the prize is awarded every 2 years.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/satter-prize`.

AMS-Simons Travel Grants

The AMS-Simons Travel Grant program acknowledges the importance of research interaction and collaboration in mathematics and aims to facilitate these activities for recent PhD recipients. AMS-Simons Travel Grants are administered by the AMS with support from the Simons Foundation. These grants provide support for committed researchers who have limited opportunities for travel and conferences and for collaborative work. For the 2024–2025 award cycle, each grant will provide an early-career mathematician with $3,000 per year for two years to be used for research-related travel. Annual discretionary funds for the enhancement of a grantee's department will be available to institutions that administer the grant on behalf of the AMS. No additional institutional overhead or indirect costs will be covered with these award funds.

About this Grant

Eligible applicants for the 2024–2025 application cycle are early-career mathematicians who are located in the United States (or are US citizens employed outside the United States) and who have completed the PhD (or its equivalent) within the last four years (between April 1, 2020, and June 30, 2024, inclusive).

The applicant's research must be in a disciplinary research area supported by the Division of Mathematical Sciences at the National Science Foundation. Previous AMS-Simons Travel Grant recipients and early-career mathematicians who already receive substantial external funding for research and travel exceeding $3,000 per year (such as from the National Science Foundation) are not eligible to apply.

Recipients may use grant funds for research-related travel, such as travel to a conference, a university, or an institute, or to visit a collaborator. Funds may also be used for a collaborator to visit the grantee to engage in research activities. Other research-related travel may be supported, subject to the approval of the grantee's mentor. Detailed guidelines will be provided to the grantee. Only eligible travel expenses that have advance approval from the grantee's mentor will be reimbursed.

Application Period: Applications will be collected via MathPrograms.org February 15, 2024–March 31, 2024 (11:59 p.m. EST). Find more application information at `https://www.ams.org/AMS-SimonsTG`. For questions, contact the Programs Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294; `ams-simons@ams.org`.

AMS-Simons Research Enhancement Grants for Primarily Undergraduate Institution (PUI) Faculty

With generous funding from the Simons Foundation; the AMS; and Eve, Kirsten, Lenore, and Ada of the Menger family, the AMS-Simons Research Enhancement Grants for Primarily Undergraduate Institution (PUI) Faculty program was established in 2023 to foster and support research collaboration by mathematicians employed full-time at colleges and universities that do not award doctoral degrees in mathematics. Each year for three years, grantees will receive $3,000 to support research-related activities. Annual discretionary funds for a grantee's department and administrative funds for a grantee's institution will be available to institutions that administer the grant on behalf of the AMS. No additional institutional overhead or indirect costs will be covered with these award funds.

About this Grant

Mathematicians with an active research program employed full-time in tenured or tenure-track positions at PUIs in the United States are eligible to apply. For the purpose of this program, PUI institutions are those that do not confer doctoral degrees in mathematics. Additionally, to be eligible, applicants must have earned a PhD degree at least five years before the start of the grant. For the 2024 application cycle, applicants must have earned a PhD degree prior to August 1, 2019.

The applicant's research must be in a disciplinary research area supported by the Division of Mathematical Sciences of the National Science Foundation. Faculty with appointments solely in statistics departments are not eligible. The grantees may not concurrently hold external research funding exceeding $3,000 per year and may not be in residence at a National Science Foundation institute.

Activities that will further the grantee's research program are allowed. These expenses include but are not limited to conference participation, institute visits, collaboration travel (grantee or collaborator), computer equipment or software, family-care expenses, hiring a teaching assistant, publication expenses, stationery, supplies, books, and membership fees to professional organizations. During the three-year funding period, the grantee may spend up to $2,500 on electronic devices to support their research activities.

Application Period: Applications will be collected via MathPrograms.org January 9, 2024–March 18, 2024 (11:59 p.m. EST). Find more application information at `https://www.ams.org/AMS-Simons-PUI-Research`. For questions, contact the Programs Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294; `ams-simons-pui@ams.org`.

Fellows of the American Mathematical Society

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics.

AMS members may be nominated for this honor during the nomination period which occurs in February and March each year. Selection of new Fellows (from among those nominated) is managed by the AMS Fellows Selection Committee, comprised of 12 members of the AMS who are also Fellows. Those selected are subsequently invited to become Fellows and the new class of Fellows is publicly announced each year on November 1.

Learn more about the qualifications and process for nomination at `www.ams.org/profession/ams-fellows`.

Joint Prizes and Awards

Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student

(AMS-MAA-SIAM)

The Morgan Prize is awarded each year to an undergraduate student (or students for joint work) for outstanding research in mathematics. Any student who was enrolled as an undergraduate in December at a college or university in the United States or its possessions, Canada, or Mexico is eligible for the prize.

The prize recipient's research need not be confined to a single paper; it may be contained in several papers. However, the paper (or papers) to be considered for the prize must be completed while the student is an undergraduate. Publication of research is not required.

About this Prize

The prize was established in 1995. It is entirely endowed by a gift from Mrs. Frank (Brennie) Morgan. It is made jointly by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

The current prize amount is $1,200, awarded annually.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: To nominate a student, submit a letter of nomination, a brief description of the work that is the basis of the nomination, and complete bibliographic citations (or copies of unpublished work). All submissions for the prize must include at least one letter of support from a person, usually a faculty member, familiar with the student's research.

Information on how to nominate can be found here: `https://www.ams.org/morgan-prize`.

JPBM Communications Award

This award is given each year to reward and encourage communicators who, on a sustained basis, bring mathematical ideas and information to non-mathematical audiences.

About this Award

This award was established by the Joint Policy Board for Mathematics (JPBM) in 1988. JPBM is a collaborative effort of the American Mathematical Society, the Mathematical Association of America, the Society for Industrial and Applied Mathematics, and the American Statistical Association.

Up to two awards of US$2,000 are made annually. Both mathematicians and non-mathematicians are eligible.

Next Prize: January 2025

Nomination Period: open

Nomination Procedure: Nominations should be submitted on mathprograms.org. Note: Nominations collected before September 15th in year N will be considered for an award in year N+2.

Information on how to nominate can be found here: `https://www.ams.org/jpbm-comm-award`.

AMS-SIAM Norbert Wiener Prize in Applied Mathematics

The Wiener Prize is awarded for an outstanding contribution to "applied mathematics in the highest and broadest sense."

About this Prize

This prize was established in 1967 in honor of Professor Norbert Wiener and was endowed by a fund from the Department of Mathematics of the Massachusetts Institute of Technology. The endowment was further supplemented by a generous donor.

Since 2004, the US$5,000 prize has been awarded every three years. The American Mathematical Society and the Society for Industrial and Applied Mathematics award this prize jointly; the recipient must be a member of one of these societies.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/wiener-prize`.

2024 MOS–AMS Fulkerson Prize

The Fulkerson Prize Committee invites nominations for the Delbert Ray Fulkerson Prize, sponsored jointly by the Mathematical Optimization Society (MOS) and the American Mathematical Society (AMS). Up to three awards of US$1,500 each are presented at each (triennial) International Symposium of the MOS. The Fulkerson Prize is for outstanding papers in the area of discrete mathematics. The prize will be awarded at the 25th International Symposium on Mathematical Programming to be held in Montreal, Canada, in the summer of 2024.

Eligible papers should represent the final publication of the main result(s) and should have been published in a recognized journal or in a comparable, well-refereed volume intended to publish final publications only, during the six calendar years preceding the year of the Symposium (thus, from January 2018 through December 2023). The prizes will be given for single papers, not series of papers or books, and in the event of joint authorship the prize will be divided.

The term "discrete mathematics" is interpreted broadly and is intended to include graph theory, networks, mathematical programming, applied combinatorics, applications of discrete mathematics to computer science, and related subjects. While research work in these areas is usually not far removed from practical applications, the judging of papers will be based only on their mathematical quality and significance.

Previous winners of the Fulkerson Prize are listed here: `www.mathopt.org/?nav=fulkerson#winners`.

Further information about the Fulkerson Prize can be found at `www.mathopt.org/?nav=fulkerson` and `https://www.ams.org/fulkerson-prize`.

The Fulkerson Prize Committee consists of

- Julia Böttcher (London School of Economics), MOS Representative
- Rosa Orellana (Dartmouth College), AMS Representative
- Dan Spielman (Yale University), Chair and MOS Representative

Please send your nominations (including reference to the nominated article and an evaluation of the work) by February 15, 2024 to the chair of the committee:

Professor Daniel Spielman
Email: `daniel.spielman@yale.edu`

Credits

Photo of I. Martin Isaacs is courtesy of Yvonne Nagel.
Photo of Elias M. Stein is courtesy of William Crow/Princeton University.

American Mathematical Society

Policy on a Welcoming Environment

(as adopted by the January 2015 AMS Council and modified by the January 2019 AMS Council)

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, veteran status, or immigration status.

Harassment is a form of misconduct that undermines the integrity of AMS activities and mission.

The AMS will make every effort to maintain an environment that is free of harassment, even though it does not control the behavior of third parties. A commitment to a welcoming environment is expected of all attendees at AMS activities, including mathematicians, students, guests, staff, contractors and exhibitors, and participants in scientific sessions and social events. To this end, the AMS will include a statement concerning its expectations towards maintaining a welcoming environment in registration materials for all its meetings, and has put in place a mechanism for reporting violations. Violations may be reported confidentially and anonymously to **855.282.5703** or at **www.mathsociety.ethicspoint.com**. The reporting mechanism ensures the respect of privacy while alerting the AMS to the situation.

For AMS policy statements concerning discrimination and harassment, see the **AMS Anti-Harassment Policy.**

Questions about this welcoming environment policy should be directed to the **AMS Secretary.**

AMS American Mathematical Society
Advancing research. Creating connections.

2023 Election

Election Results

In the elections of 2023, the Society elected a president, a vice president, a trustee, five members at large of the Council, three members of the Nominating Committee, and two members of the Editorial Boards Committee. The membership also adopted bylaws changes.

President-elect

Photo credit: Rod Searcey

Ravi Vakil
Stanford University

Term is one year as President-elect
(February 1, 2024–January 31, 2025)
Term is two years as President
(February 1, 2025–January 31, 2027)
Term is one year as Immediate Past President
(February 1, 2027–January 31, 2028)

Vice President

Photo provided by Irene Fonseca

Irene Fonseca
Carnegie Mellon University

Term is three years
(February 1, 2024–January 31, 2027)

Board of Trustees

Photo provided by Duke University Communications

Jonathan Mattingly
Duke University

Term is five years
(February 1, 2024–January 31, 2029)

Members at Large of the Council

Term is three years *(February 1, 2024–January 31, 2027)*

Photo provided by Matilde Lalín

Matilde Lalín
Université de Montréal

Photo provided by Jennifer Murawski, Communications

Naiomi T. Cameron
Spelman College

Photo provided by Sara Maloni

Sara Maloni
University of Virginia

Photo provided by Andrew Putman

Andrew Putnam
University of Notre Dame

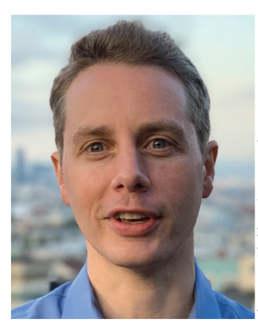


Photo provided by Brian Kennedy

John Voight
Dartmouth College

Nominating Committee

Term is three years *(January 1, 2024–December 31, 2026)*


Photo provided by Donatella Danielli

Donatella Danielli
Arizona State University


Photo provided by Idriss Njike

Michael Hill
University of California, Los Angeles


Photo provided by David Savitt

David Savitt
Johns Hopkins University

Editorial Boards Committee

Term is three years *(February 1, 2024–January 31, 2027)*


Photo provided by Ben Green

Ben Green
University of Oxford


Photo provided by Lucille Meci

Robert Lazarsfeld
Stony Brook University

All bylaws amendments passed. These are described in the election section of the September 2023 issue of the *Notices* (p. 1310, available at `https://www.ams.org/journals/notices/202308/rnoti-p1310.pdf`). The resulting current version of the bylaws is available at `https://www.ams.org/bylaws`.

2024 *Election*

Nominations by Petition

Vice President or Member at Large

One position of vice president and member of the Council *ex officio* for a term of three years is to be filled in the election of 2024. The Council intends to nominate at least two candidates, among whom may be candidates nominated by petition as described in the rules and procedures below.

Five positions of member at large of the Council for a term of three years are to be filled in the same election. The Council intends to nominate at least ten candidates, among whom may be candidates nominated by petition in the manner described in the rules and procedures below.

Petitions are presented to the Council, which, according to Section 2 of Article VII of the bylaws, makes the nominations.

Prior to presentation to the Council, petitions in support of a candidate for the position of vice president or of member at large of the Council must have at least fifty valid signatures and must conform to several rules and procedures, which are described below. Petitioners can facilitate the procedure by accompanying the petitions with a signed statement from the candidate giving consent.

Editorial Boards Committee

Two places on the Editorial Boards Committee will be filled by election. There will be four continuing members of the Editorial Boards Committee.

The president will name at least four candidates for these two places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate's assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and procedures, described below, should be followed.

Nominating Committee

Three places on the Nominating Committee will be filled by election. There will be six continuing members of the Nominating Committee.

The president will name at least six candidates for these three places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate's assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and procedures, described below, should be followed.

Rules and Procedures

Use separate copies of the form for each candidate for vice president, member at large, or member of the Nominating or Editorial Boards Committees.

1. To be considered, petitions must be addressed to Secretary, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2213, USA, and must arrive by 24 February 2024.
2. The name of the candidate must be given as it appears in the American Mathematical Society's membership records and must be accompanied by the member code. If the member code is not known by the candidate, it may be obtained by the candidate contacting the AMS headquarters in Providence (`amsmem@ams.org`).
3. The petition for a single candidate may consist of several sheets each bearing the statement of the petition, including the name of the position, and signatures. The name of the candidate must be exactly the same on all sheets.
4. On the next page is a sample form for petitions. Petitioners may make and use photocopies or reasonable facsimiles.
5. A signature is valid when it is clearly that of the member whose name and address is given in the left-hand column.
6. When a petition meeting these various requirements appears, the secretary will ask the candidate to indicate willingness to be included on the ballot.

Nominations by Petition

The undersigned members of the American Mathematical Society propose the name of

______________________________ as a candidate for the position of (check one):

☐ **Vice President** (*term beginning 02/01/2025*)
☐ **Member at Large of the Council** (*term beginning 02/01/2025*)
☐ **Member of the Nominating Committee** (*term beginning 01/01/2025*)
☐ **Member of the Editorial Boards Committee** (*term beginning 02/01/2025*)

of the American Mathematical Society.

Return petitions by February 24, 2024 to:
Secretary, AMS, 201 Charles Street, Providence, RI 02904-2213, USA

Name, address, and AMS member code, if available (printed or typed)

Name, address, and AMS member code	
	Signature
	Signature
	Signature
	Signature
	Signature
	Signature

2024 Class of Fellows of the AMS

Forty mathematical scientists from around the world have been named Fellows of the American Mathematical Society (AMS) for 2024.

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics. Among the goals of the program are to create an enlarged class of mathematicians recognized by their peers as distinguished for their contributions to the profession and to honor excellence.

Names of the individuals who are in this year's class and their institutions appear below.

The nomination period for Fellows is open each year from February 1 to March 31. For additional information about the Fellows program, as well as instructions for making nominations, visit the web page `https://www.ams.org/ams-fellows`.

Agnès Beaudry, University of Colorado, Boulder

Grigoriy Blekherman, Georgia Institute of Technology

Anders Skovsted Buch, Rutgers University

Erika Tatiana Camacho, University of Texas at San Antonio and Arizona State University

Maria Chudnovsky, Princeton University

Caterina (Katia) Consani, Johns Hopkins University

Xianzhe Dai, University of California, Santa Barbara

Athanasios Fokas, University of Cambridge and University of Southern California

Paul Hacking, University of Massachusetts, Amherst

Leslie Hogben, Iowa State University and American Institute of Mathematics

Lan-Hsuan Huang, University of Connecticut, Storrs

Daniel Isaksen, Wayne State University

Palle E. T. Jorgensen, University of Iowa

Autumn Exum Kent, University of Wisconsin, Madison

Minhyong Kim, International Centre for Mathematical Sciences, Edinburgh and Korea Institute for Advanced Study

Jean-François Lafont, Ohio State University, Columbus

Thang Le, Georgia Institute of Technology

Doron Levy, University of Maryland

Chun Liu, Illinois Institute of Technology

Eugenia Malinnikova, Stanford University and Norwegian University of Science and Technology

Michelle Manes, American Institute of Mathematics and University of Hawaiʻi at Mānoa

Javad Mashreghi, Laval University

Dorina Mitrea, Baylor University

Evgeny Mukhin, Indiana University, Indianapolis

Qing Nie, University of California, Irvine

Kasso A. Okoudjou, Tufts University

Alex Pothen, Purdue University

Eric Todd Quinto, Tufts University

Kasra Rafi, University of Toronto

Amit Sahai, University of California, Los Angeles

Christian Schnell, Stony Brook University

Luis Silvestre, University of Chicago

Slawomir Solecki, Cornell University

Jian Song, Rutgers University

Gabor Szekelyhidi, Northwestern University

Cynthia Vinzant, University of Washington

Monica Visan, University of California, Los Angeles

Jiaping Wang, University of Minnesota

Xin Zhou, Cornell University

Kehe Zhu, State University of New York at Albany

Credits

Photo of Erika Tatiana Camacho is courtesy of Erika Tatiana Camacho.
Photo of Leslie Hogben is courtesy of Iowa State University.
Photo of Palle E. T. Jorgensen is courtesy of Palle E. T. Jorgensen.
Photo of Minhyong Kim is courtesy of The KAOS Foundation, South Korea.
Photo of Doron Levy is courtesy of University of Maryland/Lisa Helfert.
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AMS COMMUNICATION

Homotopical Combinatorics

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The authors of this piece are organizers of the 2024 AMS Mathematics Research Communities summer conference *Homotopical Combinatorics*, one of four topical research conferences offered this year that are focused on collaborative research and professional development for early-career mathematicians. Additional information can be found at `https://www.ams.org/programs/research-communities/2024MRC-HomotopicalComb`. Applications are open until February 15, 2024.

Homotopical combinatorics is an emerging field that studies combinatorial structures encoding aspects of equivariant homotopy theory, equivariant algebra, and abstract homotopy theory. Its methods—a pleasant mix of enumerative combinatorics, algebraic combinatorics, and order theory—are relatively elementary, but its theorems have deep implications in homotopy theory. The youth and accessibility of homotopical combinatorics should make the field especially attractive to early career researchers, and we hope that this article and the 2024 Mathematics Research Community by the same name welcome mathematicians from many backgrounds into the field.

The central object in homotopical combinatorics is the *transfer system*. These combinatorial gadgets were originally defined in order to encode the homotopy theory of N_∞ operads, which control multiplicative structures in equivariant stable homotopy theory. Special pairs of transfer systems control the structure of bi-incomplete Tambara functors, basic objects of equivariant algebra. In a seemingly unrelated direction, pairs of transfer systems also encode model structures (presentations of $(\infty, 1)$-categories) on posets. Below, we introduce transfer systems in purely combinatorial terms, and then explore their applications.

Transfer Systems

Suppose $(P, \leq)$ is a finite partially ordered set (poset). A (*categorical*) *transfer system* on $(P, \leq)$ is a partial order $\to$ on the set P such that

- $\to$ refines $\leq$: $x \to y$ implies $x \leq y$, and
- $\to$ is closed under restriction: $x \to y$, $z \leq y$, and w maximal among $w' \leq x, z$ implies $w \to z$.

In most cases, we restrict attention to finite posets admitting greatest lower bounds (so-called *meet-semilattices*). We write $x \wedge y$ for the greatest lower bound (or *meet*) of x, y when it exists. When P is a meet-semilattice, the restriction condition becomes simpler:

- $x \to y$ and $z \leq y$ implies $x \wedge z \to z$.

Categorically inclined readers will recognize this condition as closure under pullbacks, and it is pleasant to record diagramatically, where solid arrows are relations in the transfer system, dashed arrows represent $\leq$, and the double arrow indicates logical implication; we draw the diagram

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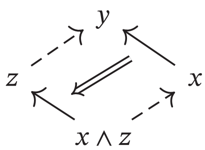
"oriented upwards" so it is also reminiscent of a Hasse diagram, see below.



We write $\operatorname{Tr} P$ for the collection of all transfer systems on P. The set $\operatorname{Tr} P$ admits a natural partial order by refinement: $\to \,\leq\, \twoheadrightarrow$ if and only if $x \to y$ implies $x \twoheadrightarrow y$. If P is a finite lattice (admits least upper and greatest lower bounds), then $\operatorname{Tr} P$ is a finite lattice as well.

One of the fundamental problems of transfer systems is to determine the structure of the lattice $\operatorname{Tr} P$ for a given lattice P or family of lattices. In [BBR21], Balchin–Barnes–Roitzheim achieve this for $P = [n] = \{0 < 1 < \cdots < n\}$ a finite chain. They prove that $\operatorname{Tr} P$ is isomorphic to the famed *Tamari lattice* $\mathcal{A}_{n+1}$ of planar rooted binary trees with $n+2$ leaves; see Figure 1. In particular, transfer systems on $[n]$ are counted by Catalan numbers, with

$$|\operatorname{Tr}[n]| = \operatorname{Cat}(n+1) = \frac{1}{2n+3}\binom{2n+3}{n+1}.$$

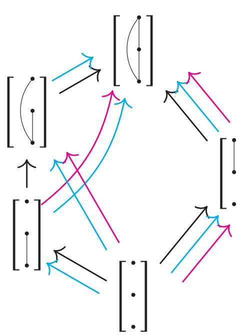


Figure 1. In brackets, we display the five elements of $\operatorname{Tr}[2]$. The elements of $[2]$ are arranged vertically as dots (0 lowest, 2 highest), and each transfer system is depicted by lines indicating relations present in the transfer system, omitting reflexive loops $x \to x$. The black arrows represent the covering (*i.e.*, minimal) relations of $\operatorname{Tr}[2]$; they assemble into a pentagon isomorphic to $\mathcal{A}_3$. The rest of the diagram should be interpreted after the reader engages with the *Model structures on posets* section. The blue arrows correspond to $\preccurlyeq$, and the magenta arrows are the covering relations of $\sqsubseteq$. Counting black, blue, and magenta intervals, we see that $|\operatorname{Pre}[2]| = 13$, $|\operatorname{Pre}^{cc}[2]| = 12$, and $|\operatorname{MS}([2])| = 10$.

There are also some general structural results on transfer systems. In Construction 2.9 of [BMO23], Balchin–MacBrough–Ormsby give a recursion for $|\operatorname{Tr} P|$ in terms of transfer systems on certain induced subposets. The recursion is based on the notion of the *minimal fibrant* element of a transfer system $\to$, *i.e.*, the (necessarily unique) minimal element m of P such that $m \to \top$, where $\top$ denotes the maximum of P. In [BHK+23], the participants in the 2023 Electronic Computational Homotopy Theory REU follow an idea of Hill to relativize minimal fibrancy, resulting in a *characteristic function* $\chi^{\to} : P \to P$ defined by $\chi^{\to}(x) = \min\{y \in P \mid y \to x\}$. This ultimately provides a strong (but far from tight) lower bound on the cardinality of transfer systems. To state the theorem, let $\operatorname{End}^{\circ} P$ denote the set of *interior operators* on P, that is, order-preserving functions $f : P \to P$ that are *contractive* ($f(x) \leq x$) and *idempotent* ($f(f(x)) = f(x)$). We give $\operatorname{End}^{\circ} P$ the pointwise ordering $f \leq g \iff f(x) \leq g(x)$ for all $x \in P$.

Theorem 1 (Theorems 2.8 and 2.12 of [BHK+23]). *The assignment*

$$\begin{aligned} \chi : \operatorname{Tr} P &\longrightarrow \operatorname{End}(P) \\ \to &\longmapsto \chi^{\to} \end{aligned}$$

is an order-reversing map with image $\operatorname{End}^{\circ} P$.

While interior operators are hard to enumerate, their asymptotic behavior is understood, and Kleitman [Kle76] proves that the base-2 logarithm of $|\operatorname{End}^{\circ}([1]^n)|$ grows like $\binom{n}{\lfloor n/2 \rfloor}$ (see OEIS A102896).

In order to prepare for applications in equivariant homotopy theory, let G be a finite group. We will take particular interest in the case $P = \operatorname{Sub} G$, the lattice of subgroups of G ordered under inclusion. (Note: If $G = C_{p^n}$, the cyclic group of order p^n, p prime, then $\operatorname{Sub} G \cong [n]$. This is the orginal context of [BBR21].) We will need, though, to introduce one additional axiom in this context: A *G-transfer system* is a categorical transfer system $\to$ on $\operatorname{Sub} G$ such that

- $\to$ is closed under conjugation: $H \to K$ implies ${}^gH \to {}^gK$,

where ${}^gH \coloneqq gHg^{-1}$ is the g-conjugate of H. We write $\operatorname{Tr} G$ for the lattice of G-transfer systems under refinement. Of course, if G is Abelian, then G-transfer systems and categorical transfer systems on $\operatorname{Sub} G$ are identical.

Despite their elementary and relatively natural definition, the authors are not aware of any appearance of such structures on posets prior to [Rub21, BBR21]. If any reader has encountered objects isomorphic to transfer systems in older (presumably combinatorial or order-theoretic) literature, we invite them to contact us.

N_∞ Operads

Transfer systems first arose through the work of Blumberg–Hill [BH15] on N_∞ operads. These are equivariant generalizations of E_∞ operads, and their algebras are equipped with both an operation that is associative and commutative up to coherent homotopies (coming from an E_∞ structure) and homotopy coherent multiplicative norm maps (encoded by the fixed points of the spaces in the operad).

Ever since their appearance in the Hill–Hopkins–Ravenel [HHR16] solution of the Kervaire invariant one problem, norms have become a critical component of contemporary equivariant homotopy theory. Each N_∞ operad encodes potentially different classes of norms, and thus we need to classify N_∞ operads if we hope to understand what norms might appear in applications.

Let G be a finite group and let $\mathfrak{S}_n$ denote the symmetric group on n letters. A *G-operad* $\mathscr{O}$ is a sequence of $G\times\mathfrak{S}_n$-spaces $\mathscr{O}(n)$, $n\geq 0$ along with an identity element $1\in\mathscr{O}(1)$ fixed by $G = G\times\mathfrak{S}_1$ and a G-equivariant composition map

$$\mathscr{O}(k)\times\mathscr{O}(n_1)\times\cdots\times\mathscr{O}(n_k)\to\mathscr{O}(n_1+\cdots+n_k)$$

satisfying the standard compatibility conditions for an operad. A *map* of G-operads is a morphism of operads in G-spaces; in particular, at level n it is $G\times\mathfrak{S}_n$-equivariant.

A *G-N_∞ operad* (or just *N_∞ operad* if G is clear from context) is a G-operad such that

- $\mathscr{O}(0)$ is G-contractible,
- the action of $\mathfrak{S}_n = e\times\mathfrak{S}_n$ on $\mathscr{O}(n)$ is free,
- for all $\Gamma\leq G\times\mathfrak{S}_n$, the Γ-fixed point space $\mathscr{O}(n)^\Gamma$ is either contractible or empty, and
- for each n, the collection of $\Gamma\leq G\times\mathfrak{S}_n$ such that $\mathscr{O}(n)^\Gamma\simeq *$ is closed under conjugacy and under passage to subgroups[1] and contains all subgroups of the form $H\times e$.

The category of G-N_∞ operads is denoted $N_\infty\text{-}\mathbf{Op}^G$.

A G-operad map $\varphi\colon\mathscr{O}_1\to\mathscr{O}_2$ of N_∞ operads is a *weak equivalence* when it induces a weak homotopy equivalence $\mathscr{O}_1(n)^\Gamma\to\mathscr{O}_2(n)^\Gamma$ for all $n\geq 0$ and all $\Gamma\leq G\times\mathfrak{S}_n$. Inverting weak equivalences in $N_\infty\text{-}\mathbf{Op}^G$ produces the *homotopy category of G-N_∞ opeards* $\mathrm{Ho}(N_\infty\text{-}\mathbf{Op}^G)$.

If $H\leq G$ and T is a finite H-set, we say that an N_∞ operad $\mathscr{O}$ *admits a T-norm* when $\mathscr{O}(|T|)^{\Gamma(T)}\simeq *$, where $\Gamma(T)\leq G\times\mathfrak{S}_{|T|}$ is the graph of some permutation representation $H\to\mathfrak{S}_{|T|}$ of T. If X is an $\mathscr{O}$-algebra[2] (say in G-spaces) and $\mathscr{O}$ admits T-norms, then we get a G-equivariant map

$$G\times_H X^T\longrightarrow X,$$

where X^T is the H-space of all functions $f\colon T\to X$ with H acting via $h\cdot f\colon t\mapsto hf(h^{-1}t)$. In particular, if $K\leq H\leq G$, then an H/K-norm induces a 'wrong-way' map

$$X^K\to X^H$$

between fixed point spaces. In an additive setting, these maps are called transfers instead of norms, leading to the nomenclature for transfer systems.

To draw out this connection further, let $\mathscr{O}$ denote a G-N_∞ operad and define a binary relation $\xrightarrow{\mathscr{O}}$ on $\mathrm{Sub}\,G$ by the rule

$$K\xrightarrow{\mathscr{O}}H \iff K\leq H \text{ and } \mathscr{O}([H:K])^{\Gamma(H/K)}\simeq *.$$

In other words, $K\xrightarrow{\mathscr{O}}H$ if and only if $\mathscr{O}$ admits H/K-norms. Of course, $\xrightarrow{\mathscr{O}}$ turns out to be a G-transfer system, and this assignment is part of a functor from G-N_∞ operads to (the category induced by) the lattie $\mathrm{Tr}\,G$. The work of many authors [BH15, GW18, BP21, Rub21, BBR21] gives the following theorem:

Theorem 2. *The assignment $\mathscr{O}\mapsto\xrightarrow{\mathscr{O}}$ induces an equivalence of categories*

$$\mathrm{Ho}(N_\infty\text{-}\mathbf{Op}^G)\xrightarrow{\simeq}\mathrm{Tr}\,G,$$

where $\mathrm{Tr}\,G$ is viewed as the category with objects G-transfer systems and a unique morphism between transfer systems if and only if the source refines the target.

This provides a first and pressing motivation for studying transfer systems: by determining the structure of $\mathrm{Tr}\,G$, we solve a classification problem for G-N_∞ operads; if we know all the G-transfer systems, then we know exactly which collections of norms are induced by N_∞ operads.

At the time of writing, the full structure of $\mathrm{Tr}\,G$ is known for the following finite groups G (p,q,r distinct primes): C_{p^n} [BBR21], C_{pq}, $C_2\times C_2$, Q_8, $\mathfrak{S}_3$ [Rub21], C_{pqr} [BBPR20], and $C_p\times C_p$ [BHK⁺23]. Additionally, Balchin–MacBrough–Ormsby [BMO23] determine elaborate interleaved recurrences which effectively compute $|\mathrm{Tr}\,C_{qp^n}|$ and $|\mathrm{Tr}\,D_{p^n}|$ but do not give closed forms.

Another motivation for acquiring structural and enumerative knowledge of $\mathrm{Tr}\,G$ is understanding and describing the complicated behavior of N_∞ structures with respect to localization. While Bousfield and finite localizations of topological spectra preserve E_∞ structures, it is *not* the case that such localizations preserve N_∞ structures. Rather, localization can destroy norms. In [Hil19], Hill has studied certain chromatic localizations of equivariant ring spectra and deduced conditions under which thick subcategories preserve $\mathscr{O}$-algebras (see Theorem 5.2 of *loc. cit.*). Despite this significant progress, much work remains if we are to fully understand how localizations act on $\mathrm{Tr}\,G$.

Equivariant Algebra

Each equivariant commutative ring spectrum R (*i.e.*, representing object for a generalized Bredon-style cohomology on G-spaces) carries a wealth of algebraic data on the level of $\underline{\pi}_0 R$. Here $\underline{\pi}_0 R$ may be viewed as a functor

$$\begin{aligned}(\mathrm{Sub}\,G)^{\mathrm{op}} &\longrightarrow \mathrm{CRing}\\ H &\longmapsto \pi_0 R^H,\end{aligned}$$

[1] *Such a collection $\mathscr{F}$ is called a* family *for the group $G\times\mathfrak{S}_n$; combining the third and fourth criteria implies that $\mathscr{O}(n)$ is a* universal space *for $\mathscr{F}$.*

[2] *For the operadically uninitiated, the n-th space $\mathscr{O}(n)$ of an operad $\mathscr{O}$ parametrizes n-ary operations. An algebra X over $\mathscr{O}$ comes equipped with maps $\mathscr{O}(n)\times X^n\to X$. Thus for each point of $\mathscr{O}(n)$ we get an n-ary operation on X.*

where R^H denotes the H-fixed points of R (viewed as a nonequivariant spectrum). The induced homomorphism $\underline{\pi}_0 R(K \leq H) =: r_K^H : \pi_0 R^H \to \pi_0 R^K$ is called *restriction* along $K \leq H$. The G-universe over which R is defined (a technical condition regarding which representation spheres R has suspension isomorphisms with respect to) further endows $\underline{\pi}_0 R$ with additive *transfer* maps $t_K^H : \pi_0 R^K \to \pi_0 R^H$. These assemble into the data of an $\overset{a}{\dashrightarrow}$-*Mackey functor*, where $\overset{a}{\dashrightarrow} \in \operatorname{Tr} G$ is a transfer system encoding which transfers are allowed in the Mackey functor. (There are also maps c_g induced by conjugation by group elements, but we omit these from our discussion.) The transfer and restriction maps satisfy compatibility axioms, including an elaborate double coset formula.

Now suppose $\mathscr{O}_m$ is an N_∞ operad with associated transfer system $\overset{m}{\longrightarrow}$, and that R is an $\mathscr{O}_m$-algebra. Then the $\overset{a}{\dashrightarrow}$-Mackey functor $\underline{\pi}_0 R$ also admits multiplicative *norm* maps $n_K^H : \pi_0 R^K \to \pi_0 R^H$ for each $K \overset{m}{\longrightarrow} H$. These maps satisfy further compatibilities involving so-called exponential diagrams which we omit from this discussion. This makes $\underline{\pi}_0 R$ a *bi-incomplete* $(\overset{a}{\dashrightarrow}, \overset{m}{\longrightarrow})$-*Tambara functor* in the sense of Blumberg–Hill [BH21].[3]

In order to phrase all of the compatibilities between restrictions, transfers, and norms, certain compatibilities are necessary between $\overset{a}{\dashrightarrow}$ and $\overset{m}{\longrightarrow}$. These are codified in the following theorem of Chan:

Theorem 3 (Theorem 4.10 of [Cha22])**.** *Bi-incomplete Tambara functors with respect to G-transfer systems* $(\overset{a}{\dashrightarrow}, \overset{m}{\longrightarrow})$ *are well-defined if and only if* $\overset{m}{\longrightarrow} \leq \overset{a}{\dashrightarrow}$ *and the following condition holds:*

- *if* $K, L \leq H \leq G$ *such that* $K \overset{m}{\longrightarrow} H$ *and* $K \cap L \overset{a}{\dashrightarrow} K$, *then* $L \overset{a}{\dashrightarrow} H$.

We call a pair of transfer systems $(\overset{a}{\dashrightarrow}, \overset{m}{\longrightarrow})$ satisfying the conditions of the theorem a *compatible pair*. We can record the final compatibility axiom diagrammatically, where the double arrow is logical implication, see below.

$$\begin{tikzcd} & H & \\ L \arrow[ur, dashed, "a"] \arrow[rr, phantom, "\Leftarrow"] & & K \arrow[ul, "m"'] \\ & K \cap L \arrow[ul, "m"] \arrow[ur, dashed, "a"'] & \end{tikzcd}$$

[3]*Tambara functors were originally introduced by Tambara in* [Tam93], *where they were referred to as* TNR-functors *for "transfer, norm, restriction." We note that equivariant ring spectra are not the only source of Tambara functors. They also appear naturally when considering representation rings and other equivariant algebraic structures.*

(Note that $K \cap L \overset{m}{\longrightarrow} L$ is forced by the restriction axiom for $\overset{m}{\longrightarrow}$.) Loosely speaking, we are looking for intervals $\overset{m}{\longrightarrow} \leq \overset{a}{\dashrightarrow}$ in $\operatorname{Tr} G$ where $\overset{a}{\dashrightarrow}$ satisfies a type of "relative saturation" condition with respect to $\overset{m}{\longrightarrow}$.

Several authors have undertaken the challenge of enumerating compatible pairs of transfer systems. We highlight the work of Hill–Meng–Li which enumerates compatible pairs for $G = C_{p^n}$ (a cyclic group of order p^n, p prime).

Theorem 4 (Theorem 1.7 of [HML24])**.** *For* $G = C_{p^n}$, *there are exactly*

$$\frac{1}{3n+4}\binom{3n+4}{n+1}$$

compatible pairs of transfer systems.

The bivariate sequences $A_n(p,r) := \frac{r}{np+r}\binom{np+r}{n}$ are known as *Fuss–Catalan numbers*. By [BBR21], we have $|\operatorname{Tr} C_{p^n}| = \operatorname{Cat}(n+1) = A_{n+1}(2,1)$, while Theorem 4 says that compatible pairs of transfer systems for C_{p^n} are enumerated by $A_{n+1}(3,1)$. We will enounter the $(3,1)$-Fuss–Catalan numbers once more when considering composition closed premodel structures on $[n] \cong \operatorname{Sub} C_{p^n}$.

Model Structures on Posets

Thus far, our applications of transfer systems have been equivariant in nature, but these structures also parametrize weak factorization systems on (categories associated with) poset lattices. Compatible pairs of weak factorization systems give rise to model structures, and this provides a link between intervals in $\operatorname{Tr} P$ and abstract homotopy theory.

The role of a weak factorization system is to axiomatize the relationship between acyclic cofibrations and fibrations (or cofibrations and acyclic fibrations) in topology. This is phrased in terms of lifting properties, which we presently define. Given morphisms $i : a \to b$ and $p : x \to y$ in a category $\mathscr{C}$, we say that i has the *left lifting property* with respect to p, or that p has the *right lifting property* with respect to i, when for all commutative squares of the form

$$\begin{tikzcd} a \arrow[r] \arrow[d, "i"'] & x \arrow[d, "p"] \\ b \arrow[r] \arrow[ur, dashed, "\exists h"] & y \end{tikzcd}$$

in $\mathscr{C}$, there exists a morphism $h : b \to x$ making the diagram commute. In this situation, we write $i \boxslash p$. Given a class M of morphisms in $\mathscr{C}$, we further define

$$M^{\boxslash} := \{g \in \operatorname{Mor} \mathscr{C} \mid f \boxslash g \text{ for all } f \in M\},$$
$${}^{\boxslash}M := \{f \in \operatorname{Mor} \mathscr{C} \mid f \boxslash g \text{ for all } g \in M\}.$$

A *weak factorization system* on $\mathscr{C}$ is a pair (L, R) of subclasses of $\operatorname{Mor}\mathscr{C}$ such that

- $R \circ L = \operatorname{Mor}\mathscr{C}$, and
- $L = {}^{\boxslash}R$ and $R = L^{\boxslash}$.

A *premodel structure* on $\mathscr{C}$ is now a pair of weak factorization systems (L, R), (L', R') such that $R \subseteq R'$ (or equivalently $L' \subseteq L$). A premodel structure is a *model structure* when the morphism set $W := R \circ L'$ satisfies the two-out-of-three property:

- if f and g are composable morphisms in $\mathscr{C}$ and two of f, g, and $g \circ f$ are in W, then so is the third.

In [JT07], Joyal–Tierney prove that this presentation of a model structure is equivalent to Quillen's, with R' playing the role of fibrations, L cofibrations, and W weak equivalences. The principal role of a model structure is to produce a nice model for the homotopy category $\operatorname{Ho}\mathscr{C} = \mathscr{C}[W^{-1}]$ in which weak equivalences are inverted.

By astounding coincidence, a weak factorization system on a finite lattice P (viewed as a category) is the same thing as a transfer system on P. Let us write $\operatorname{WFS}(P)$ for the collection of weak factorization systems on P ordered by inclusion of right morphism sets.

Theorem 5 (Theorem 4.13 of [FOO+22]). *Let P be a finite poset lattice. Then the assignment*

$$\operatorname{WFS}(P) \longrightarrow \operatorname{Tr} P$$
$$(L, R) \longmapsto \xrightarrow{R}$$

is an isomorphism of posets, where $\xrightarrow{R} \in \operatorname{Tr} P$ is the relation given by

$$x \xrightarrow{R} y \iff (x \to y) \in R.$$

Before considering the ramifications of this theorem for model structures, we note an important corollary regarding self-duality of transfer systems. Suppose that P is a self-dual lattice, *i.e.*, P admits an order-reversing bijection $\nabla : P \to P$, or, phrased categorically, ∇ is an isomorphism of categories $P^{\mathrm{op}} \to P$. Importantly, if G is Abelian, then $\operatorname{Sub} G$ is non-canonically self-dual via Pontryagin duality, so this is a case of significant interest in equivariant applications.

Theorem 6 (Theorem 4.21 of [FOO+22]). *If P is a lattice with self-duality ∇, then $\operatorname{Tr} P$ is self-dual with duality*

$$\phi\colon \operatorname{Tr} P \longrightarrow \operatorname{Tr} P$$
$$\to \longmapsto \to^{\phi} := (({}^{\boxslash}\!\to)^{\mathrm{op}})^{\nabla}.$$

Moreover, if ∇ is an involution, then so is ϕ.

The proof hinges on the fact that the assignment $({}^{\boxslash}R, R) \mapsto (R^{\mathrm{op}}, ({}^{\boxslash}R)^{\mathrm{op}})$ is an isomorphism $\operatorname{WFS}(P) \to \operatorname{WFS}(P^{\mathrm{op}})$. While it is ultimately possible to construct the duality ϕ without reference to weak factorization systems (see Corollary 4.22 of [FOO+22]), discovering and presenting this duality is much simpler when working with weak factorization systems.

We now turn to the connection between transfer systems and model structures. Any lattice P has an *interval lattice* $\operatorname{Int} P$ whose elements are intervals

$$[x, y] = \{z \in P \mid x \leq z \leq y\}$$

with $x \leq y$; the partial order is defined by $[x, y] \leq [x', y']$ if and only if $x \leq x'$ and $y \leq y'$. (In categorical language, this is the *arrow category* associated with P.) If $\operatorname{Pre} P$ denotes the collection of premodel structures on P, then it follows from Theorem 5 that $\operatorname{Pre} P \cong \operatorname{Int}(\operatorname{Tr} P)$; furthermore, the class $W = R \circ L'$ associated with a premodel structure $(L, R) \leq (L', R')$ may be identified with $\xrightarrow{R} \circ {}^{\boxslash}\!\xrightarrow{R'}$, a formula only involving transfer systems. Thus, in order to enumerate model structures on a finite lattice P, it suffices to find intervals $[\to, \dashrightarrow] \in \operatorname{Int}(\operatorname{Tr} P)$ such that $\to \circ {}^{\boxslash}\!\dashrightarrow$ satisfies the two-out-of-three property.

Balchin–Ormsby–Osorno–Roitzheim solve this problem for $P = [n]$. Let $\operatorname{MS}(P)$ denote the set of model structures on P considered as an induced subposet inside $\operatorname{Pre} P \cong \operatorname{Int}(\operatorname{Tr} P)$.

Theorem 7 (Theorems 4.10 and 4.13 of [BOOR23]). *For $n \geq 0$,*

$$|\operatorname{MS}([n])| = \binom{2n+1}{n}.$$

Each model structure on $[n]$ has homotopy category isomorphic to $[k]$ for some $0 \leq k \leq n$, and the number of model structures on $[n]$ with homotopy category isomorphic to $[k]$ is exactly

$$\frac{2(k+1)}{n+k+2}\binom{2n+1}{n-k}.$$

Despite the simple form of this enumeration, the proof in [BOOR23] passes through a convolution of Catalan numbers and enumeration in terms of north/east paths on an $(n+1) \times (n+1)$ grid with first step north. A more conceptual bijection between model structures on $[n]$ and a certain flavor of tricolored tree is given by Balchin–MacBrough–Ormsby in [BMO24].

The authors of [BMO24] achieve their results by considering an intermediate structure between premodel and model structures, which they dub *composition closed premodel structures*. These are pairs of weak factorization systems $(L, R), (L', R')$ with $R \subseteq R'$ and $R \circ L'$—the putative weak equivalences—closed under composition, but not necessarily fulfilling the full two-out-of-three property required of model structures. It turns out (Theorem 3.8 of [BMO24]) that for P a finite lattice, there is a refinement $\preccurlyeq$ of the usual order on $\operatorname{WFS}(P)$ such that $(\operatorname{WFS}(P), \preccurlyeq)$ is a lattice and intervals with respect $\preccurlyeq$ are exactly the composition closed premodel structures on P. There is also a partial ordering $\sqsubseteq$ on $\mathrm{WFS}(P)$ further refining $\preccurlyeq$ such that intervals with respect to $\sqsubseteq$ are model structures, but $(\mathrm{WFS}(P), \sqsubseteq)$ is *not* a lattice. The relations $\preccurlyeq$ and $\sqsubseteq$ on $\mathrm{Tr}[2]$ are depicted in Figure 1 in blue and magenta, respectively.

Returning to the case $P = [n]$, where the standard ordering on $\mathrm{WFS}(P) \cong \mathrm{Tr}\, P$ gives the Tamari lattice, we find (Theorem 4.6 of [BMO24]) that $(\mathrm{Tr}[n], \preccurlyeq)$ is isomorphic to the *Kreweras lattice* of noncrossing partitions on the set $[n]$, ordered by refinement of partitions. Since Kreweras intervals have already been enumerated, we find there are exactly

$$\frac{1}{3n+4}\binom{3n+4}{n+1}$$

composition closed premodel structures on $[n]$—the $(3,1)$-Fuss–Catalan numbers appear again! We rush to note, though, that the intervals encoding composition closed premodel structures on $[n] \cong \mathrm{Sub}\, C_{p^n}$ are distinct from the intervals encoding compatible pairs for bi-incomplete Tambara functors for C_{p^n}, and thus far no one has constructed a principled bijection between the two structures. For most finite groups G, composition closed premodel structures on $\mathrm{Sub}\, G$ are not equinumerous with compatible pairs of G-transfer systems.

Since Tamari intervals have also been enumerated [Cha05], we find that the sequences $|\mathrm{MS}([n])| \leq |\mathrm{Pre}^{cc}[n]| \leq |\mathrm{Pre}[n]|$ (where Pre^{cc} denotes composition closed premodel structures) take the form

$$\binom{2n+1}{n} \leq \frac{1}{3n+4}\binom{3n+4}{n+1} \leq \frac{2}{(n+1)(n+2)}\binom{4n+5}{n}.$$

Asymptotic analysis reveals that model structures on $[n]$ are vanishingly rare among composition closed premodel structures on $[n]$, which are in turn vanishingly rare among premodel structures on $[n]$.

Conclusion

While we have touched on a number of recent advances in homotopical combinatorics, it is not possible in this limited space to cover the entirety of this rapidly growing field. We hope we have conveyed a flavor of work in the area, and want to emphasize that much terrain remains unexplored and there are many ways that researchers from various backgrounds can contribute. (In fact, much of the combinatorial work on transfer systems has been undertaken in collaboration with undergraduates.) To whet the reader's appetite, we provide the following short list of open problems:

1. Explore the combinatorics of the recursive construction of transfer systems from [BMO23] for new families of lattices/groups.
2. Use multivariable generating functions to convert the recursions of [BMO23] for $|\mathrm{Tr}\, D_{p^n}|$ and $|\mathrm{Tr}\, C_{qp^n}|$ into closed formulæ.
3. Enumerate compatible pairs of transfer systems (in the sense of [Cha22]) for new families of groups.
4. After identifying the lattice of transfer systems for a (family of) poset(s) P, use the methods of [BOOR23, BMO24] to enumerate $\mathrm{Pre}\, P$, $\mathrm{Pre}^{cc}\, P$, and $\mathrm{MS}(P)$.
5. Leverage new structural results on transfer systems to extend the work of [Hil19] on the interaction between localizations and norms.
6. Lift the duality on transfer systems discovered in [FOO⁺22] to the level of N_∞ operads.

The authors—whose backgrounds are primarily in homotopy theory—are especially eager to see how more advanced tools from algebraic and analytic combinatorics might apply to these problems. We look forward to exploring these topics with participants in our 2024 Mathematics Research Community, and welcome inquiries from potential applicants.

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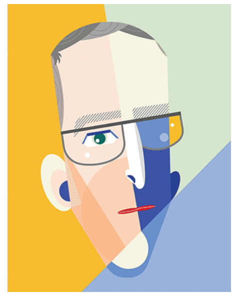

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Climate Science at the Interface Between Topological Data Analysis and Dynamical Systems Theory

Davide Faranda, Théo Lacombe, Nina Otter, and Kristian Strommen

The authors are hosting an AMS sponsored Mathematics Research Community (MRC) on novel applications of topological data analysis (TDA) and dynamical systems theory to the study of climate change and weather forecasting. In this *Notices* article we introduce some of the big challenges in climate science, and describe how methods from TDA and dynamical systems theory can help tackle these. We hope to encourage applications to the MRC from mathematicians with a background in applied algebraic topology and scientists working on climate or meteorology, both from academia and industry.

Challenges in climate science. Understanding how the climate system is responding to increasing levels of anthropogenic emissions is one of today's key scientific challenges. Some aspects of this are very well understood: most fundamentally, the atmosphere as a whole will continue to warm up, and its capacity to carry moisture will therefore also increase. When the plethora of climate models (which numerically integrate the primitive equations describing the evolution of the climate system) are asked to simulate the future climate given continued greenhouse gas emissions, they all broadly agree on these points. Such changes to temperature and humidity are often referred to by climate scientists as the "thermodynamical" response. However, when it comes to regional changes, the details of which will dominate both the decision making for adaptation strategies and people's lived experience, there is considerably more uncertainty. There is perhaps no better example of this than the question of how the variability of the northern hemisphere jetstream will change in the future. North America and Eurasia have experienced dramatic examples of extreme weather in recent years, and these events are often related to atypical jetstream configurations, such as "blocking" events, where a persistent high-pressure system forces the jet to deviate from its usual trajectory. There is naturally intense interest in understanding how the frequency and severity of such events may change with global warming: these are often viewed as part of the "dynamical" response. The "dynamical" and "thermodynamical" response give a first-order approximation of the total response. Unfortunately, climate models can often disagree quite dramatically on the "dynamical" response, and theoretical understanding is still

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lacking, leading to pervasive uncertainty, especially with regards to the boreal winter months (December–January–February) [WBM+18]. The answers to simple sounding questions such as "will western Europe see increased risk of flooding?" or "will East Coast USA see increased risk of extreme cold snaps?" therefore remain unclear.

Climate as a dynamical system. One can view the climate as a dynamical system [GL20], and thus conceptually model it using a stochastic differential equation:

$$\frac{d}{dt}X = C(X) + f + \epsilon. \tag{1}$$

Here X is a state vector encoding all the relevant physical variables (temperature, humidity, etc.) at every location in the atmosphere, land surface, and ocean; C is a smooth function which determines the evolution of X in time (and is therefore encoding things like the Navier–Stokes equations); and f is a forcing term, which in an equilibrium context would be 0, and in a global warming context would represent the effect of emissions. The ϵ term represents "noise." For a given fixed forcing term f, this equation determines the trajectory taken by X through its ambient phase space. For reasonable parameter choices, the climate system is stable, and the trajectory thereby settles onto its attractor, which in turn determines the statistics of the climate. The challenge of climate science can therefore be thought of as understanding how the attractor changes as a function of f = anthropogenic emissions. For example, does the new attractor contain a higher density of trajectories corresponding to extreme weather?

A key reason this is such a challenging question is that the dimensionality of the phase space (the size of the vector X with respect to a sufficiently accurate finite approximation of space-time) is in the billions. This is many orders of magnitude greater than both (i) the number of available independent observations; and (ii) the dimensionality of even the most state-of-the-art climate model simulations. This mismatch is exacerbated even further if one is interested in understanding extremes, which by definition make up only a tiny proportion of observed or simulated events.

Weather regimes and the Lorenz butterfly. Given the above discussion, a natural strategy is to try to artificially reduce the dimensionality in some way. In the context of the northern hemisphere jetstream, this is often done using the notion of "weather regimes" [HSF+17]. The basic idea is to find a small subset of dynamically relevant large-scale atmospheric circulation patterns (the "regimes") that dominate the low-frequency variability; one can then study the impact of forcing on the dynamical behavior of the regimes (e.g., their relative frequency of occurrence). An example is given in Figure 1, which shows the North

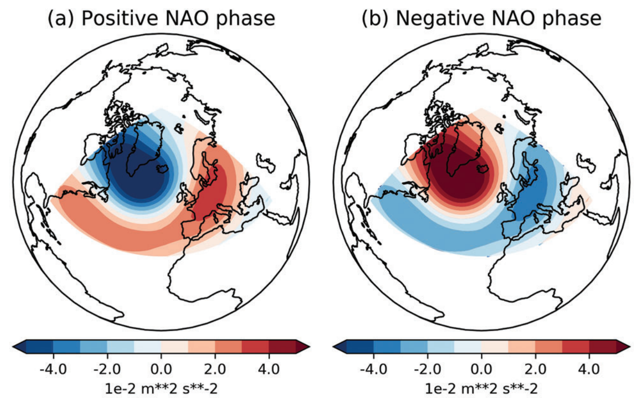

Figure 1. The positive (a) and negative (b) phases of the North Atlantic Oscillation (NAO), the dominant pattern of variability in the Euro-Atlantic wintertime circulation. The pattern is a dipole of atmospheric pressure anomalies, measured here as the first empirical orthogonal function of geopotential height at 500hPa. The two phases can be viewed as one of several possible ways to decompose the Euro-Atlantic circulation into distinct regimes.

Atlantic Oscillation, a large dipole of atmospheric pressure between Greenland and the Atlantic ocean, which is the dominant mode of variability in boreal winter. Knowing whether the NAO is in its positive or negative phase gives a good first-order approximation of the jetstream, and, consequently, of winter weather in Europe and eastern North America. It gives one possible regime-view of the Euro-Atlantic circulation. Strong NAO events are frequently linked to extreme surface weather, such as the exceptionally warm European winter of 2019–2020 [HDS+20].

A key reason why this perspective can be so powerful is that in some cases the response to a forcing f can be entirely understood in terms of the regime dynamics. The motivating example here is the "Lorenz '63" system [Lor63]. This system can be loosely thought of as an extremely severe truncation of Equation (1) to just 3 variables, with $f = \epsilon = 0$; it is famous for its "strange attractor," which, besides resembling a butterfly (see Figure 2a), encodes the first example of a chaotic dynamical system. Palmer, Corti, and Molteni [CMP99] showed that if the forcing vector f is made nonzero, the overall shape of the attractor does not change: the two "wings" remain in the same place. Instead, the system simply starts spending more time in whichever of the two wings the vector f is pointing towards, see Figure 2b. If the two wings are interpreted as two regimes, this says that in the Lorenz '63 system, the effect of forcing is determined by a single number, namely the relative frequency of occurrence of the two regimes, with the regimes themselves remaining invariant.

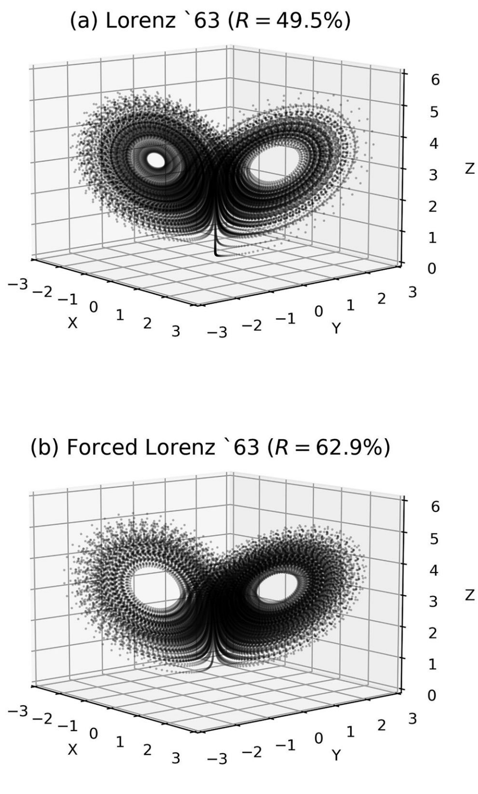

Figure 2. In (a), 30,000 samples from a long integration of the Lorenz '63 system. In (b), the same but with a nonzero constant forcing vector added which roughly points toward the right-hand "wing," following the set-up of [CMP99]. The value of R in each plot is the approximate proportion of time spent in the right-hand wing ($x > 0$). Lorenz '63 data kindly provided by J. Dorrington.

Palmer famously argued that the same may be true of real-world weather regimes [Pal99]. In other words, that the effect of global warming on the northern hemisphere jetstream may be to modulate the dynamical behavior of a small set of invariant regimes. If true, questions such as "will East Coast USA see increased risk of extreme cold snaps?" might be largely reducible to a determination of (a) how different weather regimes modulate the risk of cold snaps; (b) how the regime dynamics will change (their occurrence, temporal persistence etc.); and (c) a potential scaling factor to represent the much more well understood "thermodynamic" changes (to account for, e.g., the fact that temperatures will be warmer on average everywhere). Crucially, invariance of the regimes implies that point (a) can be estimated using *historical* observations. This strategy thereby offers a potentially dramatic simplification to the question of how midlatitude extreme weather will change.

Using persistent homology to diagnose weather regimes. It turns out that the main difficulty in implementing Palmer's strategy for understanding midlatitude climate change is actually determining what the "correct" weather regimes are. In the 3-variable Lorenz '63 system one can simply look at the attractor and identify the two distinct wings by eye. The dynamics of the jetstream is by contrast a very high-dimensional complex interplay between multiple physical variables confounded by considerable chaotic noise. A variety of strategies have been proposed [HSF+17], which typically involve using a clustering algorithm applied to a single physical variable whose dimensionality has been cut down using empirical orthogonal functions. The different strategies amount to inequivalent definitions of what a weather regime actually *is*. They all thereby suggest a different set of weather regimes that are not in general comparable and are far from being invariant under global warming [DSFM22]. Furthermore, when applied to different idealized representations of the atmosphere (of which the Lorenz '63 system is one example), they fail to identify the correct regime behavior in one or more cases [SCDO23]. It is likely that this is at least in part because these methods rely on a truncation of the true system which is far too severe. What is needed therefore is a method that allows one to efficiently "see" the structure in a more high-dimensional space. Enter persistent homology.

Persistent homology (PH) is one of the most successful methods in the field of topological data analysis [Car09, Oud15]. It is particularly useful for applications in which one is interested in studying data sets that depend on specific parameters of interest (e.g., studying points that have a specific density estimate, or thickenings of points), and for which one might not be able to accurately estimate the value of the parameter. In such cases, instead of

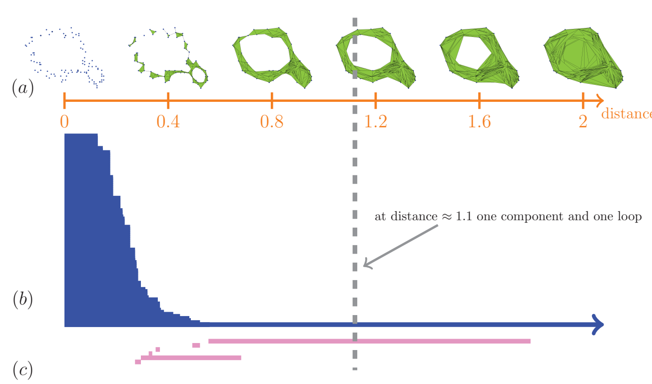

Figure 3. Persistent homology pipeline: (a) we associate a 1-parameter nested sequence of spaces ("filtration") to a metric space; barcode plots for (b) 0-cycles ("components") and (c) 1-cycles ("holes"). Each interval in a barcode plots describes the persistence of a component or hole.

estimating the optimal density value of the points, or the optimal thickening, PH allows us to study how the topological invariants (number of components, holes, voids) of the data set vary as a function of the density threshold or thickening parameter. The output of PH is a disjoint collection of intervals (collectively called the "barcode") with the length of each interval corresponding to the persistence of a specific topological invariant, see Figure 3.

PH has been shown to be able to capture local and global topological and geometric properties, as well as signals of different orders of magnitude. Crucial for applications to climate science is that main algorithms for PH do not depend on the dimension of the system that one is trying to study, and that PH is, in a precise sense, robust with respect to perturbations in the system. These properties of PH make it particularly suited for addressing the problem of determining weather regimes in a high-dimensional, noisy system, provided that the regimes correspond in some sense to topological features. A key observation of [SCDO23] is that this appears to be the case. This is already clear for the Lorenz '63 system, where the two "wings" are associated with the two holes, and PH is able to not just detect the existence of the two homological cycles but also compute "good" representatives of these that can be *visualized*, see Figure 4. Similarly, in other idealized models used to study weather regimes, and in low-dimensional truncations of atmospheric data, one finds that the regimes always correspond to a mixture of loops, holes and connected components. These can in all cases be efficiently and correctly identified with PH, whereas previously considered approaches would fail in one or more cases.

To summarize, PH represents a powerful tool for probing the high-dimensional dynamics of the jetstream, and there is good reason to believe that the topological features it might extract will encode physically meaningful weather regimes. Put informally, PH may be able to "see" the regimes of the real atmosphere as clearly as our human eyes can see the regimes in the Lorenz butterfly.

Open problems. There are several open problems that need to be addressed before techniques from topology can be used to study weather regimes in large data sets of high-dimensional climate data: these and others will be addressed at the MRC.

One such question concerns theoretical and computational improvements needed for developing algorithms for optimal PH representatives for weather data. For a recent survey on several approaches to define optimal representatives, see [LTHP+21]. Computing optimal representatives allows one to sensibly visualize the topological information that PH detects, essential for applications to climate science where one needs to verify that the

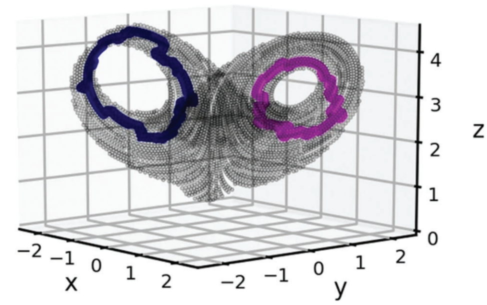

Figure 4. An illustration of two loops in the Lorenz '63 system. The homological cycles were identified using PH; optimal representatives of these were then computed, and are visualized in dark blue and purple thick lines.

topological cycles correspond to physically sensible phenomena. Having such a suitable notion of topological representatives then raises the question: how can these be used to study the dynamical properties of the weather systems? In particular, how can we best approximate the system in terms of transitions between a finite set of topologically defined states? What dynamical properties can one expect of regimes in general? Can we derive general principles for dynamical systems with arbitrary topological signals? Here we will make use of recent developments from dynamical systems theory, which allow dynamical features (such as the number of degrees of freedom or temporal persistence) to be analyzed robustly using instantaneous samples [FMY17].

Another question relates to statistical analysis. Our topological approach to regimes essentially equates the existence of regimes in a dynamical system with the existence of nontrivial topological structure. Here the term "nontrivial" is explicitly a statistical notion: a randomly drawn sample from a noisy system could produce nontrivial homology at some degree purely by chance. How can we assess the statistical significance of any regimes found using PH? Can we meaningfully perform statistical optimization across multiple topological descriptors to study weather systems? To address these questions we will rely on several different approaches developed in statistical topological data analysis, such as [BS22, BCCS+11, CFL+17].

Conclusions. We have highlighted how methods from computational topology represent a potentially very powerful tool for studying the effect of global warming on northern hemisphere weather, via the framework of weather regimes; this will be the focus for our MRC. However, there are many other potential applications of topological data analysis to climate science [MKK+18, TMD+20] and dynamical systems [YB20], including the use of more sophisticated topological information [GS23]. For example, many important components of the climate system exhibit oscillatory behavior, such as the El Niño Southern Oscillation, which has a big influence on year-to-year variations in global temperatures. Viewed appropriately, such oscillations correspond to loops through phase space which can be detected and studied using PH. The authors believe that there is great potential for the use of methods from topology in climate science and dynamical systems theory more broadly.

In order for this potential to be realized, important theoretical and algorithmic mathematical challenges in topological data analysis need to be tackled. The goal of our MRC is to begin this process.

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Credits

Mathematics for Public Service

Duncan Wright

Mathematics is our primary tool to model the world around us, and its practitioners can benefit society in a myriad of ways. Some choose to apply their expertise directly through technical work and research. For example, one could build sophisticated models to predict the effect of carbon dioxide emissions on global temperatures, interpret the decision-making process of artificial intelligence, or protect our national security at the National Security Agency. For others, mathematics education is the way. Is there a more noble pursuit than preparing the next generation to think and challenge the world head on? Today, I write about a third, less-trodden path to use mathematics for public service—policymaking.

I served alongside thirty Congressional Fellows as part of the Science & Technology Policy Fellowship program run by the American Association for the Advancement of Science (AAAS). In addition to Congressional Fellows, there are about 270 AAAS Fellows placed in the Executive Branch each year and one in the Judicial Branch. The AAAS Fellows are a cohort of driven PhD scientists from diverse backgrounds. The Congressional cohort was spread across the House and Senate, in personal offices and on committees, and worked for both Democrats and Republicans. The Congressional Fellows relied on each other's expertise and unique positions to further legislation and understand the functions of Congress in a way few are afforded.

I joined the Office of Senator Todd Young to oversee implementation of the CHIPS and Science Act including the CHIPS for America programs at the Departments of Commerce, Defense, and State. As a Fellow, I promoted federally funded research and development, developed policy informed by scientific and technical expertise, and provided critical context during engagement with stakeholders from industry, government, and civil society. My background in mathematics—rare among congressional staff—not only prepared me for work on the Hill, but allowed me to excel at many aspects of the legislative process.

No, nobody ever needed to know what a Galois group was or asked me to explain the difference between convergence at a point and uniform convergence. No, you don't need a PhD to develop the soft skills necessary to thrive on the Hill—time management, communicating effectively through speaking and writing, and foreseeing unintended consequences of potential policy decisions—but mathematicians are great problem solvers and researchers, both of which are crucial to success. As I always tell the attorneys in my family, the only difference between mathematics and law is that mathematicians agree on their definitions beforehand (and on their undefined terms).

Most importantly, as a scientist, I was able to assess the technical feasibility of new technologies and call out those that are trying to take advantage of the taxpayers' dollar. In one instance, an organization claimed they would soon be able to have an open-air breathalyzer installed in every car that would accurately test a driver's blood alcohol content (BAC) by measuring ambient ethanol concentration, forever solving the drunk driving epidemic. Forgetting the potential privacy violations or implications of any malfunction, one quickly realizes that the degrees of freedom for the input are greater than those being measured. Unlike a traditional breathalyzer which measures ethanol to volume directly providing a driver's BAC, an ambient breathalyzer measures ethanol concentration in the air, neglecting variability in breath volume and other factors. My technical expertise provided my office with crucial information and allowed the senator to make more informed policy decisions. This expertise has been invaluable in responding to the increased attention on artificial intelligence regulation, as my office plays a central role in the conversation.

The AMS Congressional Fellowship has been an incredible, life-changing experience. I have witnessed the gears of democracy, the responsibility of the citizenry, and the power of compromise. Behind the cameras and the circus

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of dysfunction that we all love to hate, there are dedicated men and women, both congressional members and staff, that are working to find solutions for the benefit of the American public. I am grateful to the AMS for granting me this amazing opportunity. I urge everyone reading this to consider how you use mathematics for public service and to consider pursuing this fellowship.

Learn about the AMS Congressional Fellowship at `https://www.ams.org/ams-congressional-fellowship`. Applications open until February 1, 2024.

Duncan Wright

Bringing Math into *The Conversation*: My Summer as a Mass Media Fellow

Maxine Calle

If you'd asked me when I was a kid what I wanted to be when I grew up, "mathematician" wouldn't have made the list. Even though I was considered "good at math" in school, I preferred to spend my time reading fantasy novels, doodling in sketchbooks, and performing silly plays with my friends. I didn't see math as providing any avenue for creativity and exploration, and I didn't know that being paid to think about math was even a possibility.

Now, as I pursue my PhD in math, I know that I was wrong on both counts. Not only do I pay my rent by thinking about abstract math, but I also have a great appreciation for the beauty and creativity that the field holds. Echoing some philosophers of science, I like to think of math as "the science of patterns"—it provides a framework within which to explore the natural human inclination to find structure in the world around us. As I dove deeper into my studies of math in college, I came to embrace the struggle of working through complicated proofs and the feeling of satisfaction when it finally all fell into place.

But as the math I studied got more and more abstract, I found it harder and harder to explain what it was that I was learning about. This wasn't for lack of trying—I spent many late nights trying to explain to my college roommates what n-dimensional holes were and how homology counts them. I felt frustrated that math was often presented in a way that made people feel like they weren't "a math person" and I became passionate about trying to make the subject more accessible and enjoyable for everyone. So when I heard about the Mass Media Fellowship, which pairs students with different news outlets across the country to work in science journalism for the summer, it seemed like the perfect opportunity to take a serious step in my mathematical communication journey.

Although I heard about the fellowship in college, I didn't apply until a few years later since starting graduate school during the Covid-19 pandemic had put enough on my plate. I finally felt ready to apply in 2023, after my third year of graduate school, and I was pretty surprised to be selected as a 2023 Mass Media Fellow. I'd never written a news story before, I didn't know anything about science journalism, and my area of research (algebraic topology) isn't exactly something that comes up in the daily news. But my summer fellowship at *The Conversation US* showed me that both math and I can have a place in the newsroom.

The Conversation's model is a bit different from a traditional news outlet. All of the articles published by *The Conversation* are written by academics and then edited by professional journalists. So during my fellowship I was mostly helping other academics write about their areas of expertise in a way that the public could understand. As an editor, I worked on a variety of articles about science, technology and, of course, math. Whether answering the age-old question "Will I ever need math in real life?"[1] or exploring the mysterious origins of X in algebra,[2] I happily snuck as much math as I could into *The Conversation*'s lineup.

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[1] `https://theconversation.com/will-i-ever-need-math-a-mathematician-explains-how-math-is-everywhere-from-soap-bubbles-to-pixar-movies-204609`

[2] `https://theconversation.com/x-marks-the-unknown-in-algebra-but-xs-origins-are-a-math-mystery-210440`

I also had the chance to experience *The Conversation*'s process from the other side, as an author. I coauthored two articles, one in celebration of the 30th anniversary of Wiles' announcement of his infamous solution to Fermat's last theorem[3] and another about how ancient ideas in Euclidean geometry have fueled centuries of mathematical research.[4] I enjoyed getting to share these small glimpses into the world of math that I had come to love, and the positive feedback I got from *The Conversation*'s staff made these articles feel like a success.

The challenge of writing these articles was figuring out how to weave complex mathematical ideas into a compelling story that anyone could read and understand. It's easy to forget that what is second-nature to you is incomprehensible jargon to someone else, and I had to be careful not to slip into exclusionary language. But *The Conversation*'s process is exactly designed to help with this issue. The editors I worked with for these articles were great at pointing out things that my academic training hid from view, like that the words "symmetry" and "finite" might need more explanation than I had originally given them.

I learned a lot over the summer, and I still have a lot to learn. Returning to the fourth year of my PhD program, I'll continue to look for creative ways to help mathematicians tell their stories—both to each other and to the broader public. Even if I don't end up becoming a science journalist, I know that the skills I gained as a Mass Media Fellow will be invaluable in my career and I'm so thankful that I got to have the experience.

More information on the AMS-AAAS Mass Media Fellowship can be found at `https://www.ams.org/massmediafellow`.

Maxine Calle

Credits

Photo of Maxine Calle is courtesy of Nina Johnson.

[3] `https://theconversation.com/proving-fermats-last-theorem-2-mathematicians-explain-how-building-bridges-within-the-discipline-helped-solve-a-centuries-old-mystery-207968`

[4] `https://theconversation.com/a-brief-illustrated-guide-to-scissors-congruence-an-ancient-geometric-idea-thats-still-fueling-cutting-edge-mathematical-research-210612`

AMS Updates

Catching Up with AMS Congressional Fellows

Since 2005, the AMS has sponsored 19 mathematicians via the AAAS (American Association for the Advancement of Science) Science and Technology Policy Fellowships program to work in Congress. Of the 19 Fellows, 13 served in the offices of US senators, while others worked for members of the House of Representatives, for the House Science Committee, and the Senate Homeland Security and Government Affairs Committee. AMS writer Elaine Beebe spoke with three past fellows to learn how the program helped advance their careers. See `https://www.ams.org/news?news_id=7225`.

Twelve Students Receive 2023 AMS Undergraduate Opportunity Awards

Twelve students received 2023 AMS Undergraduate Opportunity Awards—financial assistance designed to support students who lack adequate financial resources and who may be in danger of not completing their mathematics degree program for financial reasons. The annual awards go to students chosen at colleges and universities randomly selected from the AMS's institutional members.

Eleven of this year's awardees received Waldemar J. Trjitzinsky Memorial Awards, named in honor of the mathematician:

Colin Brennan, Northwestern University;
Marianna Gabriele, East Tennessee State University;
Paul Gongwer, Purdue University Fort Wayne;
Mia Grella, Southern Connecticut State University;
Frances Grout, University of San Francisco;
Noah Haley, University of North Carolina Asheville;
Jillian Marasco, Washington State University;
Jeffrey Nubla, University of Nevada, Reno;
Wandy Saint-Phard, University of Massachusetts Boston;
Suzanne (Suzy) Smith, Trinity University;
Nasir Wynruit, Tufts University.

In addition, the Edmund Landau Award, named in honor of the mathematician, was presented to **Megan Vitale**, Florida Institute of Technology.

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AMS Email Support for Frequently Asked Questions

A number of email addresses have been established for contacting the AMS staff regarding frequently asked questions.

The following is a list of those addresses together with a description of the types of inquiries that should be made through each address:

`abs-coord@ams.org` for questions regarding a particular abstract or abstracts questions in general.

`acquisitions@ams.org` to submit book proposals and obtain information about publishing books at the AMS.

`ams@ams.org` to contact AMS Headquarters in Providence, Rhode Island.

`amsdc@ams.org` to reach the AMS Washington Office about government relations and advocacy programs.

`amsfellows@ams.org` to inquire about the Fellows of the AMS.

`amsmem@ams.org` to request information about membership in the AMS and about dues payments or to ask any general membership questions; may also be used to submit address changes.

`ams-mrc@ams.org` for questions about the AMS Mathematics Research Communities.

`ams-simons@ams.org` for information about the AMS-Simons Travel Grants Program.

`ams-simons-pui@ams.org` for information about the AMS-Simons Research Enhancement Grants for PUI Faculty program.

`ams-survey@ams.org` for information or questions about the Annual Survey of the Mathematical Sciences or to request reprints of survey reports.

bookstore@ams.org for inquiries related to the online AMS Bookstore.

classads@ams.org to submit classified advertising for the *Notices*.

com-staff@ams.org to reach the AMS Communications Department.

consortia@ams.org for information on consortia subscriptions and MathSciNet for Developing Countries Program.

cust-serv@ams.org for general information about AMS products (including electronic products), to send address changes, or to conduct any general correspondence with the Society's Customer Service Department.

development@ams.org for information about charitable giving to the AMS.

edi-query@ams.org for questions about AMS equity, diversity, and inclusion efforts.

education@ams.org for questions and suggestions for AMS education programs and advocacy.

emp-info@ams.org for information regarding AMS employment and career services.

eprod-support@ams.org for technical questions regarding AMS electronic products and services.

exdir@ams.org to contact the AMS executive director.

gradprg-ad@ams.org to inquire about a listing or ad in the Find Graduate Programs online service.

mathcal@ams.org for questions regarding posting on the Mathematics Calendar.

mathjobs@ams.org for questions about the online job application service Mathjobs.org.

mathprograms@ams.org for questions about the online program application service Mathprograms.org.

mathrev@ams.org to send correspondence to *Mathematical Reviews* related to reviews or other editorial questions.

meet@ams.org to request general information about Society meetings and conferences.

mmsb@ams.org for information or questions about registration, housing, and exhibits for any Society meetings or conferences (Mathematics Meetings Service Bureau).

mr-exec@ams.org to contact the executive editor of the Mathematical Reviews Division regarding editorial and related questions.

mr-librarian@ams.org to contact the Mathematical Reviews Division regarding the acquisition and cataloging of the mathematical literature indexed in MathSciNet®.

msn-support@ams.org for technical questions regarding MathSciNet®.

mtgs-virtual@ams.org to request assistance during a virtual meeting.

notices@ams.org to send correspondence to the managing editor of the *Notices*.

notices-ads@ams.org to submit paid display ads for the *Notices*.

notices-booklist@ams.org to submit suggestions for books to be included in the "Bookshelf" section of the *Notices*.

notices-letters@ams.org to submit letters and opinion pieces to the *Notices*.

opportunities@ams.org for questions about submitting to the Opportunities webpage.

outreach@ams.org for questions about AMS outreach activities.

president@ams.org to contact the AMS president.

prof-serv@ams.org for correspondence with AMS Programs.

promorequests@ams.org to request AMS giveaway materials such as posters, brochures, and catalogs for use at mathematical conferences, exhibits, and workshops.

publications@ams.org for questions related to publications such as production, sales and marketing, or distribution.

publisher@ams.org to contact the AMS publisher about mathematical matters related to AMS publications.

pub-submit@ams.org to submit accepted electronic manuscripts to AMS publications (other than Abstracts). See www.ams.org/submit-book-journal to electronically submit accepted manuscripts to the AMS book and journal programs.

reprint-permission@ams.org to request permission to reprint material from Society publications.

reviewcopies@ams.org to request a book for review.

royalties@ams.org for AMS authors to direct questions about royalty payments.

sales@ams.org to inquire about reselling or distributing AMS publications or to send correspondence to the AMS Customer Service Department.

secretary@ams.org to contact the AMS secretary.

studchap-staff@ams.org for inquiries about the AMS Graduate Student Chapter Program.

tech-support@ams.org to contact the Society's typesetting Technical Support Group.

textbooks@ams.org to request examination/desk copies or to inquire about using AMS publications as course texts.

travel-grants@ams.org for information about AMS sectional and JMM travel grants.

webmaster@ams.org for general information or for assistance in accessing and using the AMS website.

Deaths of AMS Members

Bill Hassinger, Jr., of Greensboro, North Carolina, died on November 3, 2022. Born on March 25, 1929, he was a member of the Society for 46 years.

Jose L. Viviente, of Spain, died on January 25, 2023. Born on March 2, 1926, he was a member of the Society for 53 years.

B. T. Polyak, of Russia, died on February 3, 2023. Born on May 4, 1935, he was a member of the Society for 45 years.

Carroll F. Blakemore, of New Orleans, Louisiana, died on February 20, 2023. Born on November 22, 1940, he was a member of the Society for 54 years.

Harold Levine, of Asheville, North Carolina, died on March 11, 2023. Born on December 14, 1928, he was a member of the Society for 68 years.

Roger W. Brockett, of Lexington, Massachusetts, died on March 19, 2023. Born on October 22, 1938, he was a member of the Society for 53 years.

Ming-Chit Liu, of Monterey Park, California, died on March 24, 2023. Born on December 30, 1937, he was a member of the Society for 46 years.

Zvie Liberman, of Lincolnwood, Illinois, died on April 3, 2023. Born on August 1, 1935, he was a member of the Society for 38 years.

Lawrence Breen, of France, died on May 8, 2023. Born on July 18, 1944, he was a member of the Society for 35 years.

George F. McNulty, of Blythewood, South Carolina, died on June 13, 2023. Born on June 18, 1945, he was a member of the Society for 54 years.

Henry C. Pinkham, of New York, New York, died on June 24, 2023. Born on September 26, 1948, he was a member of the Society for 38 years.

Khristo N. Boyadzhiev, of Ada, Ohio, died on June 28, 2023. Born on September 4, 1948, he was a member of the Society for 33 years.

Gordon Elliott Wall, of Australia, died on July 9, 2023. Born on March 11, 1925, he was a member of the Society for 49 years.

Irving O. Bentsen, of Fairport, New York, died on July 13, 2023. Born on September 23, 1927, he was a member of the Society for 64 years.

J. D. Maitland Wright, of the United Kingdom, died on August 7, 2023. Born on May 20, 1942, he was a member of the Society for 36 years.

Mathematics People

Association for Women in Mathematics (AWM) Awardees

- Yunqing Tang, assistant professor at UC Berkeley, received the 2024 AWM Microsoft Research Prize in Algebra and Number Theory. Tang was recognized for her breakthrough work in arithmetic geometry, including results on the Grothendieck-Katz p-curvature conjecture, a conjecture of Ogus on algebraicity of cycles, arithmetic intersection theory, and the unbounded denominators conjecture of Atkin and Swinnerton-Dyer.
- Cristina Villalobos, Myles and Sylvia Aaronson Endowed Professor, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley (UTRGV), received the 2024 M. Gweneth Humphreys Award for exceptional success in mentoring and its subsequent impact on the mathematical profession as a whole.
- Trena Wilkerson, professor and interim chair in the Department of Curriculum & Instruction in the School of Education at Baylor University, received the 2024 Louise Hay Award for Contributions to Mathematics Education. Wilkerson was honored for her leadership at the national, state, and local levels in mathematics education, her transformational teaching and mentorship, and her global initiatives and programs.
- Robin Neumayer, Assistant Professor at Carnegie Mellon University, received the 2024 AWM Sadosky Research Prize in Analysis for outstanding contributions to calculus of variations, partial differential equations, and geometric analysis.
- Zoë Batterman, a senior mathematics and statistics major at Pomona College and Arianna Meenakshi McNamara, a senior mathematics and physics major at Purdue University received the 2024 Alice T. Schafer Prize for Excellence in Mathematics. Mattie Ji, a senior at Brown University majoring in mathematics-computer science and applied mathematics was named runner-up.

—Association for Women in Mathematics

Association for Women in Mathematics (AWM) Names 2024 AWM Fellows

Erika Tatiana Camacho, University of Texas at San Antonio, for leadership advancing, mentoring, and supporting women and underrepresented groups at all levels through the creation of opportunities, collaborative research, and impactful service. Her work brings sustained systemic change, diversity, equity, and inclusion in mathematics, and more broadly in STEM.

Ellen Eischen, University of Oregon, for outstanding leadership in support of women in mathematics; for sustained efforts to create new research opportunities for women at conferences, including at APAW, AWM, WIN, and MSRI/SLMath; and for her innovative approach to creating diverse communities in math with an AWM reading room and math art exhibits.

Kathryn Hess Bellwald, École polytechnique fédérale de Lausanne, for support of women in mathematics via innovative and impactful programs, including her role in founding and sustaining the Women in Topology program; for exceptional mentoring; and for a commitment to gender diversity throughout her many leadership roles in the mathematics profession.

Michael Hill, University of California, Los Angeles, for being a consistent and vocal ally for women and nonbinary researchers; for his commitment to inclusion as part of the founding board of Spectra (the association for LGBTQ+ mathematicians); for being a founding editor of *La Matematica* and for ongoing service to the AWM Mentor Network, and for working to make mathematics a welcoming and joyful place for all of us.

Christine Kelley, University of Nebraska-Lincoln, for initiating and continuing impactful efforts to encourage

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young women to pursue mathematics, including her instrumental leadership within the Nebraska Conference for Undergraduate Women in Mathematics, and for her long record of mentoring, advising, and supervising women in mathematics.

Matilde Lalin, Université de Montréal, for ongoing contributions to the AWM, most notably her leadership role in the Women in Numbers Network and considerable contributions to its growth; service to the International Mathematics Union Committee for Women; and for ardent efforts toward making conferences more welcoming and accessible for researchers by actively advocating for childcare resources.

Emille Davie Lawrence, Black Achievement Success & Engagement, University of San Francisco, for her commitment to sharing her love of mathematics with girls, women, and underrepresented groups; for creating the blog "Math Mamas" and editing the book *Living Proof*, whose goal is "to provide support and inspiration for mathematics students experiencing struggle and despair," and for extensive service to AWM, especially during its 50th Anniversary celebration.

Katharine A. Ott, Bates College, for immense dedication to outreach to girls and women, including directing GirlsGetMath at ICERM; supporting AWM through committees, grant-writing, the newsletter, and AWM's USA Science and Engineering Festival booth; and leading award-winning tutoring and volunteering initiatives in Maine.

Margaret Maher Robinson, Mount Holyoke College, for support and empowerment of several generations of women in mathematics; for mentoring within the Hudson River Undergraduate Mathematics Conference and the Carleton Summer Math Program; and for seeing the spark in each individual under her guidance and supporting them in the fulfillment of rewarding careers in mathematics.

Karen Saxe, Macalester College, American Mathematical Society, for longstanding efforts with professional societies advocating for policies—notably at the federal level—to reduce barriers and further support women and others who have had limited access to STEM careers; for mentoring women at all career stages; and for program-building to recruit and retain women in the math research ecosystem.

Christina Sormani, Lehman College and CUNY Graduate Center, for utilizing every opportunity to open pathways to mathematics for more women and students by creating and maintaining online access to advice, mathematical resources, and information about women mathematicians; for organizing the "Inspiring Talks by Mathematicians" lecture series featuring underrepresented speakers, and for her dedicated and active contributions to the AWM.

Suzanne L. Weekes, Society for Industrial and Applied Mathematics, Worcester Polytechnic Institute, for consistent and outstanding support for broadening the participation of women and girls as well as others who are underrepresented in mathematics; for award-winning teaching and mentoring; and for vision and success in co-creating and co-directing innovative programs that have improved and diversified the mathematics community.

—Association for Women in Mathematics

De Simoi Awarded Michael Brin Prize in Dynamical Systems

Jacopo De Simoi, University of Toronto, received the Michael Brin Prize in Dynamical Systems for fundamental contributions to the study of Fermi acceleration, of marked length spectrum rigidity for integrable and dispersing billiards, and entropy rigidity for conservative Anosov flows in dimension 3.

—Selection committee of the Michael Brin Prize in Dynamical Systems

Ding Wins Loève Prize

Jian Ding received the 2023 Liné and Michel Loève International Prize in Probability (Loève Prize). In 2011, Ding earned his PhD from the Department of Statistics, University of California, Berkeley, focusing on probability theory. He currently serves as chair professor with the School of Mathematical Sciences at Peking University in Beijing. His research area is probability theory, focusing on interactions with statistical physics and computer science theory. In particular, his recent research topics include random constraint satisfaction problems, random planar geometry, Anderson localization, and disordered spin models.

—Department of Statistics, University of California, Berkeley

Pronk Receives 2023 Graham Wright Award

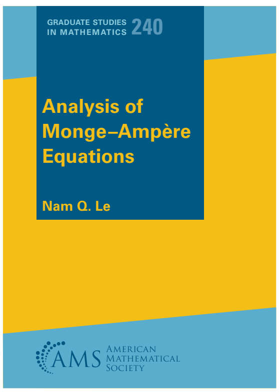
Dorette Pronk, Dalhousie University, received the 2023 Graham Wright Award for Distinguished Service from the

Canadian Mathematical Society (CMS). According to a CMS news release, Pronk has made consistent and significant contributions to the Canadian mathematical community and to the Canadian Mathematical Society. She has served as the Chair of the Math Competitions Committee since 2016, as Chair of the IMO Committee from 2014–2015, and as Chair of the EGMO committee since 2018. Additionally, she has represented Canada many times as leader and deputy leader of Math Team Canada, at the International Math Olympiad, the European Girls Math Olympiad and the Pan American Girls Math Olympiad. In 2018, she was instrumental in securing Canada's first participation in the European Girls Math Olympiad. She has also served as member of the Women in Math Committee and was part of the team organizing the first Connecting Women in Math Across Canada workshop.

—The Canadian Mathematical Society

New Books Offered by the AMS

Differential Equations



Analysis of Monge–Ampère Equations

Nam Q. Le, *Indiana University, Bloomington, IN*

This book presents a systematic analysis of the Monge–Ampère equation, the linearized Monge–Ampère equation, and their applications, with emphasis on both interior and boundary theories. Starting from scratch, it gives an extensive survey of fundamental results, essential techniques, and intriguing phenomena in the solvability, geometry, and regularity of Monge–Ampère equations. It describes in depth diverse applications arising in geometry, fluid mechanics, meteorology, economics, and the calculus of variations.

The modern treatment of boundary behaviors of solutions to Monge–Ampère equations, a very important topic of the theory, is thoroughly discussed. The book synthesizes many important recent advances, including Savin's boundary localization theorem, spectral theory, and interior and boundary regularity in Sobolev and Hölder spaces with optimal assumptions. It highlights geometric aspects of the theory and connections with adjacent research areas.

This self-contained book provides the necessary background and techniques in convex geometry, real analysis, and partial differential equations, presents detailed proofs of all theorems, explains subtle constructions, and includes well over a hundred exercises. It can serve as an accessible text for graduate students as well as researchers interested in this subject.

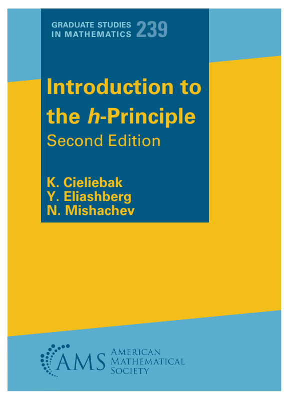
Graduate Studies in Mathematics, Volume 240
April 2024, approximately 589 pages, Hardcover, ISBN: 978-1-4704-7420-1, LC 2023046004, 2020 *Mathematics Subject Classification*: 35–XX; 52–XX, 49–XX, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code GSM/240

bookstore.ams.org/gsm-240

April 2024, Softcover, ISBN: 978-1-4704-7625-0, LC 2023046004, 2020 *Mathematics Subject Classification*: 35–XX; 52–XX, 49–XX, List US$89, **AMS members US$71.20**, **MAA members US$80.10**, Order code GSM/240.S

bookstore.ams.org/gsm-240-s

Geometry and Topology



Introduction to the *h*-Principle

Second Edition

K. Cieliebak, *University of Augsburg, Germany*, **Y. Eliashberg**, *Stanford University, CA*, and **N. Mishachev**, *Lipetsk Technical University, Russia*

In differential geometry and topology one often deals with systems of partial differential equations as well as partial differential inequalities that have infinitely many solutions whatever boundary conditions are imposed. It was discovered in the 1950s that the solvability of differential relations (i.e., equations and inequalities) of this kind can often be reduced to a problem of a purely homotopy-theoretic nature. One says in this case that the corresponding differential relation satisfies the *h*-principle. Two famous examples of the *h*-principle, the Nash–Kuiper C^1-isometric embedding theory in Riemannian geometry and the Smale–Hirsch immersion theory in differential topology, were later transformed by Gromov into powerful general methods for establishing the *h*-principle.

The authors cover two main methods for proving the *h*-principle: holonomic approximation and convex integration. The reader will find that, with a few notable exceptions, most instances of the *h*-principle can be treated by the methods considered here. A special emphasis is made on applications to symplectic and contact geometry.

The present book is the first broadly accessible exposition of the theory and its applications, making it an excellent text for a graduate course on geometric methods for solving partial differential equations and inequalities. Geometers, topologists, and analysts will also find much value in this very readable exposition of an important and remarkable topic.

This second edition of the book is significantly revised and expanded to almost twice of the original size. The most significant addition to the original book is the new part devoted to the method of wrinkling and its applications. Several other chapters (e.g., on multivalued holonomic approximation and foliations) are either added or completely rewritten.

This item will also be of interest to those working in analysis.

Graduate Studies in Mathematics, Volume 239
April 2024, 363 pages, Hardcover, ISBN: 978-1-4704-6105-8, LC 2023043257, 2020 *Mathematics Subject Classification*: 58Axx, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code GSM/239

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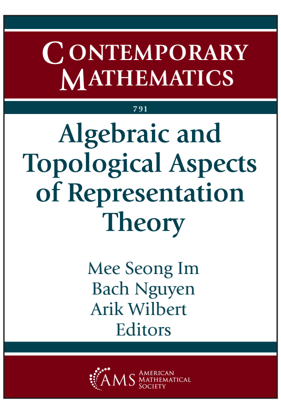
April 2024, 363 pages, Softcover, ISBN: 978-1-4704-7617-5, LC 2023043257, 2020 *Mathematics Subject Classification*: 58Axx, List US$89, **AMS members US$71.20**, **MAA members US$80.10**, Order code GSM/239.S

bookstore.ams.org/gsm-239-s

New in Contemporary Mathematics

Algebra and Algebraic Geometry



Algebraic and Topological Aspects of Representation Theory

Mee Seong Im, *United States Naval Academy, Annapolis, MD*, **Bach Nguyen**, *Xavier University of Louisiana, New Orleans, LA*, and **Arik Wilbert**, *University of South Alabama, Mobile, AL*, Editors

This volume contains the proceedings of the virtual AMS Special Session on Geometric and Algebraic Aspects of Quantum Groups and Related Topics, held from November 20–21, 2021.

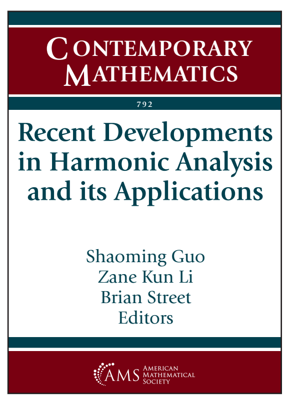
Noncommutative algebras and noncommutative algebraic geometry have been an active field of research for the past several decades, with many important applications in mathematical physics, representation theory, number theory, combinatorics, geometry, low-dimensional topology, and category theory.

Papers in this volume contain original research, written by speakers and their collaborators. Many papers also discuss new concepts with detailed examples and current trends with novel and important results, all of which are invaluable contributions to the mathematics community.

Contemporary Mathematics, Volume 791
March 2024, approximately 232 pages, Softcover, ISBN: 978-1-4704-7034-0, LC 2023048362, 2020 *Mathematics Subject Classification*: 14R15, 16E40, 16W20, 17B37, 35Q53, 18M05, 57K45, 18M30; 57K18, 57K16, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code CONM/791

bookstore.ams.org/conm-791

Analysis



Recent Developments in Harmonic Analysis and its Applications

Shaoming Guo, *The University of Wisconsin, Madison, WI*, **Zane Kun Li**, *North Carolina State University, Raleigh, NC*, and **Brian Street**, *The University of Wisconsin, Madison, WI*, Editors

This volume contains the proceedings of the virtual AMS Special Session on Harmonic Analysis, held from March 26–27, 2022.

Harmonic analysis has gone through rapid developments in the past decade. New tools, including multilinear Kakeya inequalities, broad-narrow analysis, polynomial methods, decoupling inequalities, and refined Strichartz inequalities, are playing a crucial role in resolving problems that were previously considered out of reach. A large number of important works in connection with geometric measure theory, analytic number theory, partial differential equations, several complex variables, etc., have appeared in the last few years. This book collects some examples of this work.

Contemporary Mathematics, Volume 792
March 2024, approximately 171 pages, Softcover, ISBN: 978-1-4704-7140-8, LC 2023044954, 2020 *Mathematics Subject Classification*: 11D25, 42B10, 42B20, 42B25, 32–00, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code CONM/792

bookstore.ams.org/conm-792

New in Memoirs of the AMS

Algebra and Algebraic Geometry

Total Positivity is a Quantum Phenomenon: The Grassmannian Case

S. Launois, *University of Kent, Canterbury, United Kingdom*, **T. H. Lenagan**, *University of Edinburgh, United Kingdom*, and **B. M. Nolan**, *Mitchelstown, Ireland*

Memoirs of the American Mathematical Society, Volume 291, Number 1448
January 2024, 109 pages, Softcover, ISBN: 978-1-4704-6694-7, 2020 *Mathematics Subject Classification*: 16T20, 05C10, 05E10, 14M15, 20G42, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/291/1448

bookstore.ams.org/memo-291-1448

Analysis

On the Boundary Behavior of Mass-Minimizing Integral Currents

Camillo De Lellis, *Institute for Advanced Study, Princeton, NJ*, **Guido De Philippis**, *New York University, NY, and Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy*, **Jonas Hirsch**, *Universität Leipzig, Germany*, and **Annalisa Massaccesi**, *Universitá degli Studi di Padova, Vicenza, Italy*

Memoirs of the American Mathematical Society, Volume 291, Number 1446
January 2024, 166 pages, Softcover, ISBN: 978-1-4704-6695-4, 2020 *Mathematics Subject Classification*: 49Q20, 35B65, 49Q15, 49Q05, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/291/1446

bookstore.ams.org/memo-291-1446

Differential Equations

Potential Estimates and Quasilinear Parabolic Equations with Measure Data

Quoc-Hung Nguyen, *Chinese Academy of Sciences, Beijing, People's Republic of China*

Memoirs of the American Mathematical Society, Volume 291, Number 1449
January 2024, 131 pages, Softcover, ISBN: 978-1-4704-6722-7, 2020 *Mathematics Subject Classification*: 35K55, 35K58, 35K59, 31E05; 35K67, 42B37, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/291/1449

bookstore.ams.org/memo-291-1449

Local Smoothing Estimates for Schrödinger Equations on Hyperbolic Space

Andrew Lawrie, *Massachusetts Institute of Technology, Cambridge, MA*, **Jonas Lührmann**, *Texas A&M University, College Station, TX*, **Sung-Jin Oh**, *University of California Berkeley, CA, and Korea Institute for Advanced Study, Seoul, Korea*, and **Sohrab Shahshahani**, *University of Massachusetts Amherst, MA*

Memoirs of the American Mathematical Society, Volume 291, Number 1447
January 2024, 165 pages, Softcover, ISBN: 978-1-4704-6697-8, 2020 *Mathematics Subject Classification*: 35Q41, 35R01, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/291/1447

bookstore.ams.org/memo-291-1447

Number Theory

Lattice Paths and Branched Continued Fractions: An Infinite Sequence of Generalizations of the Stieltjes–Rogers and Thron–Rogers Polynomials, with Coefficientwise Hankel-Total Positivity

Mathias Pétréolle, *University College London, United Kingdom*, **Alan D. Sokal**, *University College London, United Kingdom, and New York University, NY*, and **Bao-Xuan Zhu**, *University College London, United Kingdom, and Jiangsu Normal University, Xuzhou, People's Republic of China*

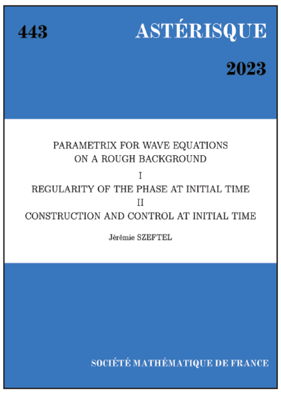
Memoirs of the American Mathematical Society, Volume 291, Number 1450
January 2024, 154 pages, Softcover, ISBN: 978-1-4704-6268-0, 2020 *Mathematics Subject Classification*: 05A15;

30B70, 15B48, 05E05, 05A20, 05A19, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/291/1450

`bookstore.ams.org/memo-291-1450`

New AMS-Distributed Publications

Differential Equations



Parametrix for Wave Equations on a Rough Background

I: Regularity of the Phase at Initial Time. II: Construction and Control at Initial Time

Jérémie Szeftel, *CNRS and Laboratoire Jacques-Louis Lions, Sorbonne Université, Paris, France*

This book is dedicated to the construction and the control of a parametrix to the homogeneous wave equation $\square_g \phi = 0$, where g is a rough metric satisfying the Einstein vacuum equations. Controlling such a parametrix as well as its error term when one only assumes L^2 bounds on the curvature tensor R of g is a major step of the proof of the bounded L^2 curvature conjecture, the latter being solved jointly with S. Klainerman and I. Rodnianski. On a more general level, this book deals with the control of the eikonal equation on a rough background, and with the derivation of L^2 bounds for Fourier integral operators on manifolds with rough phases and symbols, and as such is also of independent interest.

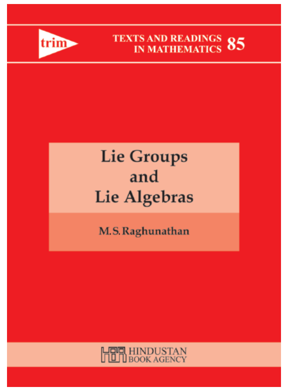
This item will also be of interest to those working in analysis.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 443
November 2023, 275 pages, Softcover, ISBN: 978-2-85629-977-7, 2020 *Mathematics Subject Classification*: 83C05, 35Q75, 58J45, 35S30, 58J40, List US$81, **AMS members US$64.80**, Order code AST/443

`bookstore.ams.org/ast-443`

Number Theory



Lie Groups and Lie Algebras

M. S. Raghunathan, *UM-DAE Centre for Excellence in Basic Sciences, Mumbai, India*

This is a textbook meant to be used at the advanced undergraduate or graduate level. It is an introduction to the theory of Lie groups and Lie algebras. The book treats real and p-adic groups in a unified manner.

The first chapter outlines preliminary material that is used in the rest of the book. The second chapter is on analytic functions and is of an elementary nature; this material is included to cater to students who may not be familiar with p-adic fields. The third chapter introduces analytic manifolds and contains standard material; the only notable feature being that it covers both real and p-adic analytic manifolds.

Chapters 4 and 5 are on Lie groups. All the standard results on Lie groups are proved here. Some of the proofs are different from those in the earlier literature. The last two chapters are on Lie algebras and cover their structure theory as found in the first of the Bourbaki volumes on the subject. Some proofs here are new.

This item will also be of interest to those working in algebra and algebraic geometry.

A publication of Hindustan Book Agency; distributed within the Americas by the American Mathematical Society. Maximum discount of 20% for all commercial channels.

Hindustan Book Agency
January 2024, 160 pages, Hardcover, ISBN: 978-81-957829-5-6, 2020 *Mathematics Subject Classification*: 11–XX, 14–XX, 17–XX, 20–XX, 22–XX, 53–XX, List US$56, **AMS members US$44.80**, Order code HIN/85

`bookstore.ams.org/hin-85`

AMS-SIMONS TRAVEL GRANTS

Beginning each February 15, the AMS will accept applications for the AMS-Simons Travel Grants program. Each grant provides an early career mathematician with $3,000 per year for two years to reimburse travel expenses related to research. Individuals who are not more than four years past the completion of their PhD are eligible. Discretionary funds may also be available for the awardee's department.

The deadline for applications is **March 31** of each year.

For complete details of eligibility and application instructions, visit:
www.ams.org/AMS-SimonsTG

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SIMONS FOUNDATION

Meetings & Conferences of the AMS

February Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. *Paid meeting registration is required to submit an abstract to a sectional meeting.*

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at `www.ams.org/meetings`.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to `https://www.ams.org/meetings/meetings-general` for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit `www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Betsy Stovall, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706; email: `stovall@math.wisc.edu`; telephone: (608) 262-2933.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: `steve.weintraub@lehigh.edu`; telephone: (610) 758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: `brian@math.uga.edu`; telephone: (706) 542-2547.

Western Section: Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: `mamanes@hawaii.edu`; telephone: (808) 956-4679.

Meetings in this Issue

2024

2025

2026

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at `https://www.ams.org/welcoming-environment-policy`.

Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `https://www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

Tallahassee, Florida

Florida State University

March 23–24, 2024

Saturday – Sunday

Meeting #1193

Southeastern Section
Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: To be announced
For abstracts: January 23, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Wenjing Liao, Georgia Institute of Technology, *Exploiting low-dimensional data structures in deep learning*.
Olivia Prosper, University of Tennessee, Knoxville, *Modeling Malaria at Multiple Scales*.
Jared Speck, Vanderbilt University, *Singularity Formation for the Equations of Einstein and Euler*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advanced Numerical Methods for Partial Differential Equations and Their Applications (Code: SS 1A), **Seonghee Jeong**, Louisiana State University, **Sanghyun Lee**, Florida State University, and **Seulip Lee**, University of Georgia.

Advances in Financial Mathematics (Code: SS 2A), **Qi Feng**, **Alec N Kercheval**, and **Lingjiong Zhu**, Florida State University.

Advances in Shape and Topological Data Analysis (Code: SS 3A), **Emmanuel L Hartman**, **Eric Klassen**, and **Ethan Semrad**, Florida State University.

Algebraic Groups and Local-Global Principles (Code: SS 4A), **Suresh Venapally**, Emory University, and **Daniel Reuben Krashen**, University of Pennsylvania.

Bases and Frames in Hilbert spaces (Code: SS 5A), **Laura De Carli**, Florida International University, and **Azita Mayeli**, City University of New York.

Combinatorics in Geometry of Polynomials (Code: SS 6A), **Papri Dey**, Georgia Institute of Technology.

Control, Inverse Problems and Long Time Dynamics of Evolution Systems (Code: SS 7A), **Shitao Liu**, Clemson University, and **Louis Tebou**, Florida International University.

Data Integration and Identifiability in Ecological and Epidemiological Models (Code: SS 8A), **Omar Saucedo**, Virginia Tech, and **Olivia Prosper**, University of Tennessee/Knoxville.

Diversity in Mathematical Biology (Code: SS 9A), **Daniel Alejandro Cruz** and **Skylar Grey**, University of Florida.

Fluids: Analysis, Applications, and Beyond (Code: SS 10A), **Aseel Farhat** and **Anuj Kumar**, Florida State University.

Geometric Measure Theory and Partial Differential Equations (Code: SS 11A), **Alexander B. Reznikov**, **John Hoffman**, and **Richard Oberlin**, Florida State University.

Geometry and Symmetry in Data Science (Code: SS 12A), **Dustin G. Mixon**, The Ohio State University, and **Thomas Needham**, Florida State University.

Homotopy Theory and Category Theory in Interaction (Code: SS 13A), **Ettore Aldrovandi** and **Brandon Doherty**, Florida State University, and **Philip John Hackney**, University of Louisiana at Lafayette.

Human Behavior and Infectious Disease Dynamics (Code: SS 14A), **Bryce Morsky**, Florida State University.

Mathematical Advances in Scientific Machine Learning (Code: SS 15A), **Wenjing Liao**, Georgia Institute of Technology, and **Feng Bao** and **Zecheng Zhang**, Florida State University.

Mathematical Modeling and Simulation in Fluid Dynamics (Code: SS 16A), **Pejman Sanaei**, Georgia State University.

Mathematical Models for Population and Methods for Parameter Estimation in Epidemiology (Code: SS 17A), **Yang LI**, Georgia State University, and **Guihong Fan**, Columbus State University.

Moduli Spaces in Algebraic Geometry (Code: SS 18A), **Jeremy Usatine**, Florida State University, **Hulya Arguz** and **Pierrick Bousseau**, University of Georgia, and **Matthew Satriano**, University of Waterloo.

Nonlinear Evolution Partial Differential Equations in Physics and Geometry (Code: SS 19A), **Jared Speck** and **Leonardo Abbrescia**, Vanderbilt University.

Numerical Methods and Deep Learning for PDEs (Code: SS 20A), **Chunmei Wang**, University of Florida, and **Haizhao Yang**, University of Maryland College Park.

PDEs in Incompressible Fluid Mechanics (Code: SS 21A), **Wojciech S. Ozanski**, Florida State University, **Stanley Palasek**, UCLA, and **Alexis F Vasseur**, The University of Texas At Austin.

Recent Advances in Geometry and Topology (Code: SS 22A), **Thang Nguyen**, **Samuel Aaron Ballas**, **Philip L. Bowers**, and **Sergio Fenley**, Florida State University.

Recent Advances in Inverse Problems for Partial Differential Equations and Their Applications (Code: SS 23A), **Anh-Khoa Vo**, Florida A&M University, and **Thuy T. Le**, North Carolina State University.

Recent Development in Deterministic and Stochastic PDEs (Code: SS 24A), **Quyuan Lin**, Clemson University, and **Xin Liu**, Texas A&M University.

Recent Developments in Numerical Methods for Evolution Partial Differential Equations (Code: SS 25A), **Thi-Thao-Phuong Hoang**, **Yanzhao Cao**, and **Hans-Werner Van Wyk**, Auburn University.

Regularity Theory and Free Boundary Problems (Code: SS 27A), **Lei Zhang**, University of Florida, and **Eduardo V. Teixeira**, University of Central Florida.

Stochastic Analysis and Applications (Code: SS 28A), **Hakima Bessaih**, Florida International University, and **Oussama Landoulsi**, FLORIDA INTERNATIONAL UNIVERSITY.

Stochastic Differential Equations: Modeling, Estimation, and Applications (Code: SS 29A), **Sher B Chhetri**, University of South Carolina Sumter, **Hongwei Long**, Florida Atlantic University, and **Olusegun M. Otunuga**, Augusta University.

Theory of Nonlinear Waves (Code: SS 30A), **Nicholas James Ossi** and **Ziad H Musslimani**, Florida State University.

Topics in Graph Theory (Code: SS 31A), **Songling Shan**, Auburn University, and **Guantao Chen**, Georgia State University.

Topics in Stochastic Analysis/Rough Paths/SPDE and Applications in Machine Learning (Code: SS 32A), **Cheng Ouyang**, University of Illinois At Chicago, **Fabrice Baudoin**, University of Connecticut, and **Qi Feng**, Florida State University.

Topological Algorithms for Complex Data and Biology (Code: SS 33A), **Henry Hugh Adams**, **Johnathan Bush**, and **Hubert Wagner**, University of Florida.

Topological Interactions of Contact and Symplectic Manifolds (Code: SS 34A), **Angela Wu**, University College of London and Louisiana State University, and **Austin Christian**, Georgia Institute of Technology.

Washington, District of Columbia

Howard University

April 6–7, 2024

Saturday – Sunday

Meeting #1194

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 13, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Ryan Charles Hynd, University of Pennsylvania, *Title to be announced*.

Jinyoung Park, Institute for Advanced Study, *Threshold Phenomena for Random Discrete Structures*.

Jian Song, Rutgers, State University of New Jersey, *Geometric Analysis on Singular Complex Spaces*.

Talitha M Washington, Clark Atlanta University & Atlanta University Center, *The Data Revolution* (Einstein Public Lecture in Mathematics).

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advanced Mathematical Methods in Naval Engineering Research (Code: SS 1A), **Michael Traweek**, Office of Naval Research, and **Anthony Ruffa**, Emeritus Naval Undersea Warfare Center.

Algebraic and Enumerative Combinatorics (Code: SS 2A), **Samuel Francis Hopkins**, Howard University, **Joel Brewster Lewis**, George Washington University, and **Peter R. W McNamara**, Bucknell University.

Analysis of PDE in Inverse Problems and Control Theory (Code: SS 3A), **Matthias Eller**, Georgetown University, and **Justin Thomas Webster**, University of Maryland, Baltimore County.

Artificial Intelligence Emergent From Mathematics and Physics (Code: SS 4A), **Bourama Toni**, Howard University, and **Artan Sheshmani**, MIT IAiFi.

Automorphic Forms and Langlands Program (Code: SS 5A), **Baiying Liu** and **Freydoon Shahidi**, Purdue University.

Automorphic Forms and Trace Formulae (Code: SS 6A), **Yiannis Sakellaridis**, Johns Hopkins University, **Bao Chau Ngo**, University of Chicago, and **Spencer Leslie**, Boston College.

Coding Theory & Applications (Code: SS 7A), **Emily McMillon**, **Eduardo Camps**, and **Hiram H. Lopez**, Virginia Tech.

Commutative Algebra and its Applications (Code: SS 8A), **Hugh Geller**, West Virginia University, and **Rebecca R.G.**, George Mason University.

Complex Systems in the Life Sciences (Code: SS 9A), **Zhisheng Shuai**, University of Central Florida, **Junping Shi**, College of William & Mary, and **Seoyun Choe**, University of Central Florida.

Computability, Complexity, and Algebraic Structure (Code: SS 10A), **Valentina S Harizanov**, George Washington University, **Keshav Srinivasan**, The George Washington University, and **Philip White** and **Henry Klatt**, George Washington University.

Computational and Machine Learning Methods for Modeling Biological Systems (Code: SS 11A), **Christopher Kim**, **Vipul Periwal**, **Manu Aggarwal**, and **Xiaoyu Duan**, National Institutes of Health.

Control of Partial Differential Equations (Code: SS 12A), **Gisele Adelie Mophou**, Universite des Antilles en Guadeloupe, and **Mahamadi Warma**, George Mason University.

Culturally Responsive Mathematical Education in Minority Serving Institutions (Code: SS 13A), **Lucretia Glover**, **Lifoma Salaam**, and **Julie Lang**, Howard University.

Elementary Number Theory and Elliptic Curves (Code: SS 14A), **Sankar Sitaraman** and **Francois Ramaroson**, Howard University.

Fresh Researchers in Algebra, Combinatorics, and Topology (FRACTals) (Code: SS 15A), **Dwight Anderson Williams II**, Morgan State University, and **Saber Ahmed**, Hamilton College.

GranvilleFest 100: A Celebration of the Legacy of Evelyn Boyd Granville (Code: SS 16A), **Edray Herber Goins**, Pomona College, **Torina D. Lewis**, National Association of Mathematicians, and **Talitha M Washington**, Clark Atlanta University & Atlanta University Center.

Interactions Between Analysis, Geometric Measure Theory, and Probability in Non-Smooth Spaces (Code: SS 17A), **Luca Capogna**, Smith College, **Jeremy Tyson**, University of Illinois at Urbana-Champaign, and **Nageswari Shanmugalingam**, University of Cincinnati.

Mathematical Modeling, Computation, and Data Analysis in Biological and Biomedical Applications (Code: SS 18A), **Maria G Emelianenko** and **Daniel M Anderson**, George Mason University.

Mathematical Modeling of Climate-Biosphere Interactions (Code: SS 19A), **Ivan Sudakow**, Department of Mathematics, Howard University.

Mathematical Modeling of Type 2 Diabetes and Its Clinical Studies (Code: SS 20A), **Joon Ha**, Howard University.

Mathematics of Infectious Diseases: A Session in Memory of Dr. Abdul-Aziz Yakubu (Code: SS 21A), **Abba Gumel**, University of Maryland, **Daniel Brendan Cooney**, University of Pennsylvania, and **Chadi M Saad-Roy**, University of California, Berkeley.

Modeling and Numerical Methods for Complex Dynamical Systems in Biology (Code: SS 22A), **Hye Won Kang** and **Bradford E. Peercy**, University of Maryland, Baltimore County.

Moduli Spaces in Geometry and Physics (Code: SS 23A), **Artan Sheshmani**, MIT IAiFi.

New Trends in Mathematical Physics (Code: SS 24A), **W. A. Zuniga-Galindo**, University of Texas Rio Grande Valley, and **Tristan Hubsch**, Howard University.

Nonlinear Hamiltonian PDEs (Code: SS 25A), **Benjamin Harrop-Griffiths**, Georgetown University, and **Maria Ntekoume**, Concordia University.

Optimization, Machine Learning, and Digital Twins (Code: SS 26A), **Harbir Antil**, **Rohit Khandelwal**, and **Sean Carney**, George Mason University.

Permutation Patterns (Code: SS 27A), **Juan B Gil**, Penn State Altoona, and **Alexander I. Burstein**, Howard University.

Post-Quantum Cryptography (Code: SS 28A), **Jason LeGrow**, Virginia Tech, **Veronika Kuchta**, Florida Atlantic University, **Travis Morrison**, Virginia Tech, and **Edoardo Persichetti**, Florida Atlantic University.

Qualitative Dynamics in Finite and Infinite Dynamical Systems (Code: SS 29A), **Roberto De Leo**, Howard University, and **Jim A Yorke**, University of Maryland.

Quantum Mathematics: Foundational Mathematics for Quantum Information Theory, Science and Communication (Code: SS 30A), **Tepper L. Gill**, Howard University.

Recent Advances in Harmonic Analysis and Their Applications to Partial Differential Equations (Code: SS 31A), **Guher Camliyurt** and **Jose Ramon Madrid Padilla**, Virginia Polytechnic Institute and State University.

Recent Advances in Optimal Transport and Applications (Code: SS 32A), **Henok Mawi**, Howard University (Washington, DC, US), and **Farhan Abedin**, Lafayette College.

Recent Advances on Machine Learning Methods for Forward and Inverse Problems (Code: SS 33A), **Haizhao Yang**, University of Maryland College Park, and **Ke Chen**, University of Maryland, College Park.

Recent Developments in Geometric Analysis (Code: SS 34A), **Yueh-Ju Lin**, Wichita State University, **Samuel Perez-Ayala**, Princeton University, and **Ayush Khaitan**, Rutgers University.

Recent Developments in Noncommutative Algebra and Tensor Categories (Code: SS 35A), **Kent B. Vashaw**, Massachusetts Institute of Technology, **Van C. Nguyen**, U.S. Naval Academy, **Xingting Wang**, Louisiana State University, and **Robert Won**, George Washington University.

Recent Developments in Nonlinear and Computational Dynamics (Code: SS 36A), **Emmanuel Fleurantin** and **Christopher K. R. T. Jones**, University of North Carolina.

Recent Developments in the Study of Free Boundary Problems in Fluid Mechanics (Code: SS 45A), **Huy Q. Nguyen**, University of Maryland, and **Ian Tice**, Carnegie Mellon University.

Recent Progress on Model-Based and Data-Driven Methods in Inverse Problems and Imaging (Code: SS 37A), **Yimin Zhong**, Auburn University, **Yang Yang**, Michigan State University, and **Junshan Lin**, Auburn University.

Recent Trends in Graph Theory (Code: SS 38A), **Katherine Perry**, Soka University of America, and **Adam Blumenthal**, Westminster College.

Riordan Arrays (Code: SS 39A), **Dennis Davenport** and **Lou Shapiro**, Howard University, and **Leon Woodson**, SPIRAL REU At Georgetown.

Skein Modules in Low Dimensional Topology (Code: SS 40A), **Jozef Henryk Przytycki**, George Washington University.

Spectral Theory and Quantum Systems (Code: SS 41A), **Laura Shou**, University of Maryland, and **Shiwen Zhang**, U Mass Lowell.

Stochastic Methods in Fluid Mechanics (Code: SS 42A), **Hussain Ibdah**, Univeristy of Maryland, **Theodore D. Drivas**, S, and **Kyle Liss**, Duke University.

Tensor Algebra & Networks (Code: SS 43A), **Giuseppe Cotardo**, **Gretchen Matthews**, and **Pedro Soto**, Virginia Tech.

Variational Problems with Lack of Compactness (Code: SS 44A), **Cheikh Birahim Ndiaye**, Howard University, and **Ali Maalaoui**, Clark University.

Contributed Paper Sessions

AMS Contributed Paper Session (Code: CP 1A), **Steven H Weintraub**, Lehigh University.

Milwaukee, Wisconsin

University of Wisconsin-Milwaukee

April 20–21, 2024

Saturday – Sunday

Meeting #1195

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 20, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Mihaela Ifrim, University of Wisconsin-Madison, *The small data global well-posedness conjecture for 1D defocusing dispersive flows*.

Lin Lin, University of California, Berkeley, *Title To Be Announced*.

Kevin Schreve, LSU, *Homological growth of groups and aspherical manifolds*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic methods in graph theory and applications I (Code: SS 1A), **Tung T. Nguyen**, University of Chicago/ Western University, **Sunil K. Chebolu**, Illinois State University, and **Jan Minac**, Western University.

Algorithms, Number Theory, and Cryptography I (Code: SS 3A), **Jonathan P Sorenson**, Butler University, **Eric Bach**, University of Wisconsin at Madison, and **Jonathan Webster**, Butler University.

Applications of Algebra and Geometry I (Code: SS 8A), **Thomas Yahl**, University of Wisconsin - Madison, and **Jose Israel Rodriguez**, University of Wisconsin Madison.

Applications of Numerical Algebraic Geometry I (Code: SS 14A), **Emma R Cobian**, University of Notre Dame.

Artificial Intelligence in Mathematics I (Code: SS 9A), **Tony Shaska**, Oakland University, **Alessandro Arsie**, The University of Toledo, **Elira Curri**, Oakland University, Rochester Hills, MI, 48126, and **Mee Seong Im**, United States Naval Academy.

Automorphisms of Riemann Surfaces and Related Topics I (Code: SS 4A), **Aaron D. Wootton**, University of Portland, **Jennifer Paulhus**, Grinnell College, **Sean Allen Broughton**, Rose-Hulman Institute of Technology (emeritus), and **Tony Shaska**, Research Institute of Science and Technology.

Cluster algebras, Hall algebras and representation theory I (Code: SS 5A), **Xueqing Chen**, University of Wisconcin, Whitewater, and **Yiqiang Li**, SUNY At Buffalo.

Combinatorial and geometric themes in representation theory I (Code: SS 23A), **Jeb F. Willenbring**, UW-Milwaukee, and **Pamela E. Harris**, University of Wisconsin, Milwaukee.

Complex Dynamics and Related Areas I (Code: SS 16A), **James Waterman**, Stony Brook University, and **Alastair N Fletcher**, Northern Illinois University.

Computability Theory I (Code: SS 25A), **Matthew Harrison-Trainor**, University of Illinois Chicago, and **Steffen Lempp**, University of Wisconsin-Madison.

Connections between Commutative Algebra and Algebraic Combinatorics I (Code: SS 10A), **Alessandra Costantini**, Oklahoma State University, **Matthew James Weaver**, University of Notre Dame, and **Alexander T Yong**, University of Illinois at Urbana-Champaign.

Developments in hyperbolic-like geometry and dynamics I (Code: SS 11A), **Jonah Gaster**, University of Wisconsin-Milwaukee, **Andrew Zimmer**, University of Wisconsin-Madison, and **Chenxi Wu**, University of Wisconsin At Madison.

Geometric group theory I (Code: SS 28A), **G Christopher Hruska**, University of Wisconsin-Milwaukee, and **Emily Stark**, Wesleyan University.

Geometric Methods in Representation Theory I (Code: SS 2A), **Daniele Rosso**, Indiana University Northwest, and **Joshua Mundinger**, University of Wisconsin - Madison.

Harmonic Analysis and Incidence Geometry I (Code: SS 17A), **Sarah E Tammen** and **Terence L. J Harris**, UW Madison, and **Shengwen Gan**, Massachusetts Institute of Technology.

Mathematical aspects of cryptography and cybersecurity I (Code: SS 24A), **Lubjana Beshaj**, Army Cyber Institute.

Model Theory I (Code: SS 15A), **Uri Andrews**, University of Wisconsin-Madison, and **James Freitag**, University of Illinois Chicago.

New research and open problems in combinatorics I (Code: SS 12A), **Pamela Estephania Harris**, University of Wisconsin, Milwaukee, **Erik Insko**, Central College, and **Mohamed Omar**, York University.

Nonlinear waves I (Code: SS 22A), **Mihaela Ifrim**, University of Wisconsin-Madison, and **Daniel I Tataru**, UC Berkeley.

Nonstandard and Multigraded Commutative Algebra I (Code: SS 13A), **Mahrud Sayrafi**, University of Minnesota, Twin Cities, and **Maya Banks** and **Aleksandra C Sobieska**, University of Wisconsin - Madison.

Panorama of Holomorphic Dynamics I (Code: SS 21A), **Suzanne Lynch Boyd**, University of Wisconsin Milwaukee, and **Rodrigo Perez** and **Roland Roeder**, Indiana University - Purdue University Indianapolis.

Posets in algebraic and geometric combinatorics I (Code: SS 26A), **Martha Yip**, University of Kentucky, and **Rafael S. González D'León**, Loyola University Chicago.

Ramification in Algebraic and Arithmetic Geometry I (Code: SS 29A), **Charlotte Ure**, Illinois State University, and **Nick Rekuski**, Wayne State University.

Recent Advances in Nonlinear PDEs and Their Applications I (Code: SS 27A), **Xiang Wan**, Loyola University Chicago, **Rasika Mahawattege**, University of Maryland, Baltimore County, and **Madhumita Roy**, Graduate Student, University of Memphis.

Recent Advances in Numerical PDE Solvers by Deep Learning I (Code: SS 7A), **Dexuan Xie**, University of Wisconsin-Milwaukee, and **Zhen Chao**, University of Michigan-Ann Arbor.

Recent Developments in Harmonic Analysis I (Code: SS 6A), **Naga Manasa Vempati**, Louisiana State University, **Nathan A. Wagner**, Brown University, and **Bingyang Hu**, Auburn University.

Recent trends in nonlinear PDE I (Code: SS 19A), **Fernando Charro** and **Catherine Lebiedzik**, Wayne State University, and **Md Nurul Raihen**, Fontbonne University.

Stochastic Control and Related Fields: A Special Session in Honor of Professor Stockbridge's 70th Birthday I (Code: SS 18A), **Chao Zhu**, University of Wisconsin-Milwaukee, and **MoonJung Cho**, U.S. Bureau of Labor Statistics.

The Algebras and Special Functions around Association Schemes I (Code: SS 20A), **Paul M Terwilliger**, U. Wisconsin-Madison, **Sarah R Bockting-Conrad**, DePaul University, and **Jae-Ho Lee**, University of North Florida.

San Francisco, California

San Francisco State University

May 4–5, 2024

Saturday – Sunday

Meeting #1196

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 3

Deadlines

For organizers: To be announced
For abstracts: March 12, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Julia Yael Plavnik, Indiana University, *Title to be announced*.

Mandi A. Schaeffer Fry, University of Denver, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Commutative and Noncommutative Algebra, Together at Last (Code: SS 1A), **Pablo S. Ocal**, University of California, Los Angeles, **Benjamin Briggs**, University of Copenhagen, and **Janina C Letz**, Bielefeld University.

Diagrammatic Algebras in Representation Theory and Beyond (Code: SS 2A), **Mee Seong Im**, United States Naval Academy, **Liron Speyer**, Okinawa Institute of Science and Technology, **Arik Wilbert**, University of Georgia, and **Jieru Zhu**, University of Queensland.

Extremal Combinatorics and Connections (Code: SS 3A), **Sam Spiro**, Rutgers University, and **Van Magnan**, University of Montana.

Geometry and Topology of Quantum Phases of Matter (Code: SS 4A), **Ralph Martin Kaufmann**, Purdue University, and **Markus J Pflaum**, University of Colorado.

Geometry, Integrability, Symmetry and Physics (Code: SS 5A), **Birgit Kaufmann** and **Sasha Tsymbaliuk**, Purdue University.

Groups and Representations (associated with Invited Address by Mandi Schaeffer Fry) (Code: SS 6A), **Nathaniel Thiem**, University of Colorado, **Mandi A. Schaeffer Fry**, University of Denver, and **Klaus Lux**, University of Arizona.

Homological Methods in Commutative Algebra & Algebraic Geometry (Code: SS 7A), **Ritvik Ramkumar**, Cornell University, **Michael Perlman**, University of Minnesota, and **Aleksandra C Sobieska**, University of Wisconsin - Madison.

Inverse Problems (Code: SS 8A), **Hanna E. Makaruk**, Los Alamos National Laboratory, Los Alamos, NM, and **Robert M. Owczarek**, University of New Mexico.

Mathematical Fluid Dynamics (Code: SS 9A), **Igor Kukavica** and **Juhi Jang**, University of Southern California, and **Wojciech S. Ozanski**, Florida State University.

Mathematical Modeling of Complex Ecological and Social Systems (Code: SS 10A), **Daniel Brendan Cooney**, University of Illinois at Urbana-Champaign, **Mari Kawakatsu**, University of Pennsylvania, and **Chadi M Saad-Roy**, University of California, Berkeley.

Partial Differential Equations and Convexity (Code: SS 11A), **Ben Weinkove**, Northwestern University, **Stefan Steinerberger**, University of Washington, Seattle, and **Albert Chau**, University of British Columbia.

Partial Differential Equations of Quantum Physics (Code: SS 12A), **Israel Michael Sigal**, University of Toronto, and **Stephen Gustafson**, University of British Columbia.

Probability Theory and Related Fields (Code: SS 13A), **Terry Soo** and **Codina Cotar**, University College London.

Random Structures, Computation, and Statistical Inference (Code: SS 14A), **Lutz Warnke**, University of California, San Diego, and **Ilias Zadik**, Yale University.

Recent Advances in Differential Geometry (Code: SS 15A), **Lihan Wang**, California State University, Long Beach, **Zhiqin Lu**, UC Irvine, and **Shoo Seto** and **Bogdan D. Suceavă**, California State University, Fullerton.

Recent Developments in Commutative Algebra (Code: SS 16A), **Arvind Kumar**, **Louiza Fouli**, and **Michael DiPasquale**, New Mexico State University.

Representations of Lie Algebras and Lie Superalgebras (Code: SS 17A), **Dimitar Grantcharov**, University of Texas At Arlington, **Daniel Nakano**, University of Georgia, and **Vera Serganova**, UC Berkeley.

Research in Combinatorics by Early Career Mathematicians (Code: SS 18A), **Nicholas Mayers**, North Carolina State University, and **Laura Colmenarejo**, NCSU.

Special Session in Celebration of Bruce Reznick's Retirement (Code: SS 19A), **Katie Anders**, University of Texas at Tyler, **Simone Sisneros-Thiry**, California State University- East Bay, and **Dana Neidmann**, Centre College.

Tensor Categories and Noncommutative Algebras, I (associated with invited address by Julia Plavnik) (Code: SS 20A), **Ellen E Kirkman**, Wake Forest University, and **Julia Yael Plavnik**, Indiana University, Bloomington.

Contributed Paper Sessions

AMS Contributed Paper Session (Code: CP 1A), **Michelle Ann Manes**, University of Hawaii.

Palermo, Italy

July 23–26, 2024

Tuesday – Friday

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

San Antonio, Texas

University of Texas, San Antonio

September 14–15, 2024

Saturday – Sunday

Meeting #1198

Central Section

Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 45, Issue 3

Deadlines

For organizers: February 13, 2024

For abstracts: July 23, 2024

Savannah, Georgia

Georgia Southern University, Savannah

October 5–6, 2024

Saturday – Sunday

Meeting #1199

Southeastern Section

Associate Secretary for the AMS: Brian D. Boe, University of Georgia

Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 5, 2024

For abstracts: August 13, 2024

Albany, New York

University at Albany

October 19–20, 2024

Saturday – Sunday

Meeting #1200

Eastern Section

Associate Secretary for the AMS: Steven H. Weintraub, Lehigh University

Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 19, 2024

For abstracts: August 27, 2024

Riverside, California

University of California, Riverside

October 26–27, 2024

Saturday – Sunday

Meeting #1201

Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: Not applicable
Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 26, 2024
For abstracts: September 3, 2024

Auckland, New Zealand

December 9–13, 2024

Monday – Friday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 8–11, 2025

Wednesday – Saturday
Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 16, 2024
For abstracts: September 10, 2024

Hartford, Connecticut

Hosted by University of Connecticut; taking place at the Connecticut Convention Center and Hartford Marriott Downtown

April 5–6, 2025

Saturday – Sunday
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center and Marriott Marquis Washington DC

January 4–7, 2026

Sunday – Wednesday
Associate Secretary for the AMS: Betsy Stovall
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Explaining Wildfires Through Curvature

In the summer of 2023, hundreds of wildfires in Canada sent smoke drifting down the North American continent, resulting in dangerous levels of air pollution in New York, Columbus, and other cities. The East Coast was finally tasting the horrific effects of wildfires. In Australia, where wildfires are a concern year-round, residents have already battled those effects for years.

Researchers in Australia have long tried to model these wildfires, hoping to learn information that can help with firefighting policy. Mathematicians like Jason Sharples and Valentina Wheeler are joining the effort. In 2012, Sharples and Wheeler began studying a particularly dangerous phenomenon: When two wildfires meet, they create a new V-shaped fire whose pointed tip races along to catch up with the two branches of the V, moving faster than either of the fires alone. This is exactly what happens in a mathematical process known as mean curvature flow. Mean curvature flow is a process in which a shape smooths out its boundaries over time. Just as with wildfires, pointed corners and sharp bumps will change the fastest.

Wheeler, Sharples, and colleagues modeled fires as curvature flows, using equations that specify how curvature affects the way a wildfire front changes in time and space. Though curvature information cannot explain everything about wildfires, it's one simple and effective tool for a problem that will only become more pressing as the climate evolves.

Wind tunnel fires meeting at an apex. Photo courtesy of Dr. Andrew L. Sullivan for Commonwealth Scientific and Industrial Research Organization.

Watch an interview with an expert!

MM/168

References:

H. Yan, A. Elamroussi, C. Tebor, M. Tirrell, T. Burnside, and E. Tucker. "Intense smoke fills NYC and forces a 'code red' in Philadelphia as millions from the East Coast to Canada suffer from Quebec's wildfires." CNN.

Reszelska. "Using geometry to fight fires and cure disease." University of Wollongong Media.

J.J. Sharples, I.N. Towers, G. Wheeler, V.-M. Wheeler, J.A. McCoy. "Modelling fire line merging using plane curvature flow." 20th International Congress on Modelling and Simulation, Adelaide, Australia, December 1–6, 2013.

V.-M. Wheeler, G. E. Wheeler, J. A. McCoy and J. J. Sharples, "Modelling dynamic bushfire spread: perspectives from the theory of curvature flow." 21st International Congress on Modelling and Simulation, Gold Coast, Australia, November 29–December 4, 2015.

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