ON THE "COLLECTIVE HAUSDORFF METHOD"

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A sequence \( \{s_n\} \) is said to be summable to \( S \) by the Hausdorff method \( T \sim \mu_n \), if \( \lim_{n \to \infty} t_n = S \), where \( \{t_n\} \) is given by

\[
\Delta^n t_0 = \sum_{k=0}^{n} (-1)^k C_n, \lf_k = \mu_n \Delta^n s_0 \quad (n = 0, 1, 2, \cdots).
\]

Hausdorff methods have been extensively investigated by F. Hausdorff and others [1, 3, 5].

A method of summation \( A \) is said to be stronger than another method \( B \) if every sequence summed by \( B \) is also summed by \( A \). Two methods of summation are consistent if any sequence which can be summed by both methods has the same limit assigned to it by both of them. I shall say that \( A \) contains \( B \) if \( A \) is stronger than \( B \) and also consistent with \( B \). A method containing ordinary convergence is called regular. F. Hausdorff proved that any two regular Hausdorff methods are consistent [3]. This enabled R. P. Agnew to introduce the "collective Hausdorff method" \( \mathcal{H} \) by the definition: \( \{s_n\} \) is summable \( \mathcal{H} \) to the sum \( S \) if \( \{s_n\} \) is summable to \( S \) by any regular Hausdorff method [1]. He raised at the same time the question whether there is a matrix method of summation containing \( \mathcal{H} \). I shall now show that the answer is in the negative. More precisely I prove the following:

**Theorem.** There is no matrix \( A = (a(m, n)) \) such that

(a) \( t_m = \sum_{n=0}^{\infty} a(m, n)s_n \quad (m = 0, 1, 2, \cdots) \) converges whenever \( \{s_n\} \) is summable \( \mathcal{H} \) and

(b) if \( \{s_n\} \) is summed to \( S \) by \( \mathcal{H} \), then \( t_m \to S \) as \( m \to \infty \).

To start with I list a few special sequences summed by \( \mathcal{H} \).

(a) Since ordinary convergence is the Hausdorff method \( C \sim 1 \), \( \mathcal{H} \) sums every convergent sequence to its ordinary limit. In other words, \( \mathcal{H} \) is regular.

(b) The sequence \( s_n = (-c)^n \quad (c > 1) \) is summed to 0 by a suitable Euler method \( E_p \sim \rho^{-n} \quad (\rho > 1) \).

(c) The sequence \( s_n = C_{n,k} \) is summed to 0 by Mercer's method \( M_k \sim (k-n)/k(n+1) \).

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1 Numbers in brackets refer to the references cited at the end of the paper.

2 To avoid double subscripts I write \( a(m, n) \) instead of the usual \( a_{mn} \).
Euler methods have been extensively studied by Knopp [4]. The special result stated as (b) can easily be verified by the use of (1). Mercer's methods are investigated in [5]; again (c) can be verified directly by means of (1).

Suppose now that \( A \) is a matrix satisfying the conditions (\( \alpha \)) and (\( \beta \)) of the theorem. By (a) this matrix defines a regular method of summation and hence, by a well known theorem [6],

\[
a(m, n) \to 0 \quad \text{as} \quad m \to \infty \quad (n = 0, 1, \ldots)
\]

and

\[
\sum_{n=0}^{\infty} a(m, n) \to 1 \quad \text{as} \quad m \to \infty.
\]

By (b) and condition (\( \alpha \)) of the theorem

\[
\sum_{n=0}^{\infty} |a(m, n)| c^n < \infty
\]

for every \( c \), since \( \sum a(m, n)c^n \) is an integral function.

By (3) there is an integer \( n(k) \) such that

\[
\sum_{l \geq n(k)} |a(k, l)| 4^l < 2^{-k}.
\]

I show next that there is a matrix \( B \) satisfying the following conditions:

(i) \( b(k, l) = 0 \) except perhaps for \( 2 \leq l < n(k) \).

(ii) Every sequence summed by \( A \) and satisfying \( |s_n| \leq 4^n \) for all large \( n \) is summed to the same sum by \( B \).

It is an immediate consequence of (2) and (4) that the matrix

\[
B = (b(k, l)) \quad \text{with}
\]

\[
b(k, l) = \begin{cases} 
  a(k, l) & \text{for } 2 \leq l < n(k) \\
  0 & \text{otherwise}
\end{cases}
\]

satisfies the requirements (i) and (ii). In particular \( B \) defines a regular method of summation and therefore

\[
\sum_{l=0}^{\infty} b(k, l) = \sum_{l=2}^{\infty} b(k, l) \to 1
\]

as \( k \to \infty \).

If we write

\[
\Delta^n s_0 = s_n,
\]
\[ \sum b(k, l)s_l = \sum c(k, h)\sigma_h, \]

then

\[ s_n = \Delta^n\sigma_0, \]

(6)

\[ b(k, l) = (-1)^l \sum_{h \geq l} c(k, h)C_{h, l}, \]

(7)

\[ c(k, h) = (-1)^h \sum_{l \geq h} b(k, l)C_{l, h}. \]

By condition (i) imposed on \( B \)

(8)

\[ c(k, h) = 0 \quad (h \geq n(k)). \]

By condition (ii), (c), and (7)

(9)

\[ c(k, h) \to 0 \quad \text{as} \quad k \to \infty. \]

I show now that for each \( k > K \), where \( K \) is a sufficiently large number, there is at least one index \( h \) such that

(10)

\[ |c(k, h)| > 3^{-h}. \]

For otherwise, by (6),

\[ |b(k, l)| \leq \sum_{h \geq l} C_{h, l}3^{-h} = 3^{-l} \left( 1 - \frac{1}{3} \right)^{-l-1} = 3 \cdot 2^{-l-1} \]

and so

\[ \left| \sum_{l=2}^{\infty} b(k, l) \right| \leq \sum_{l=2}^{\infty} 3 \cdot 2^{-l-1} = \frac{3}{4} \]

in contradiction to (5).

By (10) we can choose \( k_0 > K \), and \( h_0 \) such that \( |c(k_0, h_0)| > 3^{-h_0}. \)

Because of (8) the element in the \( k_0 \)th row of the matrix \( (c(k, h)) \) are 0 for \( h \geq m_0 \), say. Because of (9) we can find an index \( k_1 > K \) such that

(11)

\[ \sum_{h < m_0} |c(k_1, h)| 3^h < \frac{1}{2}. \]

Also, by (10), there is an index \( h_1 \) such that \( |c(k_1, h_1)| > 3^{-h_1}. \) By (11) we must have \( h_1 \geq m_0 \). Again \( c(k_1, h) = 0 \) for \( h \geq m_1 \), say. Proceeding in this way we find successively \( k_2, h_2, m_2, k_3, h_3, m_3, \ldots \) such that

(12)

\[ \sum_{h < m_{j-1}} |c(k_j, h)| 3^h < \frac{1}{2}, \]

(13)

\[ |c(k_j, h_j)| > 3^{-h_j}, \]
There is always a least possible choice of $m_j$, but we may choose a larger number, if desired, so that there is no loss of generality in supposing

$$\sum 1/m_j < \infty$$

which implies, by (14),

$$\sum 1/h_j < \infty.$$  

We define now a sequence $\{ s_n \}$ by

$$\Delta^n s_0 = \sigma_n = 0 \quad (n \neq h_i),$$

$$\sum_i b(k, l)_{s_i} = \sum_h c(k, h)_{\sigma_h} = (-1)^j \frac{1}{2} \quad (j = 0, 1, 2, \cdots).$$

By (17), (14), and (15),

$$(-1)^j \frac{1}{2} = \sum_h c(k, h)_{\sigma_h} = \sum_{i \leq j} c(k, h_i)_{\tau_i} \quad (j = 0, 1, 2, \cdots),$$

where $\tau_i$ is written for $\sigma$ with the index $h_i$. These equations can be solved successively for $\tau_0, \tau_1, \tau_2, \cdots$ and it is easily proved by means of (12) and (13) that $|\tau_i| < 3^h$. Together with (17) this shows that $|\sigma_h| < 3^h$ and, therefore,

$$|s_n| = |\Delta^n \sigma_0| \leq \sum C_{n, h} |\sigma_h| < \sum C_{n, h} 3^h = 4^n.$$

It is obvious from (18) that $\{ s_n \}$ is not summed by $B$, and so, by condition (ii) imposed on $B$, $\{ s_n \}$ is not summed by $A$. On the other hand it follows from (1) and (17) that $\{ s_n \}$ is summed to 0 by any Hausdorff method for which

$$\mu_n = 0 \quad (n = h_0, h_1, h_2, \cdots).$$

To complete the proof of the theorem it is necessary only to show that there is a regular Hausdorff method satisfying (19).

By a well known theorem [3] the method $T_{\sim \mu_n}$ is regular if

$$\mu_n = T(n) = \int_0^1 t^n d\phi(t),$$

where $\phi(t)$ is of bounded variation in $[0, 1)$,

$$\mu_0 = \phi(1) - \phi(0) = 1,$$
and

\[ \phi(0) = \phi(0^+). \]

It is an easy consequence of Mellin's formula that the function* can be written in the form \( T_1(z) = \int_0^1 \frac{h_i - z}{h_i + z} \) where \( \phi(t) \) satisfies all the conditions just stated (details of the proof of this statement can be found in [2]). This proves the existence of the regular method \( T_1 \sim \mu_n = T_1(n) \) satisfying (19) and completes the proof of the theorem.

This proof makes essential use of "unreasonable" Hausdorff methods for which the moment function \( T(z) \) has zeros in the right half-plane. It remains an open problem whether there is a matrix method containing all reasonable Hausdorff methods, for which \( T(z) \neq 0 \) in \( \Re z \geq 0 \).

**References**


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* The infinite product converges because of (16).