ON THE R_λ-ASSOCIATE OF A LINE

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1. Introduction. Popa has shown [4][1] a method of establishing a one-to-one correspondence between lines through a point of a plane net and lines not through the point. Our purpose is to extend this notion to the asymptotic net on a surface. This extension gives a new manner of describing geometrically the R_λ-associate of a line and the R_λ-derived curves as defined [1] by Bell. A simple characterization of the Darboux-Segre pencil is found.

To fix the notation let the coordinates (x^1, x^2, x^3, x^4) of a generic point x on the surface S be functions of the asymptotic parameters u, v; and let these functions be normalized so that they satisfy the differential equations [2]

$$\begin{aligned} x_{uu} &= \theta_u x_u + \beta x_v + p x, \\ x_{vv} &= \gamma x_u + \theta_v x_v + q x, \qquad \theta = \log R. \end{aligned} \tag{1.1}$$

The line joining the points x, x_{uv} is called the R-conjugate line, and that joining x_u, x_v the R-harmonic line of S at x.

Let local coordinates (x_1, x_2, x_3, x_4) referred to the tetrahedron x, x_u, x_v, x_{uv} of a general point X be defined by the formula $X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}$. Let there be given a curve C through x defined by the differential equation $dv - \lambda du = 0$. The equation of the tangent l to C at x is

$$\lambda x_2 - x_3 = 0. \tag{1.2}$$

Moreover let a point z, not in the tangent plane to S at x, be defined by the expression

$$z = x_{uv} - a x_u - b x_v - c x,$$

and denote by h_1 the line xz. The reciprocal h_2 of h_1 with respect to the quadric of Darboux joins the points r, s defined by

$$r = x_u - b x, \qquad s = x_v - a x.$$

2. The R_λ-associate of h_2. Let $y = x(u + \Delta u, v + \Delta v)$ be a point on C near x. The tangent to the asymptotic curve v = const. at y intersects the plane determined by h_1 and the tangent to v = const. at x in a point, the limit ρ of whose projection on the tangent plane to S at x

Presented to the Society, November 27, 1948; received by the editors October 2, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

from the point z as y approaches x along C has coordinates defined by the expression

$$\rho = x_u - \left(b + \frac{\beta}{\lambda}\right)x. \tag{2.1}$$

Similarly a point σ, defined by

$$\sigma = x_v - (a + \gamma\lambda)x, \tag{2.2}$$

may be defined. Let the line determined by ρ, σ be denoted by h.

From the form of (2.1), (2.2), it follows that *h is the R_μ-associate* [1] ($\mu = -\lambda$) of h_2. The equation of h is readily seen to be

$$x_1 + \left(b + \frac{\beta}{\lambda}\right)x_2 + (a + \gamma\lambda)x_3 = 0, \qquad x_4 = 0. \tag{2.3}$$

The envelope of h as l varies in the pencil on x is the conic whose equation is

$$(u_2 - bu_1)(u_3 - au_1) = \beta\gamma u_1^2. \tag{2.4}$$

This conic is tangent to the asymptotic tangents at x at their intersections with the line h_2.

3. **Applications.** The line h is tangent to the conic (2.4) at the point T whose coordinates are

$$(b\gamma\lambda^2 + 2\beta\gamma\lambda + a\beta, \ -\gamma\lambda^2, \ -\beta). \tag{3.1}$$

The equation of the line m determined by x and T is

$$\beta x_2 - \gamma\lambda^2 x_3 = 0. \tag{3.2}$$

The conjugate of m is readily seen to be the R_λ-correspondent [1] *of l*; that is, *it is the conjugate to the tangent to the R_λ-derived curve through x.*

The line m intersects the conic (2.4) *in the points T and T', the tangent at the latter point being the R_λ-associate of h_2.*

Comparing (1.2) and (3.2) we observe that l and m coincide if and only if l is a tangent of Segre, and are conjugate if and only if l is a tangent of Darboux. These theorems are equivalent to Theorems [4.1] and [4.2] of Bell's paper cited in the references.

More generally, one of the cross ratios of the asymptotic tangents and of l and m is $\beta/\gamma\lambda^3$. Hence *all of the tangents to the curves of the Darboux-Segre pencil of conjugate nets are characterized in terms of a constant cross ratio of these four lines.*

We find readily that the locus of the intersection of l and h as l varies in the pencil on x is the rational cubic:

$$x_1x_2x_3 + \beta x_2^3 + bx_2^2x_3 + ax_2x_3^2 + \gamma x_3^3 = 0. \tag{3.3}$$

This cubic is a special case of the cubic (8.7) derived in a similar manner by Bell in the paper [1] cited. *The points of inflexion of the cubic* (3.3) *lie on the Darboux tangents, and on the line* h_2. *We may readily show that the line* h *is tangent to the cubic if and only if* $\lambda^3 = -\beta/\gamma$, *that is, if and only if* l *is a tangent of Darboux. If the line* h_1 *is the R-conjugate line* ($a=b=0$), *the Hessian of the characteristic cubic defined by us* [3] *is the cubic* (3.3).

References

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