ON THE $R_\lambda$-ASSOCIATE OF A LINE

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1. Introduction. Popa has shown [4] a method of establishing a one-to-one correspondence between lines through a point of a plane net and lines not through the point. Our purpose is to extend this notion to the asymptotic net on a surface. This extension gives a new manner of describing geometrically the $R_\lambda$-associate of a line and the $R_\lambda$-derived curves as defined [1] by Bell. A simple characterization of the Darboux-Segre pencil is found.

To fix the notation let the coordinates $(x^1, x^2, x^3, x^4)$ of a generic point $x$ on the surface $S$ be functions of the asymptotic parameters $u, v$; and let these functions be normalized so that they satisfy the differential equations [2]

$$
x_{uu} = \theta_u x_u + \beta x_v + px,
$$
$$
x_{vv} = \gamma x_u + \theta_v x_v + qx,
$$
$$\theta = \log R.$$

The line joining the points $x, x_{uv}$ is called the $R$-conjugate line, and that joining $x_u, x_v$ the $R$-harmonic line of $S$ at $x$.

Let local coordinates $(x_1, x_2, x_3, x_4)$ referred to the tetrahedron $x, x_u, x_v, x_{uv}$ of a general point $X$ be defined by the formula $X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}$. Let there be given a curve $C$ through $x$ defined by the differential equation $dv - \lambda du = 0$. The equation of the tangent $l$ to $C$ at $x$ is

$$
\lambda x_2 - x_3 = 0.
$$

Moreover let a point $z$, not in the tangent plane to $S$ at $x$, be defined by the expression

$$z = x_{uv} - ax_u - bx_v - cx,$$

and denote by $h_1$ the line $xz$. The reciprocal $h_2$ of $h_1$ with respect to the quadric of Darboux joins the points $r, s$ defined by

$$r = x_u - bx, \quad s = x_v - ax.$$

2. The $R_\lambda$-associate of $h_2$. Let $y = x(u + \Delta u, v + \Delta v)$ be a point on $C$ near $x$. The tangent to the asymptotic curve $v = $const. at $y$ intersects the plane determined by $h_1$ and the tangent to $v = $const. at $x$ in a point, the limit $p$ of whose projection on the tangent plane to $S$ at $x$
from the point \( z \) as \( y \) approaches \( x \) along \( C \) has coordinates defined by the expression
\[
\rho = x_u - \left( b + \frac{\beta}{\lambda} \right) x.
\]

Similarly a point \( \sigma \), defined by
\[
\sigma = x_u - (a + \gamma \lambda) x,
\]
may be defined. Let the line determined by \( \rho, \sigma \) be denoted by \( h \).

From the form of (2.1), (2.2), it follows that \( h \) is the \( R_\mu \)-associate \([1]\) (\( \mu = -\lambda \)) of \( h_2 \). The equation of \( h \) is readily seen to be
\[
(2.3) \quad x_1 + \left( b + \frac{\beta}{\lambda} \right) x_2 + (a + \gamma \lambda) x_3 = 0, \quad x_4 = 0.
\]

The envelope of \( h \) as \( l \) varies in the pencil on \( x \) is the conic whose equation is
\[
(2.4) \quad (u_2 - bu_1)(u_2 - au_1) = \beta y_2 y_1^2.
\]

This conic is tangent to the asymptotic tangents at \( x \) at their intersections with the line \( h_2 \).

3. Applications. The line \( h \) is tangent to the conic (2.4) at the point \( T \) whose coordinates are
\[
(3.1) \quad (b\gamma \lambda^2 + 2\beta\gamma \lambda + a\beta, -\gamma \lambda^2, -\beta).
\]

The equation of the line \( m \) determined by \( x \) and \( T \) is
\[
(3.2) \quad \beta x_2 - \gamma \lambda^2 x_3 = 0.
\]

The conjugate of \( m \) is readily seen to be the \( R_\lambda \)-correspondent \([1]\) of \( l \); that is, it is the conjugate to the tangent to the \( R_\lambda \)-derived curve through \( x \).

The line \( m \) intersects the conic (2.4) in the points \( T \) and \( T' \), the tangent at the latter point being the \( R_\lambda \)-associate of \( h_2 \).

Comparing (1.2) and (3.2) we observe that \( l \) and \( m \) coincide if and only if \( l \) is a tangent of Segre, and are conjugate if and only if \( l \) is a tangent of Darboux. These theorems are equivalent to Theorems \([4.1]\) and \([4.2]\) of Bell's paper cited in the references.

More generally, one of the cross ratios of the asymptotic tangents and of \( l \) and \( m \) is \( \beta/\gamma \lambda^3 \). Hence all of the tangents to the curves of the Darboux-Segre pencil of conjugate nets are characterized in terms of a constant cross ratio of these four lines.

We find readily that the locus of the intersection of \( l \) and \( h \) as \( l \) varies in the pencil on \( x \) is the rational cubic:
This cubic is a special case of the cubic (8.7) derived in a similar manner by Bell in the paper [1] cited. The points of inflexion of the cubic (3.3) lie on the Darboux tangents, and on the line $h_2$. We may readily show that the line $h$ is tangent to the cubic if and only if $\lambda^3 = -\beta/\gamma$, that is, if and only if $l$ is a tangent of Darboux. If the line $h_1$ is the $R$-conjugate line ($a = b = 0$), the Hessian of the characteristic cubic defined by us [3] is the cubic (3.3).

References