A PROBLEM OF P. A. SMITH

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In a paper in this Bulletin,¹ P. A. Smith has mentioned the problem whether in a non-abelian Lie group $G$ there exists a non-countable proper subgroup everywhere dense in $G$. We can see that a negative answer to this problem is unlikely as the non-existence of such group implies the well known continuum hypothesis. It is the aim of the present short note to show that each separable, locally compact, non-discrete metric group has a subgroup possessing the above properties.

Let $G$ be an abstract group and $S$ a subset of $G$. The least subgroup of $G$ which contains $S$ will be called the group closure of $S$ and denoted by $\text{gcl}(S)$. Evidently, $\text{gcl}(S)$ consists of all the finite products of the elements of $S$ and their inverses. It follows immediately that $S = \text{gcl}(S)$ if and only if $S$ forms a subgroup of $G$.

Suppose $R$ to be a subset of $G$ and $p$ an element of $G$ such that $p$ does not belong to the group closure $\text{gcl}(R)$ of $R$. Using Zorn's Theorem² which is equivalent to the Axiom of Choice, we can construct a subset $H$ of $G$ having the following properties:

(i) $H$ forms a subgroup of $G$;
(ii) $H$ contains $R$ but does not contain $p$;
(iii) if $t$ is an element of $G$ not belonging to $H$, then $p$ is contained in the group closure $\text{gcl}(H \cup t)$ of the union $H \cup t$. In general, there exists more than one such subgroup $H$. We shall call each of them a maximal subgroup including $R$ but excluding $p$.

Now let us consider a separable, locally compact, nondiscrete metric group $G$. We choose, in $G$, a countable everywhere dense subset $R$. The group closure $\text{gcl}(R)$ of $R$ is also countable. However, the group $G$, being nondiscrete, is a perfect space. Therefore, $G$ must be non-countable, for otherwise it would be homeomorphic with the set of all rational numbers³ which is not locally compact. It follows that there exists an element $p$ of $G$ which does not belong to $\text{gcl}(R)$. We can construct a maximal subgroup $H$ including $R$ but excluding $p$. Evidently, $H$ forms a proper, everywhere dense subgroup of $G$.

We shall show that $H$ is non-countable. For this purpose, let us

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consider equations of the form

\[ p = z^{m_1}h_{1}z^{m_2}h_{2} \cdots h_{r}z^{m_{r+1}} \]

where \( m_i \ (i = 1, 2, \cdots) \) denote integers, \( h_i \) elements in \( H \), and \( z \) the unknown. The set \( C \) of solutions of such an equation is closed in \( G \), and from the maximal property of \( H \), \( C \) belongs to the complement \( G - H \) of \( H \). Since \( H \) is everywhere dense, \( C \) is nowhere dense.

Suppose \( H \) to be countable. Then the aggregate of all equations of the form (1) is countable as well. Thus the union \( Z = \bigcup C \) of all the possible \( C \)'s is a set of the first category. Moreover, we can easily see that \( Z \) coincides with the complement \( G - H \) of \( H \). On the other hand, \( H \) is countable and \( G \) perfect, so that \( H \) is nowhere dense in \( G \). It follows then that \( G = H + Z \) is of the first category in itself. This contradicts the fact that \( G \) is a locally compact metric space. Therefore, \( H \) cannot be countable and we arrive at the following:

In any separable, locally compact, non-discrete metric group \( G \), there always exists a non-countable proper subgroup filling \( G \) densely.\(^4\)

\(^4\) The author wishes to express his sincere thanks to Professor D. Montgomery for his valuable suggestions which enable the author to prove the theorem under much weaker condition for \( G \) than a Lie group.