NATURAL HOMOMORPHISMS IN BANACH SPACES

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1. **Introduction.**[1] In a recent paper, published in this Bulletin [7], M. E. Munroe considers the following question. Let E be a Banach space, E^* its dual, G a closed linear subspace of E, Γ the subspace of E^* orthogonal to G (that is, the space consisting of the linear functionals x' on E such that $x'(x)=0$ for every $x\in G$); there is then a natural homomorphism T of E^* onto the dual G^*, with kernel Γ; Munroe investigates the continuity and openness of T when both E^* and G^* are given one of the following topologies: norm, weak, weak*, bounded weak, and bounded weak* (the topologies on E^* and G^* being always of the same type).

The openness of T for the norm topologies is proved by Krein and Smulian [5], and is an easy consequence of the Hahn-Banach extension theorem (see [3, p. 124, Theorem 17a]). Munroe proves the openness of T for the weak* and weak topologies (Theorems 3.3 and 3.4), but he has failed to notice that both theorems are particular cases of a general result I proved some years ago [3, p. 116, Theorem 7] for linearly convex spaces, by the same argument Munroe uses in his proof. I shall prove in this note that T is also open for the *bounded weak** and *bounded weak* topologies on E^* and G^*; also I shall rectify an error in the first statement of Munroe's Theorem 3.5.

2. **Characterization of the bounded weak* topology on E^*.** For $x\in E$ and $x'\in E^*$, I shall write $x'(x)=\langle x, x'\rangle$; for any subset A of E (resp. A' of E^*), A^0 (resp. A'^0) will be the subset of E^* (resp. E) consisting of the x' such that $|\langle x, x'\rangle|\leqq 1$ for every $x\in A$ (resp. the x such that $|\langle x, x'\rangle|\leqq 1$ for every $x'\in A'$); clearly A^0 is convex and weakly* closed, A'^0 convex and closed. In the following, the word "compact" is used in the Bourbaki sense (="bicompact").

LEMMA 1. *For every subset A of E, the set A^{00} is the smallest convex and closed subset of E containing both A and $-A$.*

It is clear that $A\subset A^{00}$ and $-A\subset A^{00}$; if A_1 is the smallest convex closed set containing A and $-A$, $A_1\subset A^{00}$. On the other hand, let x be a point in E not belonging to A_1; there exists a closed hyperplane H

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

separating x and A_1, by the Hahn-Banach theorem; this means that there is an $x' \in E^*$ such that $\langle y, x' \rangle \leqq 1$ for every $y \in A_1$, and $\langle x, x' \rangle > 1$; as $-A_1 = A_1$, one has also $|\langle y, x' \rangle| \leqq 1$ for every $y \in A_1$, and as $A \subset A_1$, $x' \in A^0$, and therefore $x \notin A^{00}$, which proves the lemma.

LEMMA 2 (MAZUR). *If A is strongly compact in E, so is A^{00}.*

A^{00} is the closure of the convex hull B of $A \cup (-A)$ in E, and therefore is complete for the metric on E; it is therefore enough to prove that B is strongly precompact. For every $\epsilon > 0$, there exist a finite number of points x_i in $A \cup (-A)$ $(1 \leqq i \leqq n)$ such that any $x \in A \cup (-A)$ is contained in one of the balls $\|x - x_i\| \leqq \epsilon$; if C is the convex hull of the finite set of the x_i, every point of B is at a distance not greater than ϵ from a point of C. But C is finite-dimensional and bounded, therefore compact; there are therefore a finite number of points y_j of C $(1 \leqq j \leqq n)$ such that any point of C is at a distance not greater than ϵ from one of the y_j; therefore every point of B is at a distance not greater than 2ϵ from one of the y_j, which completes the proof.

The bounded weak* topology on E^* may be defined as the finest topology on E^* which, on every ball $\|x'\| \leqq r$, coincides with the weak* topology. The open neighborhoods of the origin for that topology are therefore the sets U containing the origin and such that, for every ball S, $U \cap S$ is open for the topology induced on S by the weak* topology. We now prove the following theorem:

THEOREM 1. *A fundamental system of neighborhoods of the origin for the bounded weak* topology consists of the sets K^0, where K is an arbitrary strongly compact subset of E (in other words, the bounded weak* topology is identical with the k-topology in the sense of Arens* [1, p. 789]).

We shall see that this theorem is practically only a new formulation of a result of Banach [2, pp. 119–120]. Let us prove first that a set K^0 is a neighborhood of O for the bounded weak* topology. Let S be any ball $\|x'\| \leqq r$; as K is compact, there exist a finite number of points x_i in K $(1 \leqq i \leqq n)$ such that any point of K is at a distance not greater than $1/2r$ from one of the points x_i. Let W be the neighborhood of O in the weak* topology, defined by the n inequalities $|\langle x_i, x' \rangle| \leqq 1/2$; for any $x' \in W \cap S$, and any $x \in K$, there is a point x_i such that $\|x - x_i\| \leqq 1/2r$, and therefore

$$|\langle x, x' \rangle| = |\langle x_i, x' \rangle + \langle x - x_i, x' \rangle| \leqq \frac{1}{2} + \frac{1}{2r}\|x'\| \leqq 1$$

which proves that $W \cap S$ is contained in K^0.

We now prove conversely that if U is any open neighborhood of the origin for the bounded weak* topology, it contains a set K^0. This will be a consequence of the following lemma:

LEMMA 3. *There exists in E a finite set A_1 of points of norm greater than 1 and, for every $n>0$, a finite set B_n of points of norm not greater than $1/n$, such that, if $A_{n+1}=A_n\cup B_n$ for $n>0$, the intersection of A_n^0 with the ball $\|x'\|\leqq n$ is contained in U.*

Suppose that lemma is proved: the union A of all sets A_n and of the point O is obviously strongly compact; let S be any ball $\|x'\|\leqq r$, and n an integer greater than r; Lemma 3 shows that the intersection $S\cap A_n^0$ is contained in U, and *a fortiori* $S\cap A^0\subset U$; as the radius of S is arbitrary, $A^0\subset U$.

Lemma 3 is substantially the result of Banach quoted above; for the sake of completeness, we reproduce its proof. The intersection of U with the ball $\|x'\|\leqq 1$ contains a weak* neighborhood of O, and therefore there exists a finite set A_1 of E such that the intersection of A_1^0 with the ball $\|x'\|\leqq 1$ is contained in U; it may obviously be supposed that the norms of the elements in A_1 are greater than 1. Now argue inductively in the following way: suppose A_n is defined as a finite set in E such that the intersection of A_n^0 with the ball S_n: $\|x'\|\leqq n$ is contained in U, and let us show that there exists a finite set B_n of points of norm $\leqq 1/n$ such that the intersection of $(A_n\cup B_n)^0$ with the ball S_{n+1}: $\|x'\|\leqq n+1$ is contained in U. Suppose that proposition were not true, and let K be the intersection of S_{n+1} with the complement of U in E^*; K is closed for the bounded weak* topology, and therefore for the weak* topology, since on S_{n+1} both topologies coincide; as K is bounded, it is weakly* compact [3, p. 128, Theorem 22]. For every finite subset B of the ball $\|x\|\leqq 1/n$ there would exist a nonvoid subset $H(B)$ of K, contained in $(A_n\cup B)^0$; the subsets $H(B)$ constitute obviously a filter base on K, having therefore a cluster point x_0' in K for the weak* topology, since K is weakly* compact. For every $\epsilon>0$ and every $x\in E$ such that $\|x\|\leqq 1/n$, there exists therefore an $x'\in K$ such that $|\langle x, x'\rangle|\leqq 1$ and $|\langle x, x'-x_0'\rangle|\leqq\epsilon$, whence $|\langle x, x_0'\rangle|\leqq 1+\epsilon$, and as ϵ is arbitrary, $|\langle x, x_0'\rangle|\leqq 1$; as that relation holds for every x such that $\|x\|\leqq 1/n$, we have $\|x_0'\|\leqq n$. The same argument shows that, for every $x\in A_n$, one has $|\langle x, x_0'\rangle|\leqq 1$, in other words $x_0'\in A_n^0\cap S_n$, and the induction hypothesis implies $x_0'\in U$, which is absurd, since x_0' is in K.

THEOREM 2. *The natural homomorphism T is open for the bounded weak* topology.*

We shall adapt to our purpose an argument of Köthe [4, p. 27].

Let K be a strongly compact subset of E; as K^{00} is strongly compact and $(K^{00})^0 = K^0$, we may suppose that K is convex and symmetric by Lemma 2. Let U be the neighborhood of O in E^* for the bounded weak* topology, consisting of the elements x' such that $|\langle x, x'\rangle| < 1$ for every $x \in K$ (Theorem 1); we want to prove that $T(U)$ contains a neighborhood of the origin in G^* for the bounded weak* topology. We shall reach that goal by proving that $T(U)$ contains all elements $u' \in G^*$ such that $|\langle x, u'\rangle| < 1$ for every $x \in K \cap G$ ($K \cap G$ being strongly compact in G). As u' may be considered as the restriction to G of a linear functional x_0' defined in E, we have to prove that if x_0' is such that $|\langle x, x_0'\rangle| < 1$ for every $x \in K \cap G$, there exists a linear functional y_0' such that $x_0' - y_0' \in \Gamma$ and that $|\langle y, y_0'\rangle| < 1$ for every $y \in K$.

Consider in E^* the closed subspace H orthogonal to all elements of K, and let ϕ be the natural homomorphism of E^* onto E^*/H; for every element $\phi(x')$ of E^*/H, $\|\phi(x')\| = \sup_{x \in K} |\langle x, x'\rangle|$ is a norm on E^*/H, as is readily verified. Moreover, if f is any linear functional on E^*/H, continuous for the topology defined by that norm, $f o \phi$ is a linear functional on E^*, such that $|f(\phi(x'))|$ is bounded on K^0; as K is strongly compact in E, it is also weakly compact, and therefore, by Arens's theorem [1, p. 790, Theorem 2], there exists an element $z \in E$ such that $f(\phi(x')) = \langle z, x'\rangle$ identically.

Let d be the distance of $\phi(x_0')$ to the linear subspace $\phi(\Gamma)$ in the normed space E^*/H and suppose first $\phi(\Gamma)$ is not everywhere dense in E^*/H. The Hahn-Banach extension theorem proves the existence of a continuous linear functional f_0 on E^*/H, having a norm equal to 1, equal to 0 in the subspace $\phi(\Gamma)$, and such that $f_0(\phi(x_0')) = d$. Let x_0 be an element of E such that $f_0(\phi(x')) = \langle x_0, x'\rangle$ identically; the definition of f_0 shows that $x_0 \in G$, and that, for every $x' \in E^*$, $|\langle x_0, x'\rangle| \leqq \sup_{x \in K} |\langle x, x'\rangle|$; in particular, if $x' \in K^0$, one has $|\langle x_0, x'\rangle| \leqq 1$ by definition, and therefore $x_0 \in K^{00} = K$; we thus see that $x_0 \in K \cap G$. Finally, the definition of the distance of a point to a set in a metric space shows that, given any $\epsilon > 0$, there exists $z' \in \Gamma$ such that $\sup_{x \in K} |\langle x, x_0' - z'\rangle| \leqq \langle x_0, x_0'\rangle + \epsilon$; since $x_0 \in K \cap G$, $\langle x_0, x_0'\rangle < 1$ by assumption; if ϵ is taken such that $\langle x_0, x_0'\rangle + \epsilon < 1$, and if $y_0' = x_0' - z'$, one has $|\langle x, y_0'\rangle| < 1$ for every $x \in K$, which is what we set out to prove.

If $\phi(\Gamma)$ is everywhere dense in E^*/H, $d = 0$ and the linear functional f_0 does not exist any more;[2] but then, one can repeat the preceding argument for $x_0 = 0$, and this completes the proof.

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[2] I am indebted to the referee for calling may attention to this possibility, and to the modification the proof requires in that case.

THEOREM 3. *T is open for the bounded weak topology.*

We shall prove that a fundamental system of neighborhoods of the origin in E^* for the bounded weak topology consists of the sets K^0, where K is an arbitrary strongly compact subset of E^{**}. Once this result is secured, the argument in Theorem 2 may be applied without change, and proves Theorem 3.

To prove our assertion, consider E^* as a subspace of E^{***}; we have only to show that the bounded weak topology on E^* is induced by the bounded weak* topology on E^{***}, by Theorem 1. This will be proved if we show that any closed set F in E^* for the bounded weak topology is the intersection of E^* with a set closed in E^{***} for the bounded weak* topology and conversely that any such intersection is closed for the bounded weak topology. This second point is obvious, since closed sets in E^{***} for the bounded weak* topology are defined as sets such that their intersection with any ball S_r: $\|x''\| \leqq r$ is weakly* closed; similarly, closed sets in E^* for the bounded weak topology are defined as sets such that their intersection with any ball $S_r \cap E^*$ is weakly closed, and the weak topology on $S_r \cap E^*$ is induced by the weak* topology on S_r. To prove the first point, let F_r be the closure in S_r of $F \cap S_r$ for the weak* topology; then $F \cap S_r = F_r \cap (S_r \cap E^*)$, since by assumption $F \cap S_r$ is closed for the weak topology on E^*. Moreover, one has $F_{r'} \cap S_r \cap E^* = F_r \cap S_r \cap E^*$ for every $r' > r$; therefore, if G is the union of all F_r, $G \cap S_r \cap E^* = F \cap S_r$ for every $r > 0$, which proves that $F = G \cap E^*$. But as every S_r is weakly* closed in E^{***}, $G \cap S_r = F_{r'} \cap S_r$ for every $r' > r$, and therefore $G \cap S_r$ is weakly* closed, which shows that G is closed for the bounded weak* topology, and completes the proof.

At the end of his paper, Munroe raises the question of the closure of $T(\Delta)$ for the norm or weak topologies on E^*, when Δ is a closed (or weakly* closed) linear subspace of E^*; this, as he shows at once, is equivalent to deciding whether the sum $\Delta + \Gamma$ is or is not a closed linear subspace for these topologies. It is well known [6, p. 174] that such is *not* the case when E is any infinite-dimensional reflexive space. This shows at the same time that statement 1 in Munroe's Theorem 3.5 is erroneous. The error in the proof comes from the tacit assumption that the sum of the unit balls in Δ and Γ is still a neighborhood of O in $\Delta + \Gamma$, which need not be the case.

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A PROOF THAT THE GROUP OF ALL HOMEOMORPHISMS OF THE PLANE ONTO ITSELF IS LOCALLY ARCWISE CONNECTED

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Let P be a plane and let $H(P)$ be the group of all homeomorphisms of the plane P onto itself. We topologize $H(P)$ by defining convergence to mean uniform convergence on each compact subset of P. The resulting topology is equivalent to the compact-open topology defined in [1][1] by Fox. It is also known (see [4]) that $H(P)$ is a topological group under this topology. The result obtained in this paper is the following theorem.

THEOREM. *$H(P)$ is locally arcwise connected.*

1. **A metric for** $H(P)$. We assume a rectangular coordinate system for P and let d be the corresponding metric for P. For each positive number r we define $S(r)$ to be the set of all points (u, v) in P such that $\max(|u|, |v|) \leq r$. If f and g are members of $H(P)$ we define

$$\rho(f, g) = \sup_{r>0} \min\left(1/r, \sup_{x \in S(r)} d(f(x), g(x))\right).$$

It is a routine matter to verify that ρ is a distance function which defines an admissible metric for $H(P)$. A metric which is essentially the same as ρ is used by M. Bebutoff in [2]. We shall make use of the fact that $\rho(f, g) < \epsilon$ if and only if $d(f(x), g(x)) < \epsilon$ for all x in $S(1/\epsilon)$.

2. **Isotopy and arcs.** By an isotopy we shall mean a homotopy

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[1] Numbers in brackets refer to the bibliography at the end of the paper.