A NOTE ON CONVERGENCE IN AREA

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Let $T_n: x = x_n(u, v), y = y_n(u, v), z = z_n(u, v), n = 0, 1, \ldots$, be a sequence of continuous transformations from the unit square $S$: $0 \leq u, v \leq 1$, into $E_3$, Euclidean 3-space. For each point $p$ on the unit sphere $U: x^2 + y^2 + z^2 = 1$, $E_2(p)$ will denote the plane through the origin which is perpendicular to the radius joining the origin and $p$; $T_p$ will denote the transformation which projects $E_3$ perpendicularly onto $E_2(p)$. Then $T_pT_n$ is a sequence of continuous transformations from $S$ into $E_2(p)$. Assume that $T_n, n = 1, 2, \ldots$, converges uniformly on $S$ to $T_0$ and that the Lebesgue area $A(T_n)$ of the $F$-surface (see [1, II.3.44]) determined by the transformation $T_n$ is finite for $n = 0, 1, \ldots$. The purpose of the present note is to call attention to the fact that recent results of Helsel [2] and Radó [2, 3] imply the following theorem:

**Theorem.** Under the assumptions stated above, a necessary and sufficient condition for $A(T_n) \rightarrow A(T_0)$ is that $A(T_pT_n) \rightarrow A(T_pT_0)$ for every position of the point $p$ on $U$.

This theorem is the analogue for the Lebesgue area of a new result on convergence in length established by Ayer and Radó [4].

**Proof.** The necessity of the condition has been proved by Radó [3]. To establish the sufficiency of the condition, use will be made of the following formula for the Lebesgue area of the $F$-surface determined by $T_n$ (see [2]):

$$A(T_n) = 2 \left[ \frac{1}{4\pi} \int \int_{U} A(T_pT_n) d\sigma \right],$$

$d\sigma$ being the area element on $U$. In [2] it is shown that the Lebesgue area $A(T_n)$ is equal to twice the integral mean value over $U$ of the lower area $a(T_pT_n)$ of the flat $F$-surface determined by the transformation $T_pT_n$; however, the Lebesgue area $A(T_pT_n)$ is equal to the lower area $a(T_pT_n)$ (see [5], [1, V.2.58], and [6]) so (1) follows. The assumption that $A(T_pT_n) \rightarrow A(T_pT_0)$ for every point $p$ on $U$ implies, in view of (1), that $A(T_n) \rightarrow A(T_0)$ if termwise integration of the sequence $A(T_pT_n)$ is permissible. To show that such is the case, a uniform bound for the functions $A(T_pT_n)$ will be displayed. First ob-
serve that

\[ A(T_p T_n) \leq A(T_n) \]

Also, by a fundamental result of Cesari [5],

\[ A(T_n) \leq A(T_{p_1} T_n) + A(T_{p_2} T_n) + A(T_{p_3} T_n), \]

where \(p_1, p_2, p_3\) are any three points on \(U\) such that the radii joining these points to the origin are mutually perpendicular. Regarding \(p_1, p_2, p_3\) as fixed, the relations \(A(T_{p_i} T_n) \to A(T_{p_i} T_0)\), \(i = 1, 2, 3\), imply the existence of a constant \(K\) such that \(A(T_{p_i} T_n) \leq K\) for \(n = 0, 1, \cdots\) and \(i = 1, 2, 3\). Hence, from (2) and (3), \(A(T_p T_n) \leq 3K\) for all \(p \in U\) and \(n = 0, 1, \cdots\), which completes the proof of the sufficiency.

**Bibliography**


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