ON FAITHFUL REPRESENTATIONS OF LIE GROUPS

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Let $G$ and $H$ be two connected Lie groups and $\phi$ a continuous homomorphism of $H$ into the group of automorphisms of $G$. Then we define a new group $G \times_\phi H$ as follows. The elements of $G \times_\phi H$ are pairs $(g, h)$ ($g \in G, h \in H$) and group multiplication is defined by

$$(g_1, h_1)(g_2, h_2) = (g_1(\phi(h_1)g_2), h_1h_2).$$

Topologically $G \times_\phi H$ is taken to be just the Cartesian product of $G$ and $H$. It is then easily proved that under this topology $G \times_\phi H$ is a Lie group. It is called the semidirect product of $G$ and $H$ under $\phi$. The object of this note is to prove the following theorem.

**Theorem.** Let $G$ be a connected, simply connected solvable Lie group and $H$ a connected Lie group which has a faithful representation. Let $\phi$ be any continuous homomorphism of $H$ into the group of automorphisms of $G$. Then $G \times_\phi H$ has a faithful representation.

The special case of this theorem when $H$ is semisimple is due to Cartan.¹

Let $R$ be the field of real numbers and $K$ the field of either real or complex numbers. Let $G$ be a connected Lie group with the Lie algebra $\mathfrak{g}$ and $\theta$ a representation of $G$ over $K$ of degree $d$. Then we denote by $d\theta$ the representation of $\mathfrak{g}$ given by²

$$d\theta(x) = \lim_{t \to 0} \frac{\theta(\exp tX) - I}{t} \quad (X \in \mathfrak{g})$$

where $t \in R$ and $I$ is the unit matrix of degree $d$. Let $GL(K, d)$ denote the group of all nonsingular matrices of degree $d$ with coefficients in $K$. Any subgroup of $GL(K, d)$ will be called a linear group of degree $d$. Let $\theta$ be the identity representation of a linear Lie group $G$ with the Lie algebra $\mathfrak{g}$ so that $\theta(x) = x$ ($x \in G$). Then $d\theta$ is a faithful representation of $\mathfrak{g}$ and $\exp d\theta(X) = \theta(\exp X) = \exp X$ for any $x \in \mathfrak{g}$. Hence we may identify $\mathfrak{g}$ with $d\theta(\mathfrak{g})$ under $d\theta$. $\mathfrak{g}$ can therefore be regarded as a linear Lie algebra. In particular the Lie algebra of $GL(K, d)$ then

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² For the precise definitions of the terms used in this paper see Chevalley, Theory of Lie groups, Princeton University Press, 1946.
consists of all matrices of degree \( d \) with coefficients in \( K \). We denote it by \( \mathfrak{gl}(K, d) \). Given any subalgebra \( \mathfrak{h} \subseteq \mathfrak{gl}(K, d) \), by the linear Lie group generated by \( \mathfrak{h} \) we mean the analytic subgroup of \( GL(K, d) \) corresponding to \( \mathfrak{h} \).

Let us call a matrix subtriangular if it has zeros on and below the diagonal. First we state the following two well known lemmas.

**Lemma 1.** Let \( \mathfrak{H} \) be the Lie algebra of all subtriangular \( d \times d \) matrices over \( K \) and let \( G \) be the linear Lie group generated by \( \mathfrak{H} \). Then \( X \mapsto \exp X \) is a topological mapping of \( \mathfrak{H} \) onto \( G \). Also \( G \) consists of all matrices which have zeros below the diagonal and 1 everywhere on the diagonal.

**Lemma 2.** Let \( G \) be a connected, simply connected solvable Lie group. Then every analytic subgroup of \( G \) is closed and simply connected.

From now on I adhere strictly to the notation of my paper, *Faithful representations of Lie algebras*, which will be quoted as FRL.

**Lemma 3.** Let \( \mathfrak{L}, \mathfrak{H}, \mathfrak{D} \) be as in Lemma 1 of FRL. We construct the faithful representation \( \theta \) of \( \mathfrak{L} + \mathfrak{D} \) as described there. Then for any \( Y_1, \ldots, Y_s \in \mathfrak{L} \) and \( D_1, \ldots, D_s \in \mathfrak{D} \),

1. \( \exp \theta(Y_1) \cdots \exp \theta(Y_s) \neq I \) unless \( \exp Y_1 \cdots \exp Y_s = I'' \),
2. \( \exp \theta(D_1) \cdots \exp \theta(D_s) = I \)

if and only if \( \exp D_1 \cdots \exp D_s = I' \),

where \( I, I', \) and \( I'' \) are unit matrices of suitable degrees.

If we use the notation of the proof of Lemma 1 of FRL, (1) follows immediately from the fact that \( \mathfrak{L}/\mathfrak{L} \cong \mathfrak{L}/\mathfrak{L}_0 \) where \( \mathfrak{L}_0 = \mathfrak{L} \cap \mathfrak{H}_0 \). Now we prove (2). Put \( \omega^*(X) = (\omega(X))^* \) for \( X \in \mathfrak{L} \). Then it is easily proved by induction on \( s \) that for any \( D \in \mathfrak{D} \)

\[
\left\{ \theta(D) \right\}^s \omega^*(X) = \omega^*(D^s X), \quad s \geq 1.
\]

Hence \( \exp \theta(D) \omega^*(X) = \omega^*(\exp D X) \). Also since \( \theta(D) = d_{D\mathfrak{L}}^\theta \) is a derivation of \( \mathfrak{L}^* = \mathfrak{L}/\mathfrak{L}_0 \), \( \exp \theta(D) \) is an automorphism of \( \mathfrak{L}^* \). Put \( Y_X = (\exp D_1) \cdots (\exp D_s) X (X \in \mathfrak{L}) \). Then if \( \exp \theta(D_1) \cdots \exp \theta(D_s) = I \),

\[
\omega^*(Y_X) - \omega^*(X) = 0 \quad \text{for every } X \in \mathfrak{L}.
\]

Hence \( \omega(Y_X - X) \in \mathfrak{L}_0 \subseteq \mathfrak{L} \). Therefore \( \pi \omega(Y_X - X) = Y_X - X = 0 \). Since this is true for every \( X \),

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Let $G$ be a connected, simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{m}$ be the maximal nilpotent ideal of $\mathfrak{g}$. Then $G$ has a faithful representation $\psi$ such that $d\psi(X)$ is nilpotent for every $X \in \mathfrak{m}$.

By Corollary 1 of FRL we can find a faithful representation $\rho_0$ of $\mathfrak{g}$ such that $\rho_0(X)$ is nilpotent for every $X \in \mathfrak{m}$. We can therefore choose, if necessary, a new base in our representation space such that with respect to this base the matrix representing $\rho_0(X)$ is subtriangular for every $X \in \mathfrak{m}$. Now consider the factor algebra $\mathfrak{g}/\mathfrak{m}$ which is abelian and hence nilpotent. By the same corollary it follows that $\mathfrak{g}/\mathfrak{m}$ has a faithful representation by nilpotent matrices. Hence $\mathfrak{g}$ has a representation $\rho_1$ such that the kernel of $\rho_1$ is $\mathfrak{m}$ and $\rho_1(X)$ is nilpotent for all $X \in \mathfrak{g}$. We can again arrange that $\rho_1(X)$ is subtriangular for all $X$. Put $\rho = \rho_0 + \rho_1$, where $+$ denotes direct sum. Since $G$ is simply connected, there exist representations $\psi_0$ and $\psi_1$ of $G$ such that $d\psi_0 = \rho_0$, $d\psi_1 = \rho_1$. Put $\psi = \psi_0 + \psi_1$. Then $d\psi = \rho$. Let $N$ be the analytic subgroup of $G$ corresponding to $\mathfrak{m}$. Then from Lemma 2, $N$ is a closed invariant subgroup. Consider $\psi(N)$. It is clear that $d\psi(Y) = \rho_0(Y) + \rho_1(Y)$ is subtriangular for all $Y \in \mathfrak{m}$. Hence from Lemmas 1 and 2 it follows that the linear Lie group $\psi(N)$ generated by $d\psi(\mathfrak{m})$ is simply connected. Since $d\psi$ is an isomorphism, $\psi(N)$ is locally isomorphic to $N$. Therefore since $\psi(N)$ is simply connected, $\psi$ maps $N$ isomorphically. Similarly we prove that $\psi_1(G)$ is simply connected. It is clear that the kernel of $\psi_1$ contains $N$. Hence $\psi_1$ defines a representation $\psi^*$ of $G/N$ given by $\psi^*(x) = \psi_1(x)$ where $x \to x^*$ is the natural homomorphism of $G$ onto $G/N = G^*$. Since $d\psi^*$ is an isomorphism, the kernel of $d\psi_1 = \rho_1$ being $\mathfrak{m}$, it follows from the simple connectivity of $\psi^*(G^*) = \psi_1(G)$ that $\psi^*$ is an isomorphism. Hence the kernel of $\psi_1$ is exactly $N$. Let $D$ be the kernel of $\psi$. Then $D$ is contained in the kernel of $\psi_1$ which is $N$. Also since $\psi$ is faithful on $N$, $D \cap N = \{e\}$ where $e$ is the unit element of $N$. Hence $D = \{e\}$ and $\psi$ is a faithful representation. Also $d\psi(X)$ is nilpotent for every $X \in \mathfrak{m}$.

Now we come to the proof of the theorem. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{m}$ the maximal nilpotent ideal of $\mathfrak{g}$. By Lemma 4, $G$ has a faithful representation. Hence we may assume that $G$ is a linear Lie group such that every element of $\mathfrak{m}$ is nilpotent. We keep to the notation of Lemma 3 except that $\mathfrak{g}$ is replaced by $\mathfrak{g}$. Let $\mathfrak{h}$ be the Lie algebra of $H$. Define a homomorphism $d\tau$ of $\mathfrak{h}$ into $\mathfrak{d}$ as follows. Let

$$(\exp D_1) \cdots (\exp D_n) = I'.$$ The converse is obvious. Hence the lemma is proved.
Aut (G) be the group of automorphisms of G. Then Aut (G) is a Lie group with a Lie algebra $\mathfrak{g}$. It is well known that there exists an isomorphism $\lambda$ of $\mathfrak{g}$ onto $\mathfrak{g}$ such that

$$(\exp A) \exp X = \exp ((\exp \lambda(A))X)$$

for any $A \in \mathfrak{g}$ and $X \in \mathfrak{g}$. We put $d\tau = \lambda d\phi$ where $d\phi$ is the homomorphism of $\mathfrak{h}$ into $\mathfrak{g}$ induced by $\phi$. Then for any $P_i \in \mathfrak{h}$, $1 \leq i \leq r$,

$$\phi(\exp P_1 \cdots \exp P_r) \exp X = \exp ((\exp d\tau(P_1) \cdots \exp d\tau(P_r))X).$$

Let $e$ and $e'$ denote the identity elements of $G$ and $H$ respectively. Suppose $\exp P_1 \cdots \exp P_r = e'$. Then clearly

$$\exp X = \phi(\exp P_1 \cdots \exp P_r) \exp X = \exp ((\exp d\tau(P_1) \cdots \exp d\tau(P_r))X).$$

Since this is true for every $X \in \mathfrak{g}$,

$$\exp d\tau(P_1) \cdots \exp d\tau(P_r) = I'.$$

Hence we can define a representation $\tau$ of $H$ by the rule

$$\tau(\exp P_1 \cdots \exp P_r) = \exp d\tau(P_1) \cdots \exp d\tau(P_r) \quad (P_1, \cdots, P_r \in \mathfrak{h}).$$

Put $d\psi = \theta d\tau$. Then $d\psi$ is a representation of $\mathfrak{h}$. Suppose $\exp P_1 \cdots \exp P_r = e' \quad (P_1, \cdots, P_r \in \mathfrak{h})$. Then

$$\tau(\exp P_1 \cdots \exp P_r) = \exp d\tau(P_1) \cdots \exp d\tau(P_r) = I'$$

and from Lemma 3

$$\exp d\psi(P_1) \cdots \exp d\psi(P_r) = I.$$

Hence we can again define a representation $\psi$ of $H$ by putting

$$\psi(\exp P_1 \cdots \exp P_r) = \exp d\psi(P_1) \cdots \exp d\psi(P_r) \quad (P_1, \cdots, P_r \in \mathfrak{h}).$$

Also since $G$ is simply connected there exists a representation $\chi$ of $G$ such that $d\chi(X) = \theta(X)$ for every $X \in \mathfrak{g}$. Hence

$$\chi(\exp Y_1 \cdots \exp Y_r) = \exp \theta(Y_1) \cdots \exp \theta(Y_r) \quad (Y_i \in \mathfrak{g}, 1 \leq i \leq r).$$

From Lemma 3 it follows that $\chi$ is faithful.

Consider the mapping $\mu$ of $G \times_H H$ defined by $\mu(g, h) = \chi(g)\psi(h)$. We claim that $\mu$ is a representation. For any $P \in \mathfrak{h}$ and $X \in \mathfrak{g}$ consider

$$\psi(\exp P)\chi(\exp X)(\psi(\exp P))^{-1} = \exp d\psi(P) \exp \theta(X) \exp (-d\psi(P)) = \exp \theta(D) \exp \theta(X) \exp (-\theta(D))$$

where $D = d\tau(P)$. Now for any two elements $A$, $B \in \mathfrak{gl}(K, d)$,
\[ \exp A \exp B \exp (-A) = \exp (\exp \text{ad} A)B \]

where \( \text{ad} A \) is defined as in FRL. Since \([\theta(D), \theta(Y)] = \theta([D, Y]) = \theta(DY)\) for any \( Y \in g \), it follows immediately that

\[ \exp \theta(D) \exp \theta(X) \exp (-\theta(D)) = \exp \theta((\exp X)Y) \]

\[ = \exp \theta(\tau(\exp P)X) \]

\[ = \chi(\exp \tau(\exp P)X) \]

Since any \( h \in H \) can be written in the form \( \exp P_1 \cdots \exp P_r \), \( P_i \in \mathfrak{h}, 1 \leq i \leq r, r \geq 1 \), we get

\[ \psi(h) \chi(\exp X)(\psi(h))^{-1} = \chi(\phi(h) \exp X). \]

Similarly since every \( g \in G \) can be written as \( \exp Y_1 \cdots \exp Y_r \), \( Y_i \in g, 1 \leq i \leq r, r \geq 1 \), we have

\[ \psi(h) \chi(g)(\psi(h))^{-1} = \chi(\phi(h)g). \]

Therefore

\[ \mu((g_1, h_1)(g_2, h_2)) = \mu(g_1 \phi(h_1)g_2, h_1h_2) = \chi(g_1 \phi(h_1)g_2)\psi(h_1h_2) \]

\[ = \chi(g_1)\chi(\phi(h_1)g_2)\psi(h_1)\psi(h_2) \]

\[ = \chi(g_1)\psi(h_1)\chi(g_2)\psi(h_2) = \mu(g_1, h_1)\mu(g_2, h_2). \]

Since \( \mu \) is clearly a continuous mapping it is a representation of \( G \times_{\phi} H \). By hypothesis \( H \) has a faithful representation \( \nu_0 \). Define a representation \( \nu \) of \( G \times_{\phi} H \) by \( \nu(g, h) = \nu_0(h) \) and put \( \xi = \mu + \nu \). Suppose \( (g, h) \) belongs to the kernel of \( \xi \). Then since \( \xi(g, h) = \mu(g, h) + \nu_0(h) = \chi(g)\psi(h) + \nu_0(h) \) and since \( \nu_0 \) is faithful on \( H \), \( h = e' \). Hence \( g \) belongs to the kernel of \( \chi \). But as \( \chi \) is faithful on \( G \), \( g = e \). Therefore \( (g, h) = (e, e') \) and \( \xi \) is faithful on \( G \times_{\phi} H \).

**Corollary (Malcev).** A connected solvable Lie group \( G \) has a faithful representation if and only if \( G = NA \), where \( N \) is a closed, connected, simply connected invariant subgroup and \( A \) is a connected, compact abelian subgroup such that \( N \cap A = \{e\} \).

Suppose \( G = NA \). For any \( a \in A \) let \( \phi(a) \) denote the automorphism of \( N \) given by \( \phi(a)n = ana^{-1} \) (\( n \in N \)). Then it is easily seen that \( (n, a) \rightarrow na \) is an isomorphism of \( N \times_{\phi} A \) onto \( G \). Since \( A \) is compact, it has a faithful representation. Hence by the above theorem it follows immediately that \( G \) has a faithful representation.

In order to establish the converse we make use of the following lemma which follows easily from the results of Chevalley.\(^4\)
Lemma 5. If $G$ is a connected solvable Lie group and $N$ a closed, connected invariant subgroup such that $G/N$ is compact, then there exists a compact connected abelian subgroup $A$ of $G$ such that $G=AN$ and $A\cap N$ is finite.

Returning to the corollary, suppose $G$ is linear. Since $\mathfrak{g}$ is solvable, we deduce in the usual way that every element $X \in [\mathfrak{g}, \mathfrak{g}]=\mathfrak{g}'$ is nilpotent and therefore may be assumed to be subtriangular. Therefore by Lemmas 1 and 2 the group $G'$ generated by $\mathfrak{g}'$ is simply connected. Let $\hat{d}$ be the degree of $G$ and $G_0$ the group of all matrices in $GL(K, \hat{d})$ which have zero below the diagonal and 1 everywhere on the diagonal. By Lemma 2, $G'$ is closed in $G_0$. However, since $G_0$ is clearly closed in $GL(K, \hat{d})$, $G'$ is closed in $GL(K, \hat{d})$ and therefore in $G$. Let $x \rightarrow x^*$ denote the natural homomorphism of $G$ onto $G/G'=G^*$. Since $G^*$ is abelian, $G^*=T^*V^*$ where $T^*$ and $V^*$ are connected subgroups, $T^*$ being compact and $V^*$ simply connected and $T^*\cap V^* = \{e^*\}$. Let $N$ be the complete inverse image of $V^*$ in $G$. Since $N/G'=V^*$ and $G'$ are both simply connected, $N$ is simply connected and $G/N\cong T^*$ is compact. Therefore by Lemma 5, $G=AN$ where $A$ is compact, connected, and abelian and $A\cap N$ is finite. Let $\sigma \in A\cap N$. Then $\sigma^r=e$ for some $r \geq 1$. Then $\sigma^* \in V^*$ and $(\sigma^*)^r = e^*$. Since $V^*$ is simply connected and abelian, $\sigma^* = e^*$. Hence $\sigma \in A\cap G'$. Since $X \rightarrow \exp X$ is a topological mapping of $\mathfrak{g}'$ onto $G'$, it follows that $\sigma \in G'$, $\sigma^r=e$ implies $\sigma=e$. Hence $A\cap N=\{e\}$. The corollary is therefore proved.

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*This lemma was pointed out to me by Dr. G. D. Mostow.*