ON FAITHFUL REPRESENTATIONS OF LIE GROUPS

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Let $G$ and $H$ be two connected Lie groups and $\phi$ a continuous homomorphism of $H$ into the group of automorphisms of $G$. Then we define a new group $G \times_\phi H$ as follows. The elements of $G \times_\phi H$ are pairs $(g, h)$ $(g \in G, h \in H)$ and group multiplication is defined by

$$(g_1, h_1)(g_2, h_2) = (g_1(\phi(h_1)g_2), h_1h_2).$$

Topologically $G \times_\phi H$ is taken to be just the Cartesian product of $G$ and $H$. It is then easily proved that under this topology $G \times_\phi H$ is a Lie group. It is called the semidirect product of $G$ and $H$ under $\phi$. The object of this note is to prove the following theorem.

**Theorem.** Let $G$ be a connected, simply connected solvable Lie group and $H$ a connected Lie group which has a faithful representation. Let $\phi$ be any continuous homomorphism of $H$ into the group of automorphisms of $G$. Then $G \times_\phi H$ has a faithful representation.

The special case of this theorem when $H$ is semisimple is due to Cartan.¹

Let $R$ be the field of real numbers and $K$ the field of either real or complex numbers. Let $G$ be a connected Lie group with the Lie algebra $g$ and $\theta$ a representation of $G$ over $K$ of degree $\hat{d}$. Then we denote by $d\theta$ the representation of $g$ given by²

$$d\theta(x) = \lim_{t \to 0} \frac{\theta(exp tx) - I}{t} \quad (X \in g)$$

where $t \in R$ and $I$ is the unit matrix of degree $\hat{d}$. Let $GL(K, \hat{d})$ denote the group of all nonsingular matrices of degree $\hat{d}$ with coefficients in $K$. Any subgroup of $GL(K, \hat{d})$ will be called a linear group of degree $\hat{d}$. Let $\theta$ be the identity representation of a linear Lie group $G$ with the Lie algebra $g$ so that $\theta(x) = x$ $(x \in G)$. Then $d\theta$ is a faithful representation of $g$ and $\exp d\theta(X) = \theta(\exp X) = \exp X$ for any $x \in g$. Hence we may identify $g$ with $d\theta(g)$ under $d\theta$. $g$ can therefore be regarded as a linear Lie algebra. In particular the Lie algebra of $GL(K, \hat{d})$ then


² For the precise definitions of the terms used in this paper see Chevalley, *Theory of Lie groups*, Princeton University Press, 1946.

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consists of all matrices of degree $d$ with coefficients in $K$. We denote it by $gl(K, d)$. Given any subalgebra $\mathfrak{h} \subset gl(K, d)$, by the linear Lie group generated by $\mathfrak{h}$ we mean the analytic subgroup of $GL(K, d)$ corresponding to $\mathfrak{h}$.

Let us call a matrix subtriangular if it has zeros on and below the diagonal. First we state the following two well known lemmas.

**Lemma 1.** Let $\mathfrak{g}$ be the Lie algebra of all subtriangular $d \times d$ matrices over $K$ and let $G$ be the linear Lie group generated by $\mathfrak{g}$. Then $X \mapsto \exp X$ is a topological mapping of $\mathfrak{g}$ onto $G$. Also $G$ consists of all matrices which have zeros below the diagonal and 1 everywhere on the diagonal.

**Lemma 2.** Let $G$ be a connected, simply connected solvable Lie group. Then every analytic subgroup of $G$ is closed and simply connected.

From now on I adhere strictly to the notation of my paper, Faithful representations of Lie algebras, which will be quoted as FRL.

**Lemma 3.** Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{d}$ be as in Lemma 1 of FRL. We construct the faithful representation $\theta$ of $\mathfrak{g} \oplus \mathfrak{d}$ as described there. Then for any $Y_1, \cdots, Y_s \in \mathfrak{g}$ and $D_1, \cdots, D_s \in \mathfrak{d}$,

1. $\exp \theta(Y_1) \cdots \exp \theta(Y_s) \neq I$ unless $\exp Y_1 \cdots \exp Y_s = I'$,
2. $\exp \theta(D_1) \cdots \exp \theta(D_s) = I$ if and only if $\exp D_1 \cdots \exp D_s = I'$,

where $I, I', I''$ are unit matrices of suitable degrees.

If we use the notation of the proof of Lemma 1 of FRL, (1) follows immediately from the fact that $\mathfrak{a}/\mathfrak{x} \cong \mathfrak{a}^*/\mathfrak{x}^*$, where $\mathfrak{x}^* = \mathfrak{x}/\mathfrak{x}_0$. Now we prove (2). Put $\omega^*(X) = (\omega(X))^*$ for $X \in \mathfrak{g}$. Then it is easily proved by induction on $s$ that for any $D \in \mathfrak{d}$

$$\{\theta(D)\}^s \omega^*(X) = \omega^*(D^s X), \quad s \geq 1.$$ 

Hence $(\exp \theta(D)) \omega^*(X) = \omega^*((\exp D) X)$. Also since $\theta(D) = d^b_D$ is a derivation of $\mathfrak{a}^* = \mathfrak{a}/\mathfrak{x}_0$, $\exp \theta(D)$ is an automorphism of $\mathfrak{a}^*$. Put $Y_X = (\exp D_1) \cdots (\exp D_s) X$ $(X \in \mathfrak{g})$. Then if

$$\exp \theta(D_1) \cdots \exp \theta(D_s) = I,$$

$$\omega^*(Y_X) - \omega^*(X) = 0$$

for every $X \in \mathfrak{g}$. Hence $\omega(Y_X - X) \in \mathfrak{x}_0 \subset \mathfrak{x}$. Therefore $\pi \omega(Y_X - X) = Y_X - X = 0$. Since this is true for every $X$,

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LEMMA 4. Let $G$ be a connected, simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{M}$ be the maximal nilpotent ideal of $\mathfrak{g}$. Then $G$ has a faithful representation $\psi$ such that $d\psi(X)$ is nilpotent for every $X \in \mathfrak{M}$.

By Corollary 1 of FRL we can find a faithful representation $\rho_0$ of $\mathfrak{g}$ such that $\rho_0(X)$ is nilpotent for every $X \in \mathfrak{M}$. We can therefore choose, if necessary, a new base in our representation space such that with respect to this base the matrix representing $\rho_0(X)$ is subtriangular for every $X \in \mathfrak{M}$. Now consider the factor algebra $\mathfrak{g}/\mathfrak{M}$ which is abelian and hence nilpotent. By the same corollary it follows that $\mathfrak{g}/\mathfrak{M}$ has a faithful representation by nilpotent matrices. Hence $\mathfrak{g}$ has a representation $\rho_1$ such that the kernel of $\rho_1$ is $\mathfrak{M}$ and $\rho_1(X)$ is nilpotent for all $X \in \mathfrak{g}$. We can again arrange that $\rho_1(X)$ is subtriangular for all $X$. Put $\rho = \rho_0 + \rho_1$, where $\rho$ denotes direct sum. Since $G$ is simply connected, there exist representations $\psi_0$ and $\psi_1$ of $G$ such that $d\psi_0 = \rho_0$, $d\psi_1 = \rho_1$. Put $\psi = \psi_0 + \psi_1$. Then $d\psi = \rho$. Let $N$ be the analytic subgroup of $G$ corresponding to $\mathfrak{M}$. Then from Lemma 2, $N$ is a closed invariant subgroup. Consider $\psi(N)$. It is clear that $d\psi(Y) = \rho_0(Y) + \rho_1(Y)$ is subtriangular for all $Y \in \mathfrak{M}$. Hence from Lemmas 1 and 2 it follows that the linear Lie group $\psi(N)$ generated by $d\psi(\mathfrak{M})$ is simply connected. Since $d\psi$ is an isomorphism, $\psi(N)$ is locally isomorphic to $N$. Therefore since $\psi(N)$ is simply connected, $\psi$ maps $N$ isomorphically. Similarly we prove that $\psi_1(G)$ is simply connected. It is clear that the kernel of $\psi_1$ contains $N$. Hence $\psi_1$ defines a representation $\psi^*$ of $G/N$ given by $\psi^*(x^*) = \psi_1(x)$ where $x \to x^*$ is the natural homomorphism of $G$ onto $G/N = G^*$. Since $d\psi^*$ is an isomorphism, the kernel of $d\psi_1 = \rho_1$ being $\mathfrak{M}$, it follows from the simple connectivity of $\psi^*(G^*) = \psi_1(G)$ that $\psi^*$ is an isomorphism. Hence the kernel of $\psi_1$ is exactly $N$. Let $D$ be the kernel of $\psi$. Then $D$ is contained in the kernel of $\psi_1$ which is $N$. Also since $\psi$ is faithful on $N$, $D \cap N = \{e\}$ where $e$ is the unit element of $N$. Hence $D = \{e\}$ and $\psi$ is a faithful representation. Also $d\psi(X)$ is nilpotent for every $X \in \mathfrak{M}$.

Now we come to the proof of the theorem. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{M}$ the maximal nilpotent ideal of $\mathfrak{g}$. By Lemma 4, $G$ has a faithful representation. Hence we may assume that $G$ is a linear Lie group such that every element of $\mathfrak{M}$ is nilpotent. We keep to the notation of Lemma 3 except that $\mathfrak{Q}$ is replaced by $\mathfrak{g}$. Let $\mathfrak{h}$ be the Lie algebra of $H$. Define a homomorphism $d\tau$ of $\mathfrak{h}$ into $\mathfrak{D}$ as follows. Let

$$(\exp D_1) \cdots (\exp D_n) = I'.$$ The converse is obvious. Hence the lemma is proved.
Aut (G) be the group of automorphisms of G. Then Aut (G) is a Lie group with a Lie algebra \( \mathfrak{g} \). It is well known that there exists an isomorphism \( \lambda \) of \( \mathfrak{g} \) onto \( \mathfrak{h} \) such that

\[
\exp A \exp X = \exp ((\exp \lambda(A))X)
\]

for any \( A \in \mathfrak{g} \) and \( X \in \mathfrak{g} \). We put \( d\tau = \lambda d\phi \) where \( d\phi \) is the homomorphism of \( \mathfrak{h} \) into \( \mathfrak{g} \) induced by \( \phi \). Then for any \( P_i \in \mathfrak{h} \), \( 1 \leq i \leq r \),

\[
\phi(\exp P_1 \cdots \exp P_r) \exp X = \exp ((\exp d\tau(P_1) \cdots \exp d\tau(P_r))X).
\]

Let \( e \) and \( e' \) denote the identity elements of \( G \) and \( H \) respectively. Suppose \( \exp P_1 \cdots \exp P_r = e' \). Then clearly

\[
\exp X = \phi(\exp P_1 \cdots \exp P_r) \exp X = \exp ((\exp d\tau(P_1) \cdots \exp d\tau(P_r))X).
\]

Since this is true for every \( X \in \mathfrak{g} \),

\[
\exp d\tau(P_1) \cdots \exp d\tau(P_r) = I'.
\]

Hence we can define a representation \( \tau \) of \( H \) by the rule

\[
\tau(\exp P_1 \cdots \exp P_r) = \exp d\tau(P_1) \cdots \exp d\tau(P_r) \quad (P_1, \cdots, P_r \in \mathfrak{h}).
\]

Put \( d\psi = \theta d\tau \). Then \( d\psi \) is a representation of \( \mathfrak{h} \). Suppose \( \exp P_1 \cdots \exp P_r = e' \) \( (P_1, \cdots, P_r \in \mathfrak{h}) \). Then

\[
\tau(\exp P_1 \cdots \exp P_r) = \exp d\tau(P_1) \cdots \exp d\tau(P_r) = I'
\]

and from Lemma 3

\[
\exp d\psi(P_1) \cdots \exp d\psi(P_r) = I.
\]

Hence we can again define a representation \( \psi \) of \( H \) by putting

\[
\psi(\exp P_1 \cdots \exp P_r) = \exp d\psi(P_1) \cdots \exp d\psi(P_r) \quad (P_1, \cdots, P_r \in \mathfrak{h}).
\]

Also since \( G \) is simply connected there exists a representation \( \chi \) of \( G \) such that \( d\chi(X) = \theta(X) \) for every \( X \in \mathfrak{g} \). Hence

\[
\chi(\exp Y_1 \cdots \exp Y_r) = \exp \theta(Y_1) \cdots \exp \theta(Y_r) \quad (Y_i \in \mathfrak{g}, 1 \leq i \leq r).
\]

From Lemma 3 it follows that \( \chi \) is faithful.

Consider the mapping \( \mu \) of \( G \times H \) defined by \( \mu(g, h) = \chi(g)\psi(h) \).

We claim that \( \mu \) is a representation. For any \( P \in \mathfrak{h} \) and \( X \in \mathfrak{g} \) consider

\[
\psi(\exp P)\chi(\exp X)(\psi(\exp P))^{-1} = \exp d\psi(P) \exp \theta(X) \exp (-d\psi(P)) = \exp \theta(D) \exp \theta(X) \exp (-\theta(D))
\]

where \( D = d\tau(P) \). Now for any two elements \( A, B \in \mathfrak{gl}(K, d) \),
\[ \exp A \exp B \exp (-A) = \exp ((\exp \text{ad} A)B) \]

where \( \text{ad} A \) is defined as in FRL. Since \( [\theta (D), \theta (Y)] = \theta ([D, Y]) = \theta (DY) \) for any \( Y \in g \), it follows immediately that

\[
\exp \theta(D) \exp \theta(X) \exp (-\theta(D)) = \exp \theta((\exp D)X)
= \exp \theta(\tau (\exp P) X)
= \chi (\exp \tau (\exp P) X)
= \chi (\phi (\exp P) \exp X).
\]

Since any \( h \in H \) can be written in the form \( \exp P_1 \cdots \exp P_r \), \( P_i \in h, 1 \leq i \leq r, r \geq 1, \) we get

\[
\psi (h) \chi (\exp X) (\psi (h))^{-1} = \chi (\phi (h) \exp X).
\]

Similarly since every \( g \in G \) can be written as \( \exp Y_1 \cdots \exp Y_r \), \( Y_i \in g, 1 \leq i \leq r, r \geq 1, \) we have

\[
\psi (h) \chi (g) (\psi (h))^{-1} = \chi (\phi (h) g).
\]

Therefore

\[
\mu ((g_1, h_1) (g_2, h_2)) = \mu (g_1 \phi (h_1) g_2, h_1 h_2) = \chi (g_1 \phi (h_1) g_2) \psi (h_1 h_2)
= \chi (g_1) \chi (\phi (h_1) g_2) \psi (h_1) \psi (h_2)
= \chi (g_1) \psi (h_1) \chi (g_2) \psi (h_2) = \mu (g_1, h_1) \mu (g_2, h_2).
\]

Since \( \mu \) is clearly a continuous mapping it is a representation of \( G \times \phi H \). By hypothesis \( H \) has a faithful representation \( \nu_0 \). Define a representation \( \nu \) of \( G \times \phi H \) by \( \nu (g, h) = \nu_0 (h) \) and put \( \xi = \mu + \nu \). Suppose \( (g, h) \) belongs to the kernel of \( \xi \). Then since \( \xi (g, h) = \mu (g, h) + \nu_0 (h) = \chi (g) \psi (h) + \nu_0 (h) \) and since \( \nu_0 \) is faithful on \( H, h = e' \). Hence \( g \) belongs to the kernel of \( \chi \). But as \( \chi \) is faithful on \( G, g = e. \) Therefore \( (g, h) = (e, e') \) and \( \xi \) is faithful on \( G \times \phi H \).

**Corollary (Malcev).**

A connected solvable Lie group \( G \) has a faithful representation if and only if \( G = NA \), where \( N \) is a closed, connected, simply connected invariant subgroup and \( A \) is a connected, compact abelian subgroup such that \( N \cap A = \{ e \} \).

Suppose \( G = NA \). For any \( a \in A \) let \( \phi (a) \) denote the automorphism of \( N \) given by \( \phi (a) n = ana^{-1} (n \in N) \). Then it is easily seen that \( (n, a) \rightarrow na \) is an isomorphism of \( N \times A \) onto \( G \). Since \( A \) is compact, it has a faithful representation. Hence by the above theorem it follows immediately that \( G \) has a faithful representation.

In order to establish the converse we make use of the following lemma which follows easily from the results of Chevalley.
Lemma 5. If $G$ is a connected solvable Lie group and $N$ a closed, connected invariant subgroup such that $G/N$ is compact, then there exists a compact connected abelian subgroup $A$ of $G$ such that $G=AN$ and $A\cap N$ is finite.

Returning to the corollary, suppose $G$ is linear. Since $\mathfrak{g}$ is solvable, we deduce in the usual way that every element $X \in [\mathfrak{g}, \mathfrak{g}]=\mathfrak{g}'$ is nilpotent and therefore may be assumed to be subtriangular. Therefore by Lemmas 1 and 2 the group $G'$ generated by $\mathfrak{g}'$ is simply connected. Let $\hat{d}$ be the degree of $G$ and $G_0$ the group of all matrices in $GL(K, \hat{d})$ which have zero below the diagonal and 1 everywhere on the diagonal. By Lemma 2, $G'$ is closed in $G_0$. However, since $G_0$ is clearly closed in $GL(K, \hat{d})$, $G'$ is closed in $GL(K, \hat{d})$ and therefore in $G$. Let $x \rightarrow x^*$ denote the natural homomorphism of $G$ onto $G/G' = G^*$. Since $G^*$ is abelian, $G^* = T^*V^*$ where $T^*$ and $V^*$ are connected subgroups, $T^*$ being compact and $V^*$ simply connected and $T^* \cap V^* = \{e^*\}$. Let $N$ be the complete inverse image of $V^*$ in $G$. Since $N/G' = V^*$ and $G'$ are both simply connected, $N$ is simply connected and $G/N \cong T^*$ is compact. Therefore by Lemma 5, $G = AN$ where $A$ is compact, connected, and abelian and $A \cap N$ is finite. Let $\sigma \in A \cap N$. Then $\sigma^r = e$ for some $r \geq 1$. Then $\sigma^* \in V^*$ and $(\sigma^*)^r = e^*$. Since $V^*$ is simply connected and abelian, $\sigma^* = e^*$. Hence $\sigma \in A \cap G'$. Since $X \rightarrow \exp X$ is a topological mapping of $\mathfrak{g}'$ onto $G'$, it follows that $\sigma \in G'$, $\sigma^r = e$ implies $\sigma = e$. Hence $A \cap N = \{e\}$. The corollary is therefore proved.

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* This lemma was pointed out to me by Dr. G. D. Mostow.