A NOTE ON PEANO SPACES
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In three-dimensional space set up a cylindrical coordinate system \((r, \theta, z)\). The Hahn-Mazurkiewicz theorem characterizes Peano spaces (locally connected metric (compact) continua) as the continuous images of the closed unit interval \(I\) on the \(z\)-axis. In this note we obtain an extension theorem for Peano spaces (henceforth called P-spaces) based upon this characterization.

We first define a dendrite \(L\). To this end order the rationals in the interior of \(I\) into a sequence \(\{0, 0, r_i\}\). For each pair of positive integers \(i, j\), let \(L_{i,j}\) denote the closed line segment joining \((0, 0, r_i)\) and \((1/i+j, 1/i+j, r_i)\). Let \(L_{-1,j}, L_{0,j}\) denote line segments from \((0, 0, 0)\) and \((0, 0, 1)\) parallel to and the same length as \(L_{1,j}\) for every \(j\). Then the dendrite \(L\) is defined to be the union of \(I\) and all the segments \(L_{i,j}\). Let \(a_{i,j}\) be the end point of \(L_{i,j}\) which is not on \(I\). We shall refer to \(a_{i,j}\) as the free end of \(L_{i,j}\).

Denote by \(D\) the sequence \(\{(0, 0, d_i)\}\) consisting of the dyadic rational points interior to \(I\) enumerated in the usual way: \(d_1=1/2, d_2=1/2^2, d_3=3/2^2, \ldots\). The following lemma is then easily established.

**Lemma.** Let \(Q^*\) be any finite or countable subset of \(I\). Then there exists a homeomorphism \(h(I)=I\) such that \(h(0)=0, h(1)=1, h(Q^*-0-1) \subset D\).

The result of this note may now be stated as follows.

**Theorem.** Let \(N\) be a P-space properly contained in the P-space \(P\), and let \(g(I)=N\) be any continuous transformation. Then there exists a homeomorphism \(h(I)=I\) and a continuous transformation \(f(L)=P\) with the following properties:

(i) on \(I, f=gh\)

(ii) \(f^{-1}(P-N) \subset L-I\).

In addition there exists a subset \(L*\) of \(L\), consisting of the union of \(I\) and a certain subcollection \(L^*_{i,j}\) of the line segments \(L_{i,j}\) such that

(iii) \(f(L^* - I) = P - N\),

and

(iv) if \(a^*_i\) is the free end of \(L^*_{i,j}\), for each \(i, j\) while \(A^*\) is the union of the points \(a^*_i,j\), then \(f^{-1}(P-N) = L^* - I - A^*\).

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Proof. The proof will be given only for the nontrivial case in which $N$ is nondegenerate. Let $T(I) = P$ be any continuous transformation. Then the open set $G = T^{-1}(P - N)$ is the union of a collection of open intervals $\{\omega_i\}$ having the respective left and right end points $\{q_i\}$ and $\{\ell_i\}$. It is convenient to assume that this sequence of intervals has been enumerated in order of decreasing diameters.

For each integer $i$ for which a point $q_i$ has been defined we choose a point $q_i^*$ in $g^{-1}(T(q_i))$. It does not follow that $q_i \neq q_j$ implies $q_i^* \neq q_j^*$. However, there are at most a countable number of $q_i$ which have the same $q_i^*$, We denote these by $q_{i,1}$, $q_{i,2}$, $\cdots$ in order of decreasing diameters of the corresponding $\omega_i$. Thus $q_i^* = q_j^*$ for all $i$. Denote by $Q^*$ the set of all points $q_i^*$ thus defined, and let $k(I) = I$ be the homeomorphism given by the lemma, and $h(I) = I$ the homeomorphism given by $h = k^{-1}$. For each pair of integers $i, j$ for which a point $q_{i,j}$ has been defined, let $\omega_{i,j}$ be the component of $G$ which has $q_{i,j}$ as its left end point. Set $d_i^* = k(q_i^*)$ for each $i$. Then, for each pair of integers $i, j$ for which a point $q_{i,j}$ has been defined, denote by $L_{i,j}^*$ the line segment of $L$ which has the point $d_i^*$ as its foot, and the point $a_{i,j}^*$ as its free end.

The desired transformation $f(L) = P$ is now defined as follows:

(a) $f(I) = gh(I) = N$. Thus (i) is satisfied.

(b) $f(L_{i,j}) = f(d_i^*)$ for every $L_{i,j}$ which is not an $L_{i,j}^*$.

(c) $f(L_{i,j}^*) = T(\omega_{i,j})$ in such a way that $f(d_i^*)$ agrees with the definition in (a), $f(a_{i,j}^*) = T(\ell_{i,j})$, and $f(L_{i,j}^* - d_i^* - a_{i,j}^*) = T(\omega_{i,j})$. Here, of course, $\ell_{i,j}$ denotes the right end point of $\omega_{i,j}$.

It is easily seen that the transformation $f(L) = P$ satisfies all of the required conditions.

In the important special case where $N$ is a simple arc we may choose $g(I) = N$ as a homeomorphism and obtain the following corollary.

**Corollary.** If $N$ is a simple arc in $P$, then $f(L) = P$ may be so defined that $f(I) = N$ is topological.

This theorem may be restated as a decomposition theorem for the $P$-space $P$, where each of the sets $f(L_{i,j}^*)$ is regarded as a $P$-space. Recent results of O. G. Harrold should not be overlooked in the light of this theorem.

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