ON A PROBLEM OF E. ČECH¹

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Let \( P \) be an abstract set and let \( \mathfrak{B} \) be the system of subsets of \( P \). An additive, single-valued set-function \( u \) taking \( \mathfrak{B} \) into \( \mathfrak{B} \) and satisfying the conditions \( u(0) = 0 \) and \( u(M) \subseteq M \) for each subset \( M \) of \( P \) is called a topology in \( P \), more precisely, an additive topology² in \( P \). The space \( P \) with the topology \( u \) is denoted by \((P, u)\) or, briefly, \( P \). Instead of \( u(M) \) we shall write simply \( uM \). We define:

\[
\begin{align*}
    u^0M &= M, \\
u^1M &= uM, \\
u^\xi M &= u(u^{\xi - 1}M) \text{ for isolated ordinals } \xi, \\
u^\xi M &= \bigcup_{\alpha < \xi} u^\alpha M \text{ for non-isolated } \xi.
\end{align*}
\]

The set \( M \) is closed if \( M = uM \). \( \phi(M) \) is the least ordinal number \( \xi \) for which the set \( u^\xi M \) is closed, that is, \( u^\xi M = u^{\xi + 1}M \). \( G(P, u) \) is the set of all ordinal numbers \( \phi(M) \) where \( M \subseteq P \).

E. Čech³ has posed the following problem: What are necessary and sufficient conditions on a set \( H \) of ordinals in order that there exist a topology \( u \) in a countable set \( P \) for which \( H = G(P, u) \)? V. Jarník⁴ solved the generalization of the problem for a set of arbitrary cardinal number \( \aleph_0 \); his topology, however, is not an additive one, as it satisfies only the axiom of monotony. The present paper contains a solution to this problem when \( u \) is to be additive; this solution is stated in Theorem 1. The proof is carried through for a class of cardinals including \( \aleph_0 \).

First we have to prove the following lemma.

**Lemma.** If the space \((P, u)\) contains a non-closed subset, then there exists a non-closed subset \( A \subseteq P \) such that \( uA = u^2A \).

Let \( M \) be a non-closed subset. The set \( uM - M \) is nonvoid: choose \( x \) in \( uM - M \) and let \( A = P - (x) \). By monotony of \( u \), \( A \neq uA = P \), so \( A \)

¹ This paper was rewritten for the Bulletin; the transcriber takes full responsibility for any errors that may have been introduced by this process.
² Every set function \( f \) satisfies the axiom of monotony: if \( M \subseteq N \), then \( f(M) \subseteq f(N) \).
is nonclosed but \( uA \) is closed.\(^6\)

**Theorem 1.** Let \( H \) be a set of ordinal numbers. There exists an additive topology \( u \) in a countable set \( P \) (different from \( uM = M \) for all \( M \subseteq P \)) for which \( H = G(P, u) \) if and only if: \(^6\)

1°. \( H \) contains only countable ordinals (that is, ordinals of the first and second number classes) including 0 and 1, and

2°. If \( \alpha \) and \( \beta \) are ordinal numbers such that \( \alpha + \beta \in H \), then \( \beta \in H \).

**Necessity.** Obviously \( 0 \in H \); by the lemma, 1 also is in \( H \). Every \( \phi(M) \) is countable since \( P \) can contain no uncountable ascending family of distinct sets.\(^7\) 2° follows from the equality \( u^{\alpha+\beta} M = u^{\alpha+\beta+1} M \), which can be rewritten as \( u^\beta (u^\alpha M) = u^\beta+1 (u^\alpha M) \). (If \( P \) is an arbitrary set of cardinal \( \aleph_p \), the same necessity proof works with 1° changed to allow only ordinals corresponding to cardinals not greater than \( \aleph_p \).)

**Sufficiency.** This proof will be carried through for cardinals \( \aleph_p \), satisfying the following condition: If \( R \) is a set of cardinal number \( \aleph_p \), there exists a family \( S \) of cardinal number \( \aleph_{p+1} \) whose elements are subsets \( P_\lambda \) of \( R \), \( \lambda < \aleph_{p+1} \), such that each \( P_\lambda \) is of cardinal \( \aleph_p \) while

\[
P_\lambda \cap P_\mu, \quad \lambda \neq \mu, \text{ has cardinal less than } \aleph_p.
\]

Such cardinals do exist; in particular, taking \( R \) to be the set of rational numbers, well-ordering the irrationals, and defining \( P_\lambda \) to be a sequence of rationals converging to the \( \lambda \)-th irrational, shows that \( \aleph_0 \) is such a cardinal. Define \( M \sim 0 \) to mean that the cardinal of \( M < \aleph_p \).

We assume that \( H \) is a set of ordinals satisfying 1°, generalized to allow ordinals less than \( \omega_{p+1} \), and 2°. To topologize a set \( R \) of the given cardinal \( \aleph_p \), assume that \( H \) is well-ordered by magnitude, \( H = \alpha_0, \alpha_1, \ldots, \alpha_\lambda, \ldots \), with \( \alpha_\lambda < \omega_{p+1} \) and \( \lambda < \omega_p, \sigma \leq p+1 \). Then, by 1°, \( \alpha_0 = 0 \) and \( \alpha_1 = 1 \). For each \( \lambda < \omega_p \) and each \( \xi \leq \alpha_p \) let \( P_\xi \) be subsets of \( P_\lambda \) such that

\[
P_\lambda \subset P_\mu \subset \cdots \subset P_\xi \subset \cdots \subset P_\lambda \alpha_\lambda,
\]

\[
P_\gamma \xi \text{ and } P_\lambda \xi + 1 - P_\lambda \xi \text{ are of cardinal number } \aleph_p,
\]

and

\[
P_\lambda \alpha_\lambda = P_\lambda \text{ and } P_\lambda \xi = \bigcup_{\alpha_\lambda \xi < \xi} P_\lambda \xi \text{ for non-isolated } \xi.
\]

\(^6\) By footnote 2, \( u \) is a monotone.

\(^7\) The condition 2° is essentially the same as Jarník's condition 3 in the paper cited under footnote 4.

\(^7\) V. Jarník defines \( \phi(0) = \phi(P) = 1 \); therefore according to Jarník, \( \phi = 1 \in H \).
Set $P_{\lambda-1} = 0$ and assume for convenience that $-1$ is less than all ordinals.

For each subset $M$ of $R$ and each $\lambda < \omega_\omega$ define the symbol $(\lambda M)$ to be the least index $\xi$ such that $M \cap (P_\lambda - P_{\lambda+1}) \sim 0$. Then $-1 \leq (\lambda M) \leq \alpha_\lambda$, and $(\lambda M) = -1$ if and only if $M \cap P_\lambda \sim 0$. We now prove

\begin{equation}
(\lambda [M \cup N]) = \max [(\lambda M), (\lambda N)].
\end{equation}

Indeed it follows from the definition that $(\lambda A) \leq (\lambda B)$ if $A \subseteq B$. Therefore $(\lambda [M \cup N]) \geq \max [(\lambda M), (\lambda N)]$. Suppose that, on the other hand, $(\lambda M) \leq (\lambda N)$; then the set $(M \cup N) \cap (P_\lambda - P_{\lambda+1}) \sim 0$ so that $(\lambda [M \cup N]) \leq \max [(\lambda M), (\lambda N)]$.

For each $\lambda < \omega_\omega$ define $f_\lambda(\xi) = \xi + 1$ for $0 \leq \xi < \alpha_\lambda$, and $f_\lambda(-1) = -1$, and $f_\lambda(\alpha_\lambda) = \alpha_\lambda$. Say that a subset $M$ of $R$ is of kind $\alpha$ if there exist a finite set of indices, $\lambda_1, \ldots, \lambda_n$, such that $M = \bigcup \{P_\lambda \mid \lambda_1 \leq \lambda \leq \lambda_n\}$; otherwise say that $M$ is of kind $\beta$. If $F$ is a set of kind $\alpha$, put

$$F(M) = \bigcup_{i \leq n} P_{f_\lambda(\lambda_i, M)}$$

where, for brevity, $P_{\lambda_i}$ and $f_\lambda$ have been replaced by $P_i$ and $f_i$. If $M$ is of kind $\alpha$, define $uM = M \cup F(M)$; if $M$ is of kind $\beta$, define $uM = R$.

If $M$ is of kind $\alpha$, the set $F(M)$ is uniquely defined. Indeed, let $\bigcup_{i \leq n} P_{\lambda_i} \supseteq M \subseteq \bigcup_{i \leq n} P_{\mu_j}$ and take $i \leq n$; then $M \cap P_{\lambda_i} - \bigcup_{i \leq n} P_{\mu_j} \sim 0$. By (1) either $\lambda_i$ is a $\mu_j$ or $(\lambda_i M) = -1$ and $P_{f_\lambda(\lambda_i, M)} = 0$. Hence every contribution to $F(M)$ by a $\lambda_i$ is supplied by a $\mu_j$, and conversely.

$M \cup N$ is of kind $\alpha$ if and only if both $M$ and $N$ are of kind $\alpha$. Also for sets of kind $\alpha$ we have $F(M \cup N) = F(M) \cup F(N)$. Indeed, let $\bigcup_{i \leq n} P_{\lambda_i} \supseteq M \cup N \supseteq \bigcup_{i \leq n} P_{\mu_j}$ and take $i \leq n$. By (5) the contribution, $P_{i, f_\lambda(\lambda_i, M \cup N)}$, of the $i$th term to $F(M \cup N)$ comes either from $F(M)$ or $F(N)$. From this it follows that $u$ is an additive topology.

First, $u$ is uniquely defined for each subset of $R$ and $M \subseteq uM$; also $u0 = 0$ since the void set is of kind $\alpha$. $u$ is additive for sets of kind $\alpha$ for

$$u(M \cup N) = F(M \cup N) \cup F(N) \cup M \cup N = uM \cup uN.$$ 

Every set $M \sim 0$ is of kind $\alpha$ and $F(M) = 0$ so $uM = M$ if $M \sim 0$. If either $M$ or $N$ is of kind $\beta$, $u[M \cup N] = R = uM \cup uN$.

Since $(\lambda P_{\lambda \xi}) = \xi$, (3) and (4) imply that $uP_{\lambda \xi} = P_{\lambda \xi+1}$ for $0 \leq \xi < \alpha_\lambda$ and $uP_{\lambda \xi} = P_{\lambda \xi}$ for $\xi = \alpha_\lambda$. By transfinite induction the reader can easily verify this from (4) that

\begin{equation}
u^r P_{\lambda \xi} = P_{\lambda \xi+r} \text{ for } \xi + r \leq \alpha_\lambda; \quad u^r P_{\lambda \xi} = P_{\lambda \xi} \text{ for } \xi + r \geq \alpha_\lambda.
\end{equation}

It remains to show that $G(R, u) = H$. By (6) with $\xi = 0$, $\eta = \alpha_\lambda$, it follows that $\phi(P_{\lambda 0}) = \alpha_\lambda$. Hence $H \subseteq G(R, u)$. To establish the opposite
inequality we must prove $\phi(M) \in H$ if $M \subset R$. If $uM = u^2M$, $\phi(M) = 0$ or 1, so it is in $H$ by 1º. If $uM \neq u^2M$, then $M$ is of kind $\alpha$, for $uM = R = u^4M$ if $M$ is of kind $\beta$. We note also that if $uA = uB$ and $uA \neq u^2A$, from the equalities $u^A = u^B$, it follows that $\phi(A) = \phi(B)$. Hence for $M - \bigcup_{i \leq n} P_{\lambda_i} \sim 0$ we have

$$uM = \bigcup_{i \leq n} P_{u^\lambda_i M} \cup K = u\left( \bigcup_{i \leq n} P_{u^\lambda_i M} \cup K \right),$$

where $K \sim 0$. By the remarks above, since $uM \neq u^2M$, we have

$$\phi M = \phi\left( \bigcup_{i \leq n} P_{u^\lambda_i M} \cup K \right).$$

By (6) and additivity of $u$, $\phi(M) = \max_{i \leq n} \phi(P_{u^\lambda_i M})$ so it suffices to see that every $\phi(P_{\lambda_0})$ is in $H$. However if $\eta = \phi(P_{\lambda_0})$, then by (6) $\xi + \eta = \phi(P_{\lambda_0}) = \alpha \in H$; by 2º, $\eta \in H$. This completes the proof.

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