ON A PROBLEM OF E. ČECH[1]

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Let P be an abstract set and let $\mathfrak{P}$ be the system of subsets of P. An additive, single-valued set-function u taking $\mathfrak{P}$ into $\mathfrak{P}$ and satisfying the conditions $u(0)=0$ and $u(M)\supset M$ for each subset M of P is called a topology in P, more precisely, an additive topology[2] in P. The space P with the topology u is denoted by (P, u) or, briefly, P. Instead of $u(M)$ we shall write simply uM. We define:

$$u^0M = M,\ u^1M = uM,\ u^\xi M = u(u^{\xi-1}M) \text{ for isolated ordinals } \xi,$$

and

$$u^\xi M = \bigcup_{0\leqq\eta<\xi} u^\eta M \text{ for non-isolated } \xi.$$

The set M is *closed* if $M=uM$. $\phi(M)$ is the least ordinal number ξ for which the set $u^\xi M$ is closed, that is, $u^\xi M=u^{\xi+1}M$. $G(P, u)$ is the set of all ordinal numbers $\phi(M)$ where $M\subset P$.

E. Čech[3] has posed the following problem: What are necessary and sufficient conditions on a set H of ordinals in order that there exist a topology u in a countable set P for which $H=G(P, u)$? V. Jarník[4] solved the generalization of the problem for a set of arbitrary cardinal number $\aleph_\rho$; his topology, however, is not an additive one, as it satisfies only the axiom of monotony. The present paper contains a solution to this problem when u is to be additive; this solution is stated in Theorem 1. The proof is carried through for a class of cardinals including $\aleph_0$.

First we have to prove the following lemma.

LEMMA. *If the space (P, u) contains a non-closed subset, then there exists a non-closed subset $A\subset P$ such that $uA=u^2A$.*

Let M be a non-closed subset. The set $uM-M$ is nonvoid: choose x in $uM-M$ and let $A=P-(x)$. By monotony of u, $A\neq uA=P$, so A

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[1] This paper was rewritten for the Bulletin; the transcriber takes full responsibility for any errors that may have been introduced by this process.

[2] Every set function f satisfies the axiom of monotony: If $M\subset N$, then $f(M)\subset f(N)$.

[3] E. Čech, *Topologické prostory*, Časopis pro Pěstování Matematiky a Fysiki vol. 66 (1937) pp. D225–D264, p. D264.

[4] V. Jarník, *Sur un problème de M. Čech*, Věstník královské české společnosti nauk (1938) pp. 1–7.

is nonclosed but uA is closed.[5]

THEOREM 1. *Let H be a set of ordinal numbers. There exists an additive topology u in a countable set P (different from $uM=M$ for all $M\subset P$) for which $H=G(P, u)$ if and only if:*[6]

1°. *H contains only countable ordinals (that is, ordinals of the first and second number classes) including* 0 *and* 1, *and*

2°. *If α and β are ordinal numbers such that $\alpha+\beta\in H$, then $\beta\in H$.*

Necessity. Obviously $0\in H$; by the lemma, 1 also is in H. Every $\phi(M)$ is countable since P can contain no uncountable ascending family of distinct sets.[7] 2° follows from the equality $u^{\xi+\beta}M=u^{\xi+\beta+1}M$, which can be rewritten as $u^{\beta}(u^{\xi}M)=u^{\beta+1}(u^{\xi}M)$. (If P is an arbitrary set of cardinal $\aleph_\rho$, the same necessity proof works with 1° changed to allow only ordinals corresponding to cardinals not greater than $\aleph_\rho$.)

Sufficiency. This proof will be carried through for cardinals $\aleph_\rho$ satisfying the following condition: If R is a set of cardinal number $\aleph_\rho$, there exists a family $\mathfrak{S}$ of cardinal number $\aleph_{\rho+1}$ whose elements are subsets P_λ of R, $\lambda<\omega_{\rho+1}$, such that each P_λ is of cardinal $\aleph_\rho$ while

$$(1)\qquad P_\lambda\cap P_\mu,\quad \lambda\neq\mu,\quad \text{has cardinal less than } \aleph_\rho.$$

Such cardinals do exist; in particular, taking R to be the set of rational numbers, well-ordering the irrationals, and defining P_λ to be a sequence of rationals converging to the λth irrational, shows that $\aleph_0$ is such a cardinal. Define $M\sim 0$ to mean that the cardinal of $M<\aleph_\rho$.

We assume that H is a set of ordinals satisfying 1°, generalized to allow ordinals less than $\omega_{\rho+1}$, and 2°. To topologize a set R of the given cardinal $\aleph_\rho$, assume that H is well-ordered by magnitude, $H=\alpha_0, \alpha_1, \cdots, \alpha_\lambda, \cdots$, with $\alpha_\lambda<\omega_{\rho+1}$ and $\lambda<\omega_\sigma$, $\sigma\leqq\rho+1$. Then, by 1°, $\alpha_0=0$ and $\alpha_1=1$. For each $\lambda<\omega_\sigma$ and each $\xi\leqq\alpha_\lambda$ let $P_{\lambda\xi}$ be subsets of P_λ such that

$$(2)\qquad P_{\lambda 0}\subset P_{\lambda 1}\subset\cdots\subset P_{\lambda\xi}\subset\cdots\subset P_{\lambda\alpha_\lambda},$$

$$(3)\qquad P_{\gamma\xi} \text{ and } P_{\lambda\xi+1}-P_{\lambda\xi} \text{ are of cardinal number } \aleph_\rho,$$

and

$$(4)\qquad P_{\lambda\alpha_\lambda}=P_\lambda \text{ and } P_{\lambda\xi}=\bigcup_{0\leqq\eta<\xi}P_{\lambda\eta} \text{ for non-isolated } \xi.$$

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[5] By footnote 2, u is a monotone.

[6] The condition 2° is essentially the same as Jarník's condition 3 in the paper cited under footnote 4.

[7] V. Jarník defines $\phi(0)=\phi(P)=1$; therefore according to Jarník ex definitione $1\in H$.

Set $P_{\lambda-1}=0$ and assume for convenience that -1 is less than all ordinals.

For each subset M of R and each $\lambda<\omega_\sigma$ define the symbol (λM) to be the least index ξ such that $M\cap(P_\lambda-P_{\lambda\xi})\sim 0$. Then $-1\leqq(\lambda M)\leqq\alpha_\lambda$, and $(\lambda M)=-1$ if and only if $M\cap P_\lambda\sim 0$. We now prove

$$(5) \qquad (\lambda[M\cup N]) = \max\,[(\lambda M),\,(\lambda N)].$$

Indeed it follows from the definition that $(\lambda A)\leqq(\lambda B)$ if $A\subset B$. Therefore $(\lambda[M\cup N])\geqq\max\,[(\lambda M),\,(\lambda N)]$. Suppose that, on the other hand, $(\lambda M)\leqq(\lambda N)$; then the set $(M\cup N)\cap(P_\lambda-P_{\lambda(\lambda N)})\sim 0$ so that $(\lambda[M\cup N])\leqq\max\,[(\lambda M),\,(\lambda N)]$.

For each $\lambda<\omega_\sigma$ define $f_\lambda(\xi)=\xi+1$ for $0\leqq\xi<\alpha_\lambda$, $f_\lambda(-1)=-1$, and $f_\lambda(\alpha_\lambda)=\alpha_\lambda$. Say that a subset M of R is of kind α if there exist a finite set of indices, $\lambda_1,\cdots,\lambda_n$, such that $M-\bigcup_{i\leqq n}P_{\lambda_i}\sim 0$; otherwise say that M is of kind β. If F is a set of kind α, put

$$F(M) = \bigcup_{i\leqq n} P_{if_i(\lambda_i M)},$$

where, for brevity, P_{λ_i} and f_{λ_i} have been replaced by P_i and f_i. If M is of kind α, define $uM=M\cup F(M)$; if M is of kind β, define $uM=R$.

If M is of kind α, the set $F(M)$ is uniquely defined. Indeed, let $\bigcup_{i\leqq n}P_{\lambda_i}\supset M\subset\bigcup_{j\leqq m}P_{\mu_j}$ and take $i\leqq n$; then $M\cap P_{\lambda_i}-\bigcup_{j\leqq m}P_{\mu_j}\sim 0$. By (1) either λ_i is a μ_j or $(\lambda_i M)=-1$ and $P_{if_i(\lambda_i M)}=0$. Hence every contribution to $F(M)$ by a λ_i is supplied by a μ_j, and conversely.

$M\cup N$ is of kind α if and only if both M and N are of kind α. Also *for sets of kind α we have $F(M\cup N)=F(M)\cup F(N)$.* Indeed, let $M\cup N-\bigcup_{i\leqq n}P_{\lambda_i}\sim 0$. By (5) the contribution, $P_{if_i(\lambda_i[M\cup N])}$, of the ith term to $F(M\cup N)$ comes either from $F(M)$ or $F(N)$. From this it follows that *u is an additive topology.*

First, u is uniquely defined for each subset of R and $M\subset uM$; also $u0=0$ since the void set is of kind α. u is additive for sets of kind α for $u(M\cup N)=F(M\cup N)\cup M\cup N=F(M)\cup F(N)\cup M\cup N=uM\cup uN$. Every set $M\sim 0$ is of kind α and $F(M)=0$ so $uM=M$ if $M\sim 0$. If either M or N is of kind β, $u[M\cup N]=R=uM\cup uN$.

Since $(\lambda P_{\lambda\xi})=\xi$, (3) and (4) imply that $uP_{\lambda\xi}=P_{\lambda\xi+1}$ for $0\leqq\xi<\alpha_\lambda$ and $uP_{\lambda\xi}=P_\lambda$ for $\xi=\alpha_\lambda$. By transfinite induction the reader can easily verify from this and (4) that

$$(6) \quad u^\eta P_{\lambda\xi} = P_{\lambda\xi+\eta} \text{ for } \xi+\eta\leqq\alpha_\lambda;\quad u^\eta P_{\lambda\xi} = P_\lambda \text{ for } \xi+\eta\geqq\alpha_\lambda.$$

It remains to show that $G(R,u)=H$. By (6) with $\xi=0$, $\eta=\alpha_\lambda$, it follows that $\phi(P_{\lambda 0})=\alpha_\lambda$. Hence $H\subset G(R,u)$. To establish the opposite inequality we must prove $\phi(M)\in H$ if $M\subset R$. If $uM=u^2M$, $\phi(M)=0$ or 1, so it is in H by 1°. If $uM\neq u^2M$, then M is of kind α, for $uM=R=u^2M$ if M is of kind β. We note also that if $uA=uB$ *and* $uA\neq u^2A$, from the equalities $u^\eta A=u^\eta B$, it follows that $\phi(A)=\phi(B)$. Hence for $M-\bigcup_{i\leq n}P_{\lambda_i}\sim 0$ we have

$$uM = \bigcup_{i\leq n} P_{if_i(\lambda_i M)} \cup K = u\left(\bigcup_{i\leq n} P_{i(\lambda_i M)} \cup K\right),$$

where $K\sim 0$. By the remarks above, since $uM\neq u^2M$, we have

$$\phi M = \phi\left(\bigcup_{i\leq n} P_{i(\lambda_i M)} \cup K\right).$$

By (6) and additivity of u, $\phi(M)=\max_{i\leq n}\phi(P_{i(\lambda_i M)})$ so it suffices to see that every $\phi(P_{\lambda\xi})$ is in H. However if $\eta=\phi(P_{\lambda\xi})$, then by (6) $\xi+\eta=\phi(P_{\lambda 0})=\alpha_\lambda\in H$; by 2°, $\eta\in H$. This completes the proof.

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