INEQUALITIES CONCERNING ULTRASPHERICAL POLYNOMIALS AND BESSEL FUNCTIONS

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1. **Introduction.** Ultraspherical polynomials are defined by the recurrence formula

$$(1.1)\quad (n+1)P_{n+1}^{\lambda}(x) = 2(n+\lambda)xP_n^{\lambda}(x) - (n+2\lambda-1)P_{n-1}^{\lambda}(x),$$

with $P_0^{\lambda}=1$, $P_1^{\lambda}=2\lambda x$. Thus we have

$$2P_2^{\lambda} = 2\lambda[2(1+\lambda)x^2 - 1],$$

and so on.

For $\lambda=1/2$ we get the Legendre polynomials $P_n(x)$. They satisfy the following interesting inequality:

$$(1.2)\quad \begin{aligned} \Delta_n(x) \equiv\ & [P_n(x)]^2 \\ & - [P_{n-1}(x)][P_{n+1}(x)] \geqq 0, \qquad n \geqq 1,\ -1 \leqq x \leqq 1, \end{aligned}$$

with equality for $x=\pm 1$. This result is due to P. Turán, as stated in a recent paper by G. Szegö,[2] where four elegant proofs of the theorem are given. The first proof (similar to that of Turán) is based on Mehler's formula, while the third proof is based on the identity

$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} z^n = e^{xz} J_0[(1-x^2)^{1/2}z],$$

and on the fact that the Bessel function J_0 has all its roots real.

Szegö remarks that the formula

$$\sum_{n=0}^{\infty} \frac{P_n^{\lambda}(x)}{P_n^{\lambda}(1)} \cdot \frac{z^n}{n!} \cdot$$
$$= 2^{\lambda-1/2}\Gamma(\lambda+1/2)e^{xz}[(1-x^2)^{1/2}z]^{1/2-\lambda}J_{\lambda-1/2}[(1-x^2)^{1/2}z],\ \lambda > -1/2,$$

and analogous formulas for Laguerre and Hermite polynomials yield inequalities analogous to (1.2) for ultraspherical, Laguerre, and

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[2] G. Szegö, *On an inequality of P. Turán concerning Legendre polynomials*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 401–405.

Hermite polynomials. Thus

$$\left[\frac{P_n^\lambda(x)}{P_n^\lambda(1)}\right]^2 - \frac{P_{n-1}^\lambda(x)P_{n+1}^\lambda(x)}{P_{n-1}^\lambda(1)P_{n+1}^\lambda(1)} \geqq 0, \qquad \lambda > -1/2. \tag{1.3}$$

We have,[3] for $\lambda > -1/2$,

$$P_n^\lambda(1) = C_{n+2\lambda-1,n} = \frac{2\lambda\Gamma(n+2\lambda)}{n!\Gamma(2\lambda+1)}, \quad n = 0, 1, \cdots, \tag{1.4}$$

so that (1.3) can be restated as

$$n(n+2\lambda)[P_n^\lambda(x)]^2 \geqq (n+1)(n+2\lambda-1)P_{n-1}^\lambda(x)P_{n+1}^\lambda(x);$$

with $\lambda = 1/2$, this reduces to the inequality (1.2).

We write $F_n(x)$ for $P_n^\lambda(x)/P_n^\lambda(1)$; then (1.3) becomes $\Delta_n^\lambda(x) \equiv [F_n(x)]^2 - F_{n-1}(x)F_{n+1}(x) \geqq 0$, and equality holds for $x = 1$. We have

$$F_0(x) \equiv 1, \quad F_1(x) = x, \quad F_2(x) = \frac{2(1+\lambda)x^2 - 1}{1+2\lambda}, \cdots.$$

For $0 < \lambda < 1$ we derive a sharper inequality, employing essentially the recursion formula (1.1). The result is

$$\frac{\lambda[1 - F_n^2(x)]}{(n+\lambda-1)(n+2\lambda)} \leqq \Delta_n^\lambda(x) < \frac{2\lambda\Gamma(2\lambda)}{\lambda+1}\,\frac{\Gamma(n)}{\Gamma(n+2\lambda)}, \tag{1.5}$$

$-1 \leqq x \leqq 1$. In particular, for $\lambda = 1/2$,

$$\frac{1 - [P_n(x)]^2}{(2n-1)(n+1)} \leqq \Delta_n(x) < \frac{2n+1}{3n(n+1)};$$

the smallness of Δ_n indicates that the inequality (1.2) is rather deep.

For Bessel functions we shall establish the inequality

$$J_\mu^2(t) - J_{\mu-1}(t)J_{\mu+1}(t) > \frac{1}{\mu+1}J_\mu^2(t), \qquad \mu > 0,\ t \text{ real}. \tag{1.6}$$

It seems particularly interesting that the procedure is very much the same in deriving (1.5) and (1.6), and is based mainly on the respective recurrence relations.

2. **Ultraspherical polynomials**; $\lambda < 1$. From (1.1) and (1.4), we obtain

[3] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939, pp. 80, 166. Referred to as O. P.

$$(n+1)F_{n+1}(x) = 2(n+\lambda)xF_n(x)\frac{\Gamma(n+2\lambda)(n+1)}{\Gamma(n+1+2\lambda)}$$
$$-(n+2\lambda-1)F_{n-1}(x)\frac{\Gamma(n+2\lambda-1)n(n+1)}{\Gamma(n+1+2\lambda)},$$

or

$$F_{n+1}(x) = \frac{2(n+\lambda)x}{n+2\lambda}F_n(x) - \frac{n}{n+2\lambda}F_{n-1}(x), \qquad n \geqq 1. \tag{2.1}$$

It follows that

$$F_{n+1}(x)F_n(x) = \frac{2(n+\lambda)}{n+2\lambda}xF_n^2(x) - \frac{n}{n+2\lambda}F_{n-1}(x)F_n(x). \tag{2.2}$$

From (2.1) we get

$$F_n(x) = \frac{2(n+\lambda-1)x}{n+2\lambda-1}F_{n-1}(x) - \frac{(n-1)}{n+2\lambda-1}F_{n-2}(x), \qquad n \geqq 2,$$

whence substitution into (2.2) yields

$$\begin{aligned}
&\frac{2x(n+\lambda)}{n+2\lambda}F_n^2(x)\\
&= F_{n+1}(x)\left\{\frac{2(n+\lambda-1)}{n+2\lambda-1}xF_{n-1}(x) - \frac{n-1}{n+2\lambda-1}F_{n-2}(x)\right\}\\
&\quad + \frac{n}{n+2\lambda}F_{n-1}(x)\left\{\frac{2(n+\lambda-1)}{n+2\lambda-1}xF_{n-1}(x) - \frac{n-1}{n+2\lambda-1}F_{n-2}(x)\right\}\\
&= \frac{2nx(n+\lambda-1)}{(n+2\lambda-1)(n+2\lambda)}F_{n-1}^2(x) + F_{n+1}(x)\frac{2(n+\lambda-1)}{n+2\lambda-1}xF_{n-1}(x)\\
&\quad - \frac{(n-1)}{n+2\lambda-1}F_{n-2}(x)\left\{\frac{2(n+\lambda)x}{n+2\lambda}F_n(x) - \frac{n}{n+2\lambda}F_{n-1}(x)\right\}\\
&\quad - \frac{(n-1)n}{(n+2\lambda-1)(n+2\lambda)}F_{n-2}(x)F_{n-1}(x)\\
&= \frac{2n(n+\lambda-1)x}{(n+2\lambda-1)(n+2\lambda)}F_{n-1}^2(x) + 2x\frac{n+\lambda-1}{n+2\lambda-1}F_{n-1}(x)F_{n+1}(x)\\
&\quad - \frac{2(n-1)(n+\lambda)x}{(n+2\lambda-1)(n+2\lambda)}F_{n-2}(x)F_n(x), \qquad n \geqq 2.
\end{aligned}$$

If we write

$$(2.3)\quad \begin{aligned}(n+\lambda)(n+2\lambda-1)F_n^2(x)\\ -(n+\lambda-1)(n+2\lambda)F_{n-1}(x)F_{n+1}(x) = D_n^\lambda(x),\end{aligned}$$

then

$$\begin{aligned}
D_n^\lambda(x) &= n(n+\lambda-1)F_{n-1}^2(x) - (n-1)(n+\lambda)F_{n-2}(x)F_n(x)\\
&= \{(n+\lambda-1)(n+2\lambda-2)F_{n-1}^2(x)\\
&\quad -(n+\lambda-2)(n+2\lambda-1)F_{n-2}(x)F_n(x)\}\\
&\quad \cdot\frac{(n-1)(n+\lambda)}{(n+\lambda-2)(n+2\lambda-1)} + F_{n-1}^2(x)\\
&\quad \cdot\left\{n(n+\lambda-1) - \frac{(n+\lambda-1)(n+2\lambda-2)(n-1)(n+\lambda)}{(n+\lambda-2)(n+2\lambda-1)}\right\}\\
&= D_{n-1}^\lambda(x)\frac{(n-1)(n+\lambda)}{(n+\lambda-2)(n+2\lambda-1)} + (n+\lambda-1)F_{n-1}^2(x)\\
&\quad \cdot\frac{n(n+\lambda-2)(n+2\lambda-1)-(n-1)(n+\lambda)(n+2\lambda-2)}{(n+\lambda-2)(n+2\lambda-1)}\\
&= \frac{(n-1)(n+\lambda)}{(n+\lambda-2)(n+2\lambda-1)}D_{n-1}^\lambda(x)\\
&\quad + \frac{2\lambda(\lambda-1)(n+\lambda-1)F_{n-1}^2(x)}{(n+\lambda-2)(n+2\lambda-1)}.
\end{aligned}$$

Let

$$(2.4)\qquad g_n = g_{n-1}\frac{(n+\lambda-2)(n+2\lambda-1)}{(n-1)(n+\lambda)},\quad n \geqq 2,\ g_1 = 1;$$

then

$$(2.5)\quad g_nD_n^\lambda(x) = g_{n-1}D_{n-1}^\lambda(x) + 2\lambda(\lambda-1)\frac{n+\lambda-1}{(n-1)(n+\lambda)}g_{n-1}F_{n-1}^2(x),$$

where, from (2.3),

$$(2.6)\quad \begin{aligned}D_1^\lambda(x) &= 2\lambda(1+\lambda)F_1^2(x) - \lambda(1+2\lambda)F_0F_2(x)\\ &= \lambda\{2(1+\lambda)x^2 - 2(1+\lambda)x^2 + 1\}\\ &= \lambda.\end{aligned}$$

From (2.4), we have

$$g_n = \prod_2^n \frac{(\nu+\lambda-2)(\nu+2\lambda-1)}{(\nu-1)(\nu+\lambda)} = \frac{\lambda(\lambda+1)\prod_2^n(\nu+2\lambda-1)}{(n-1)!(n+\lambda-1)(n+\lambda)}$$

$$(2.7) \qquad = \frac{\lambda(\lambda+1)}{(n+\lambda-1)(n+\lambda)}\prod_1^{n-1}\left(1+\frac{2\lambda}{\nu}\right) = O(n^{2\lambda-2}),$$

as $n \to \infty$.

It is well known [cf. O. P. pp. 80, 166] that, for $\lambda > 0$,

$$\max_{-1 \leqq x \leqq 1} |P_n^\lambda(x)| = P_n^\lambda(1) = C_{n+2\lambda-1,n},$$

so that

$$(2.8) \qquad |F_n(x)| \leqq F_n(1) = 1.$$

Hence

$$(2.9) \qquad \frac{n+\lambda}{n(n+\lambda+1)} g_n F_n^2(x) \leqq \frac{g_n}{n} = O(n^{2\lambda-3}).$$

We now consider the case $0 < \lambda < 1$; it follows from (2.9) that the series

$$\sum_2^\infty \{g_{n-1}D_{n-1}^\lambda(x) - g_n D_n^\lambda(x)\}$$

is uniformly convergent in $-1 \leqq x \leqq 1$, and from (2.5), (2.6) we have

$$\lambda - \lim_{n\to\infty} g_n D_n^\lambda(x) = 2\lambda(1-\lambda)\sum_1^\infty \frac{(n+\lambda)g_n}{n(n+\lambda+1)} F_n^2(x).$$

We write

$$(2.10) \qquad \lim_{n\to\infty} g_n D_n^\lambda(x) = \lambda\left\{1 - 2(1-\lambda)\sum_1^\infty \frac{(n+\lambda)g_n}{n(n+\lambda+1)} F_n^2(x)\right\}$$
$$\equiv \chi(x),$$

a continuous function of x in $-1 \leqq x \leqq 1$. It follows from (2.8) and (2.10) that

$$(2.11) \qquad \chi(x) \geqq \chi(1) = \lambda\left\{1 - 2(1-\lambda)\sum_1^\infty \frac{(n+\lambda)g_n}{n(n+\lambda+1)}\right\}.$$

We next show that

$$\chi(1) = \lim g_n D_n^\lambda(1) = 0. \tag{2.12}$$

From (2.3), we have

$$D_n^\lambda(1) = (n+\lambda)(n+2\lambda-1) - (n+\lambda-1)(n+2\lambda) = \lambda, \tag{2.13}$$

and from (2.7), for $0<\lambda<1$, we see that $g_n \to 0$; this proves (2.12).

It now follows from (2.11) that

$$\chi(x) \geqq 0 = 1 - 2(1-\lambda)\sum_1^\infty \frac{(n+\lambda)g_n}{n(n+\lambda+1)},$$

so that

$$\sum_1^\infty \frac{(n+\lambda)g_n}{n(n+\lambda+1)} = \frac{1}{2(1-\lambda)}, \qquad 0<\lambda<1.$$

Note that by (2.7) we have

$$g_n = \frac{\lambda(\lambda+1)\Gamma(n+2\lambda)}{\Gamma(1+2\lambda)(n+\lambda-1)(n+\lambda)\Gamma(n)} \sim \frac{\lambda(\lambda+1)}{\Gamma(1+2\lambda)} n^{2\lambda-2}.$$

From (2.5) and (2.6) we get

$$g_n D_n^\lambda(x) = \lambda - 2\lambda(1-\lambda)\sum_{\nu=1}^{n-1} \frac{g_\nu(\nu+\lambda)F_\nu^2(x)}{\nu(\nu+\lambda+1)}; \tag{2.14}$$

furthermore

$$g_n D_n^\lambda(x) = \chi(x) + 2\lambda(1-\lambda)\sum_n^\infty \frac{g_\nu(\nu+\lambda)}{\nu(\nu+\lambda+1)} F_\nu^2(x).$$

Now from (2.8), (2.12), and (2.13) it follows that

$$\begin{aligned} g_n D_n^\lambda(x) &= 2\lambda(1-\lambda)\sum_1^{n-1} \frac{g_\nu(\nu+\lambda)}{\nu(\nu+\lambda+1)}\{1 - F_\nu^2(x)\} \\ &\quad + \sum_n^\infty \frac{g_\nu(\nu+\lambda)}{\nu(\nu+\lambda+1)} \geqq \sum_n^\infty \frac{g_\nu(\nu+\lambda)}{\nu(\nu+\lambda+1)} \\ &= g_n D_n^\lambda(1) = \lambda g_n. \end{aligned}$$

It follows from (2.14) that, uniformly in $-1 \leqq x \leqq 1$,

$$g_n D_n^\lambda(x) \downarrow \chi(x), \qquad \text{as } n \to \infty; \tag{2.15}$$

furthermore,

$$D_n^\lambda(x) \geqq D_n^\lambda(1) = \lambda. \tag{2.16}$$

From (2.15) we have

$$g_n D_n^\lambda(x) < g_1 D_1^\lambda = \lambda.$$

From (2.3), (2.13), and (2.16) we obtain

$$D_n^\lambda(x) = (n + \lambda - 1)(n + 2\lambda)\{F_n^2(x) - F_{n-1}(x)F_{n+1}(x)\} + \lambda F_n^2(x) \geqq \lambda,$$

so that

$$\Delta_n^\lambda(x) = F_n^2(x) - F_{n-1}(x)F_{n+1}(x) \geqq \frac{\lambda(1 - F_n^2(x))}{(n + \lambda - 1)(n + 2\lambda)},$$

and

$$n^{2\lambda}\Delta_n^\lambda(x) \to \frac{\Gamma(1 + 2\lambda)}{\lambda(\lambda + 1)}\chi(x),$$

uniformly in $-1 \leqq x \leqq 1$.

Summarizing, we have the following theorem:

THEOREM 1. *Denote by $P_n^\lambda(x)$ the ultraspherical polynomial and let*

$$F_n(x) = \frac{P_n^\lambda(x)}{P_n^\lambda(1)}, \qquad \Delta_n^\lambda(x) = F_n^2(x) - F_{n-1}(x)F_{n+1}(x).$$

If $0 < \lambda < 1$, then $n^{2\lambda}\Delta_n^\lambda(x)$ tends uniformly to a non-negative function, and

$$\frac{\lambda(1 - F_n^2(x))}{(n + \lambda - 1)(n + 2\lambda)} \leqq \Delta_n^\lambda(x) < \frac{\Gamma(1 + 2\lambda)}{\lambda + 1} \cdot \frac{\Gamma(n)}{\Gamma(n + 2\lambda)}, \qquad -1 \leqq x \leqq 1.$$

In particular for $\lambda = 1/2$ (the Legendre polynomial), we have

$$\frac{1 - (P_n(x))^2}{(2n - 1)(n + 1)} \leqq (P_n(x))^2 - P_{n-1}(x)P_{n+1}(x) < \frac{2n + 1}{3n(n + 1)}. \tag{2.17}$$

3. **The case $\lambda > 1$.** Let now $\lambda > 1$; from (2.14) we have

$$g_n D_n^\lambda(x) = \lambda + 2\lambda(\lambda - 1)\sum_1^{n-1} \frac{g_\nu(\nu + \lambda)}{\nu(\nu + \lambda + 1)} F_\nu^2(x),$$

so that $g_n D_n^\lambda(x)$ increases as n increases. Hence

$$g_n D_n^\lambda(x) \geqq \lambda,$$

or

$$(n + \lambda - 1)(n + 2\lambda) g_n \Delta_n^\lambda(x) + \lambda g_n F_n^2(x) \geqq \lambda = D_n^\lambda(1).$$

Furthermore

$$g_n D_n^\lambda(x) \leqq g_n D_n^\lambda(1) = \lambda + 2\lambda(\lambda - 1) \sum_1^{n-1} \frac{g_\nu(\nu + \lambda)}{\nu(\nu + \lambda + 1)} = g_n \lambda.$$

Thus

$$\Delta_n^\lambda(x) \geqq \frac{\lambda}{(n + \lambda - 1)(n + 2\lambda) g_n} - \frac{\lambda F_n^2(x)}{(n + \lambda - 1)(n + 2\lambda)},$$

and

$$\Delta_n^\lambda(x) \leqq \frac{\lambda(1 - F_n^2(x))}{(n + \lambda - 1)(n + 2\lambda)}.$$

In this case, from (2.7), we have $g_n \uparrow \infty$, and

$$\Delta_n^\lambda(x) = O(n^{-2}), \qquad \text{as } n \to \infty.$$

Finally, for $\lambda = 1$, we have

$$F_n(x) = \frac{\sin (n + 1)\theta}{(n + 1) \sin \theta}, \qquad x = \cos \theta,$$

$$g_n D_n^\lambda(x) = g_1 D_1^\lambda = 1, \qquad g_n = 1, \qquad D_n^\lambda(x) = 1,$$

$$n(n + 2)\Delta_n(x) = 1 - F_n^2(x).$$

4. **An application.** If $P_{n+1}(x) = 0$, then from (2.17) we obtain

$$\frac{1 - P_n^2(x)}{(n + 1)(2n - 1)} \leqq P_n^2(x) < \frac{2n + 1}{3n(n + 1)},$$

or

$$\frac{1}{n(2n + 1)} \leqq P_n^2(x) < \frac{2n + 1}{3n(n + 1)}.$$

If $P_n(t) = 0$, then

$$\frac{1}{(n(2n+1))^{1/2}} \leqq |P_n(x)| = \left| \int_t^x P_n'(u)du \right|,$$

which yields a lower bound for $[x-t]$. To this end we employ the formula [O. P. p. 83]

$$\frac{d}{dx} P_n^{\lambda}(x) = 2\lambda P_{n-1}^{\lambda+1}(x),$$

so that, for $\lambda = 1/2$,

$$P_n'(x) = P_{n-1}^{3/2}(x);$$

hence, from (1.5), for $\lambda = 3/2$,

$$|P_n'(x)| \leqq C_{n+1,n-1} = (3/2)n(n+1);$$

equality holds for $x = \pm 1$.

It follows that

$$\frac{1}{(n(2n+1))^{1/2}} < |x-t| (3/2)n(n+1),$$

or

$$|x-t| > 2/3n^{-3/2}(2n+1)^{-1/2}(n+1)^{-1},$$

where $P_{n+1}(x) = 0$, $P_n(t) = 0$.

5. **Hermite polynomials.** The situation is simpler for Hermite polynomials $H_n(x)$, defined by

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n = 1, 2, 3, \cdots,$$
$$H_0 = 1, \qquad H_1 = 2x.$$

In this case [see Amer. Math. Monthly vol. 55 (1948) solution of problem 4215, pp. 34–35],

$$\frac{1}{2^{n+1}n!} \{H_{n+1}^2 - H_n H_{n+2}\} = \sum_{\nu=0}^{n} \frac{1}{\nu! 2^\nu} H_\nu^2,$$

hence monotone increasing, as n increases.

6. **Bessel functions.** The Bessel function of order μ is defined for $\mu > -1$ by the power series

$$J_\mu(t) = (t/2)^\mu \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (t/2)^{2\nu}}{\nu! \Gamma(\mu+\nu+1)}. \tag{6.1}$$

It satisfies the recursion formula

$$J_{\mu+1}(t) = \frac{2\mu}{t} J_\mu(t) - J_{\mu-1}(t), \qquad \mu > 0. \tag{6.2}$$

We write

$$\Lambda_\mu(t) = (2/t)^\mu \Gamma(\mu + 1) J_\mu(t) = \Lambda_\mu(-t). \tag{6.3}$$

It then follows from (6.2) that

$$\frac{t^2 \Lambda_{\mu+1}(t)}{4\mu(\mu + 1)} = \Lambda_\mu(t) - \Lambda_{\mu-1}(t), \tag{6.4}$$

and

$$\begin{aligned} \Lambda_\mu^2(t) &= \frac{t^2}{4\mu(\mu + 1)} \Lambda_\mu(t)\Lambda_{\mu+1}(t) + \Lambda_{\mu-1}(t)\Lambda_\mu(t) \\ &= \frac{t^2}{4\mu(\mu + 1)} \Lambda_{\mu+1}(t) \left\{ \frac{4\mu(\mu - 1)}{t^2} (\dot{\Lambda}_{\mu-1}(t) - \Lambda_{\mu-2}(t)) \right\} \\ &\quad + \Lambda_{\mu-1}(t)\Lambda_\mu(t). \end{aligned}$$

Thus, using (6.4) again, we have

$$\begin{aligned} \Lambda_\mu^2 = \frac{\mu - 1}{\mu + 1} (\Lambda_{\mu+1}\Lambda_{\mu-1} - \Lambda_{\mu+1}\Lambda_{\mu-2}) \\ + \frac{4\mu(\mu - 1)}{t^2} \Lambda_{\mu-1}(\Lambda_{\mu-1} - \Lambda_{\mu-2}), \end{aligned}$$

and

$$\begin{aligned} \Lambda_\mu^2 - \frac{\mu - 1}{\mu + 1} \Lambda_{\mu-1}\Lambda_{\mu+1} &= \frac{4\mu(\mu - 1)}{t^2} \Lambda_{\mu-1}(\Lambda_{\mu-1} - \Lambda_{\mu-2}) \\ &\quad - \frac{\mu - 1}{\mu + 1} \Lambda_{\mu-2}\{\Lambda_\mu - \Lambda_{\mu-1}\} \frac{4\mu(\mu + 1)}{t^2} \\ &= \frac{4\mu(\mu - 1)}{t^2} \Lambda_{\mu-1}^2 - \frac{4\mu(\mu - 1)}{t^2} \Lambda_{\mu-2}\Lambda_\mu. \end{aligned} \tag{6.5}$$

If we put

$$D_\mu(t) = \Lambda_\mu^2 - \frac{\mu - 1}{\mu + 1} \Lambda_{\mu-1}\Lambda_{\mu+1},$$

and

$$\Delta_\mu(t) = \Lambda_\mu^2 - \Lambda_{\mu-1}\Lambda_{\mu+1},$$

then we have

$$D_\mu = \frac{4\mu(\mu-1)}{t^2}\Delta_{\mu-1}, \tag{6.6}$$

and

$$\frac{\mu-1}{\mu+1}\Delta_\mu = D_\mu - \frac{2}{\mu+1}\Lambda_\mu^2.$$

Now from (6.5) we get

$$D_\mu(t) = \frac{4\mu^2(\mu-1)}{(\mu-2)t^2}D_{\mu-1}(t) - \frac{8\mu(\mu-1)}{(\mu-2)t^2}\Lambda_{\mu-1}^2 \qquad (\mu \neq 2);$$

hence

$$\frac{t^{2\mu}D_\mu(t)}{4^\mu(\mu-1)\Gamma^2(\mu+1)} = \frac{t^{2(\mu-1)}D_{\mu-1}(t)}{4^{\mu-1}(\mu-2)\Gamma^2(\mu)} - \frac{2t^{2(\mu-1)}\Lambda_{\mu-1}^2}{4^{\mu-1}\mu(\mu-2)\Gamma^2(\mu)}. \tag{6.7}$$

It follows that the sequence

$$Q_{\mu+\nu}(t) = \frac{t^2(\mu+\nu)D_{\mu+\nu}(t)}{4^{\mu+\nu}(\mu+\nu-1)\Gamma^2(\mu+\nu+1)}, \qquad \nu = 0, 1, 2, \cdots,$$

is monotone decreasing as ν increases. Furthermore

$$D_1(t) = \Lambda_1^2(t), \qquad D_2(t) = \frac{8}{t^2}\Delta_2(t).$$

It is known[4] that

$$|\Delta_\mu(t)| \leqq \Delta_\mu(0) = 1, \qquad t \text{ real};$$

hence

$$Q_\mu(t) \to 0 \qquad \text{as } \mu \to \infty,$$

and, from (6.7),

$$Q_\mu(t) = 2\sum_{\nu=0}^{\infty} \frac{t^{2(\mu+\nu)}\Lambda_{\mu+\nu}^2(t)}{4^{\mu+\nu}(\mu+\nu-1)\Gamma^2(\mu+\nu+1)}, \qquad \mu > 1,$$

[4] See S. Minakshisundaram and Otto Szász, *On absolute convergence of multiple Fourier series*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 36–53, in particular formula (2.3).

the series being uniformly convergent. It now follows that

$$D_\mu(t) > 0, \qquad \text{for } \mu > 1,\ t \text{ real};$$

hence, from (6.6), we have

$$\Delta_{\mu-1}(t) > 0, \qquad \mu > 1,$$

or, in terms of the Bessel functions,

$$J_\mu^2(t) - J_{\mu-1}(t)J_{\mu+1}(t) > \frac{1}{\mu+1} J_\mu^2(t), \qquad \mu > 0,\ t \text{ real}.$$

If, for example, $\mu = 3/2$, we get

$$t \sin^2 t - 3 \cos t(\sin t - t \cos t) > 0 \qquad \text{for } t > 0.$$

We state the main result of this section as:

THEOREM 2. *If $J_\mu(t)$ denotes the Bessel function of index μ, defined by* (6.1), *then, for $\mu > 0$ and all real values of t,*

$$J_\mu^2(t) - J_{\mu-1}(t)J_{\mu+1}(t) > \frac{1}{\mu+1} J_\mu^2(t).$$

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