A THEOREM IN FINITE PROJECTIVE GEOMETRY AND AN APPLICATION TO STATISTICS[1]

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R. C. Bose showed that certain statistical problems of design of experiments can be attacked fruitfully by interpreting the statistical terms involved in terms of finite geometries. In particular, the geometrical interpretation proved useful when applied to the problem of determining the maximum number of factors which can be accommodated in symmetrical factorial design without confounding the degrees of freedom belonging to interactions of a given order or lesser.

In a paper *Mathematical theory of factorial design* (Sankhya vol. 8 (1947) pp. 107–166) R. C. Bose proved that the maximum number, $m_t(r, s)$, of factors which can be accommodated in symmetrical factorial design in which each factor is at $s=p^n$ levels (p being a positive prime and n a positive integer) and each block is of size s^r, without confounding any degrees of freedom belonging to any interaction involving t or lesser number of factors, is given by the maximum number of points of a finite projective space $PG(r-1, s)$ (of r homogenous co-ordinates each of which is capable of s values) such that no t of the chosen points are conjoint (lie in a space of $t-2$ dimensions). Furthermore, R. C. Bose proved in the above mentioned paper that $m_3(4, s)=s^2+1$ when s is a power of an odd prime and $s^2+1 \leqq m_3(4, s) \leqq s^2+s+2$ when $s=2^n$, $n>1$. This inequality gives, for $s=4$, $17 \leqq m_3(4, 4) \leqq 22$. It is the purpose of this paper to prove that $m_3(4, 4)=17$. In other words: the maximum number of points of $PG(3, 4)$ such that no 3 of them are collinear cannot exceed 17 (Theorem 5).

In order to make the reading of this paper independent of the literature on finite geometry we explain shortly the $PG(3, 4)$. The points in $PG(3, 4)$ can be represented by attaching to it a Galois field with 4 elements, $GF(4)$, built with the help of an irreducible polynomial, $\epsilon^2+\epsilon+1=0$, where a point is understood to be an ordered quadruple of elements of $GF(4)$, not all zero. Two points (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are defined to be identical if there exists a nonzero element of $GF(4)$, say ρ, such that $x_i=\rho y_i$, $i=1, 2, 3, 4$. Thus, the total number of points is seen to be $(4^4-1)/3=85$. By a line passing through two different points (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4)

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is understood the set of points $(\mu x_1+\lambda y_1, \mu x_2+\lambda y_2, \mu x_3+\lambda y_3, \mu x_4+\lambda y_4)$ where μ, λ are elements of $GF(4)$. It is seen that the number of points on a line is 5. Defining, similarly, what is meant by a plane, the number of planes passing through a line will be 5 and the number of points on a plane 21. Finally, the remark that the observations usual in projective geometry concerning the choice of the coordinate system hold true in $PG(3, 4)$.

The following two theorems are known to hold in $PG(3, 4)$.

THEOREM 1. *The three diagonal points of a nondegenerate plane quadrangle are distinct and lie on a line, say L.*

The proof follows by taking coordinates such that the points of the quadrangle are $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(0, 1, 1, 1)$. In this case the diagonal points are $(0, 1, 1, 0)$, $(0, 1, 0, 1)$, $(0, 0, 1, 1)$ and the equation of L becomes evidently $x_1=0$, $x_2+x_3+x_4=0$.

THEOREM 2. *In every plane there are 6 "no three collinear" points.*

$(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(0, 1, 1, 1)$, $(0, 1, \epsilon, \epsilon^2)$, $(0, 1, \epsilon^2, \epsilon)$ are six such points of the plane $x_1=0$. Evidently the last two points lie on L and differ from the above-mentioned three diagonal points. Moreover, together with them they exhaust the points of L.

THEOREM 3. *Any 4 no three collinear points Q_1, Q_2, Q_3, Q_4 in a plane determine uniquely 2 more points, say Q_5, Q_6, such that $Q_1, Q_2, \cdots, Q_6$ are no three collinear points.*

Let us consider the five lines in the plane passing through Q_1. They exhibit all the points of the plane. Three of them coincide with the lines determined by the pairs of points Q_1Q_2, Q_1Q_3, Q_1Q_4. Each of the remaining two lines meets the lines Q_2Q_3, Q_2Q_4, Q_3Q_4 in three distinct points. Thus on each of these two lines there remains just one point, say Q_5, Q_6, respectively which form together with $Q_1, \cdots, Q_4$ a set of no three collinear points.

DEFINITION. Any set of 6 no three collinear points of a plane will be called 6-basic.

THEOREM 4. *Consider two planes p_1 and p_2 intersecting in a line determined by two points U_1, U_2 and 4 more points Q_1, Q_2, Q_3, Q_4 of p_1 and R_1, R_2, R_3, R_4 of p_2 so that U_1, U_2, Q_1, Q_2, Q_3, Q_4 as well as U_1, U_2, R_1, R_2, R_3, R_4 are 6-basic. Then,*

1. *The 16 lines Q_iR_j $(i, j=1, 2, 3, 4)$ are distinct.*
2. *They can be distributed into four sets of 4 lines so that the 4 lines of each set intersect in exactly one point.*

3. *If Q_iQ_j and R_lR_m are coplanar, then they meet the line U_1U_2 in the same point.*

4. *The four points of intersection of the lines Q_iR_j, say O_1, O_2, O_3, O_4, lie on a plane, say p_3, passing through U_1U_2.*

5. *U_1, U_2, O_1, O_2, O_3, O_4 are 6-basic and the role of the sets of Q's or R's or O's can be interchanged.*

PROOF. 1. If a line λ given by Q_iR_j were identical with that given by Q_kR_l, then, assuming without loss of generality $i\neq k$, it would follow that points Q_i, Q_k, R_j lie on λ and consequently that R_j is in p_1 against the assumption.

2. There exists a two-to-two correspondence between the 6 lines Q_iQ_j and the 6 lines R_kR_l if we associate with line Q_iQ_j any of the two lines R_kR_l which pass through the same one of the three points of L other than U_1, U_2. By this correspondence the points of a given triangle $Q_iQ_jQ_s$ are made to correspond to those of an arbitrary triangle $R_kR_lR_m$ in a unique way. For Q_iQ_j corresponds to at most one side, say R_kR_l; Q_iQ_s corresponds to another side, say R_kR_m; hence the intersection Q_i of Q_iQ_j and Q_iQ_s corresponds to the intersection of R_kR_l and R_kR_m which is R_k; and accordingly Q_j must correspond to R_l, Q_s to R_m; it is seen that also line Q_jQ_s corresponds to line R_lR_m because these lines must pass through the remaining one of the three points of L other than U_1, U_2. Now we apply the converse of Desargue's theorem to the triangles $Q_iQ_jQ_s$ and $R_kR_lR_m$ and obtain that the lines Q_iR_k, Q_jR_l, Q_sR_m pass through one point, say O; furthermore, if Q' is the fourth point of the Q set, R' the fourth point of the R set, then $Q'R'$ passes also through O; for the triangle Q_iQ_jQ' corresponds to R_kR_lR' necessarily point by point in the same order since line Q_iQ' is different from Q_iQ_j and Q_iQ_s, hence these three lines intersect with L at three distinct points other than U_1, U_2, thus determining the intersection of L with Q_iQ' by exclusion, and the same holds for R_kR'. Applying the converse of Desargue's theorem to the triangles Q_iQ_jQ' and R_kR_lR', we find that Q_iR_k, Q_jQ_l, $Q'R'$ pass through a point which necessarily is O. Making any of the four triangles formed by 3 out of the four points R_1, R_2, R_3, R_4 correspond to the triangle Q_1, Q_2, Q_3 in the indicated manner, we obtain 4 points O_1, O_2, O_3, O_4 as points of perspectivity. They are distinct, for if, for example, the lines Q_1R_1, Q_2R_2, Q_3R_3, Q_4R_4 meet at O_1, then a line Q_iR_j, where $i\neq j$, cannot pass through O_1 because then R_i, R_j, Q_i would be collinear and consequently Q_i would lie on the plane p_2 against the assumption.

3. Two coplanar lines Q_iQ_j and R_lR_m meet the line L (intersection of the planes p_1 and p_2) in the same point, because otherwise the planes p_1 and p_2 would be identical against the assumption.

4. Given any two points of the set O_1, O_2, O_3, O_4, say O_1 and O_2, there exist two points Q_i, Q_j and two points R_l, R_m such that the line Q_iR_l meets Q_jR_m in O_1 and Q_iR_m meets Q_jR_l in O_2. Take Q_i arbitrary, determine R_l so that the line Q_iR_l passes through O_1; determine R_m so that Q_iR_m passes through O_2; then O_2R_l must pass through another Q, say Q_j. Q_i, Q_j, R_l, R_m form evidently a nondegenerate plane quadrangle and thus the line Q_iQ_j meets the line R_lR_m in a point P and, by Theorem 1, O_1, O_2, P are collinear. But P lies, by 3, on the line L, thus O_1, O_2, U_1, U_2 are coplanar. The same holds for any O_i, O_j, U_1, U_2, showing that all O_1, O_2, O_3, O_4 lie in a plane through U_1, U_2, say p_3.

5. Line U_1U_2 does not contain an O; furthermore any line O_iO_j meets the line U_1U_2 at a point P different from U_1 and U_2, so that no U lies on a line joining two O's and no O lies on a line joining two U's. It remains to show that no three of the O's are collinear. Assume without loss of generality that

$$O_1 \text{ is intersection of } Q_1R_1 \text{ with } Q_2R_2,$$

$$O_2 \text{ is intersection of } Q_1R_2 \text{ with } Q_2R_1,$$

$$O_3 \text{ is intersection of } Q_1R_3 \text{ with } Q_2R_4.$$

Now if O_1, O_2, O_3 were collinear, then R_1, R_2, R_3 would be obtained as intersections of their plane (which does not contain Q_1) with the plane $Q_1O_1O_2O_3$; hence R_1, R_2, R_3 would be collinear against assumption. Thus no three O's are collinear. It is seen that, under the assumption made above, Q_1 is the intersection of the lines $O_1R_1, O_2R_2, O_3R_3, O_4R_4$ and R_1 is the intersection of $O_1Q_1, O_2Q_2, O_3Q_3, O_4Q_4$. Evidently similar results follow for the remaining 3 groups of 4 lines O_iR_j or O_iQ_j respectively. Thus the rôles of the O's, Q's, and R's can be interchanged.

THEOREM 5. *The maximum number of points which can be chosen in $PG(3, 4)$ such that no three of the chosen points are collinear does not exceed* 17.

PROOF. Let C be the set consisting of the maximum number of points no three collinear in $PG(3, 4)$. Assume without loss of generality that U_1, U_2 belong to C. The proof is trivial in case when, among the 5 planes passing through U_1U_2, there do not exist 2 planes such that one of them contains 4 points, other than U_1, U_2, belonging to C and the other plane at least 3 such points, because in this case the total number of points belonging to C would readily be seen to be at most 17. Hence assume without loss of generality that Q_1, Q_2, Q_3, Q_4 as well as R_1, R_2, R_3 belong to C. Consider two cases (1) R_4 belongs to C, (2) R_4 does not belong to C.

In case 1 the 16 distinct lines R_iQ_j $(i, j=1, 2, 3, 4)$ contain 16 points of each of the two planes, say p_4, p_5, other than p_1, p_2, p_3. Since the remaining 5 points of these planes belong to the line L, no points of these planes other than U_1, U_2 belong to C. C can contain at most 4 additional points of p_3; accordingly, the total number of points in C amounts to 14 $(2+3\times 4=14)$.

In case 2, the points of p_4 and p_5 which belong to C are on the lines Q_iR_4 $(i=1, 2, 3, 4)$. Since the Q's belong to C by assumption, there are at most 4 more points of these lines belonging to C. In addition at most 4 points other than U_1, U_2 of p_3 belong to C. Thus in this case the total number of points belonging to C is at most 17 $(2+3+3\times 4=17)$.

REMARK. It can be proved that in case 2 the total number of points belonging to C is 13.

Theorem 5 together with the inequality $m_3(4, 4)\geqq 17$ proved by R. C. Bose gives the required equality, namely $m_3(4, 4)=17$.

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It was proved after the paper was submitted to the editors that every set of 17 no three collinear points in $PG(3, 4)$ forms a quadric surface.

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