ON UPPER SEMICONTINUOUS FUNCTIONS

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1. Introduction. In his important paper [1] (see Bibliography) on the theory of the Lebesgue area of surfaces, L. Cesari established the following interesting result. Let $X$ be the unit cube $0 \leq x_j \leq 1$, $j = 1, \ldots, m$, of Euclidean $m$-space and let $Y$ be the unit cube $0 \leq y_i \leq 1$, $i = 1, \ldots, n$, of Euclidean $n$-space. Let $F(x_1, \ldots, x_m, y_1, \ldots, y_n)$ be a real-valued, bounded, upper semicontinuous function in the product space $X \times Y$. Then for each point $(x_1, \ldots, x_m) \in X$, $F$ is a bounded, upper semicontinuous function of $(y_1, \ldots, y_n) \in Y$, and hence $F$ attains, for each fixed $(x_1, \ldots, x_m) \in X$, a maximum value $M(x_1, \ldots, x_m)$, while $(y_1, \ldots, y_n)$ varies in $Y$. According to the theorem of Cesari, there exists then in $X$ Borel measurable functions $y_i = \Phi_i(x_1, \ldots, x_m)$, $i = 1, \ldots, n$, such that identically

$$F[x_1, \ldots, x_m, \Phi_1(x_1, \ldots, x_m), \ldots, \Phi_n(x_1, \ldots, x_m)] = M(x_1, \ldots, x_m).$$

The proof given by Cesari is essentially an induction proof. The purpose of this note is to give a direct proof for a more general result (see our Theorem 3.1) which contains the theorem of Cesari as a very special case. In this generalized result, $X$ is any metric space and $Y$ is any compact metric space. In this general setting, the concept of a real-valued Borel measurable function is replaced by the following concept (cf. Kuratowski [2, p. 177]). A single-valued transformation $T$ from a metric space $S_1$ into a metric space $S_2$ is termed a Borel transformation if for every closed set $F_2 \subseteq S_2$ the inverse set $T^{-1}(F_2)$ is a Borel set in $S_1$ (and, hence, if the inverse of every Borel set in $S_2$ is a Borel set in $S_1$).

2. Preliminaries. Throughout this section $X$ will denote a metric space and $I$ will denote the unit interval $0 \leq t \leq 1$. Points in $X$ will be denoted by $x$.

**Lemma 2.1.** In the product space $X \times I$ let $E$ be a set which satisfies the following conditions.

(a) For each $t_0$ the set

$$E \left[ (x, t) \in E, \ t = t_0 \right]$$

is a Borel set.

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(b) For each $x_0$ the set
\[ E \left[ (x_0, t) \in E \right] \]
is a closed interval containing 0. Then $E$ is a Borel set.

**Proof.** Let $r_1, r_2, \ldots$ be the set of rational numbers in $I$. For each pair of integers $n$ and $m$ we set
\[ E_{m}^{n} = \bigcup_{(x, t) \in P, 0 \leq t \leq r_n + 1/m, t \leq 1} (x, t) \in E. \]
It follows from condition (a) that each set $E_{m}^{n}$ is a Borel set. Thus the sets
\[ E_{m} = \sum_{n} E_{m}^{n}, \quad m = 1, 2, \ldots, \]
are Borel sets. The reader will verify readily that $E = \prod_{m} E_{m}$. Therefore $E$ is a Borel set.

**Lemma 2.2.** In the product space $X \times I$ let $E$ be a set which satisfies the following conditions.

(a) For each $t_0$ the set
\[ E \left[ (x, t) \in E, t = t_0 \right] \]
is a Borel set.

(b) For each $x_0$ the set
\[ E \left[ (x_0, t) \in E \right] \]
is a closed interval containing 1. Then $E$ is a Borel set.

**Proof.** The proof is similar to that used in Lemma 2.1.

**Lemma 2.3.** Let $t=f(x)$ be a single-valued transformation from $X$ into $I$ and in the product space $X \times I$ let
\[ E_{*} = \bigcup_{(x, t) \in E} [0 \leq t \leq f(x) \] \[ E^{*} = \bigcup_{(x, t) \in E} [f(x) \leq t \leq 1]. \]
A necessary and sufficient condition that $t=f(x)$ be a Borel transformation is that $E_{*}$ be a Borel set. The same statement holds with $E_{*}$ replaced by $E^{*}$.

**Proof.** Assume that $t=f(x)$ is a Borel transformation. Then $E_{*}$ and $E^{*}$ clearly satisfy the conditions of Lemma 2.1 and Lemma 2.2 respectively. Hence $E_{*}$ and $E^{*}$ are Borel sets.
Assume that $E_*$ is a Borel set. We assert that $f^{-1}(B)$ is a Borel set whenever $B$ is a Borel set in $I$. By well known results it is sufficient to prove this assertion for the case where

\[(1)\quad B = E_{t} \left[ t_0 \leq t \leq 1 \right]. \]

Let $B$ be a Borel set as given in (1). Set

\[ X^0 = \left\{ (x, t) \in X \times I, t = t_0 \right\}, \quad B_* = E_* X^0. \]

Let $P: (x, t) \rightarrow x$ be the projection of $X \times I$ onto $X$. From the definition of $E_*$ and by well known properties of Borel sets we have the following facts concerning the sets $B$ and $B_*$. (i) $f^{-1}(B) = P(B_*).$ (ii) Since $E_*$ is assumed to be a Borel set, $B_*$ is a Borel subset of $X^0.$ (iii) Since $P$ is a bicontinuous transformation from $X^0$ onto $X$ and $B_*$ is a Borel subset of $X^0$, $P(B_*)$ is a Borel set. From (i) and (iii) it follows that $f^{-1}(B)$ is a Borel set.

A similar reasoning applies if $E_*$ is assumed to be a Borel set.

Let $g(x, t)$ be a real-valued, bounded, upper semicontinuous function defined on the product space $X \times C$ where $C$ is a closed subset of $I$ and $C$ contains 0. Set

\[ M(x) = \max_{t \in C} g(x, t) \quad \text{for } x \text{ fixed, } t \in C. \]

Since $g(x, t)$ is an upper semicontinuous function and $C$ is compact, the set

\[ E(x) = E_{t} \left[ g(x, t) = M(x), \quad x \text{ fixed} \right] \]

is a closed subset of $C$ for each $x$. Set

\[(2)\quad t = f(x) = \min_{\tau \in E(x)} \tau \]

**Lemma 2.4.** Under the above conditions the transformation $t = f(x)$ defined in (2) is a Borel transformation from $X$ into $C$ and for each $x \in X$

\[(3)\quad g[x, f(x)] = M(x). \]

**Proof.** The relation (3) follows from the definition of $t = f(x)$ in (2). To prove that $t = f(x)$ is a Borel transformation we set

\[ g_1(x, t) = \max_{\tau \in C} g(x, \tau) \quad \text{for } 0 \leq \tau \leq t, \tau \in C. \]

Since 0 is in $C$, $g_1(x, t)$ is defined on $X \times I$. It is easily shown that $M(x)$ and $g_1(x, t)$ are bounded, upper semicontinuous functions on $X \times I$ and $g_1(x, t) \leq M(x)$. Thus the function

\[ \phi(x, t) = M(x) - g(x, t) \geq 0. \]
is a Borel measurable function. Hence the set

\[ E_0 = \{ (x, t) : \phi(x, t) = 0 \} \]

is a Borel set. It is easily shown that the set \( E_0 \) is equal to the set

\[ E^* = \{ (x, t) : f(x) \leq t \leq 1 \} \]

Therefore, by the result in Lemma 2.3, \( t = f(x) \) is a Borel transformation.

The reader can easily verify the following result on Borel transformations.

**Lemma 2.5.** Let \( Y \) be a metric space which is the product of \( n \) metric spaces \( Y_1, \ldots, Y_n \) and let

\[ T: \Phi_i(x), \quad i = 1, \ldots, n, \quad x \in X, \quad y_i \in Y_i, \]

be a single-valued transformation from \( X \) into \( Y \). A necessary and sufficient condition that \( T \) be a Borel transformation from \( X \) into \( Y \) is that each of the transformations \( y_i = \Phi_i(x) \) be a Borel transformation from \( X \) into \( Y_i \).

3. The main results.

**Theorem 3.1.** Let \( X \) be a metric space, let \( Y \) be a compact metric space, and let \( F(x, y) \) be a real-valued, bounded, upper semicontinuous function defined on the product space \( X \times Y \). Then there exists a Borel transformation

\[ y = \Phi(x) \]

from \( X \) into \( Y \) such that for each \( x \in X \)

\[ F[x, \Phi(x)] = M(x), \quad \text{where} \quad M(x) = \max F(x, y) \quad \text{for} \quad x \text{ fixed, } y \in Y. \]

**Proof.** Since \( Y \) is compact, \( Y \) is the continuous image of a closed subset \( C \) of the unit interval \( I: 0 \leq t \leq 1 \), where \( C \) contains 0 (see Whyburn [3, p. 34]). Let us denote this continuous transformation by

\[ \Phi(t), \quad t \in C. \]

Then

\[ F[x, \Phi(t)] = M(x), \quad \text{for} \quad x \text{ fixed, } t \in C. \]
The transformation \( t = f(x) \) defined in Lemma 2.4 in terms of \( g(xt) \) is a Borel transformation from \( X \) into \( C \) for which

\[
g[x, f(x)] = M(x).
\]

Set

\[
y = \Phi(x) = \phi[f(x)].
\]

From (6), (7), and (8)

\[
F[x, \Phi(x)] = F[x, \phi(f(x))] = g[x, f(x)] = M(x).
\]

Thus the transformation defined in (8) satisfies the relation (5). Let \( B \) be a Borel set in \( Y \). Then

\[
\Phi^{-1}(B) = f^{-1}[\phi^{-1}(B)].
\]

Since \( y = \phi(t) \) is a continuous transformation from \( C \) onto \( Y \), \( \phi^{-1}(B) \) is a Borel subset of \( C \), and since \( t = f(x) \) is a Borel transformation from \( X \) into \( C \), \( f^{-1}[\phi^{-1}(B)] \) is a Borel set. Thus the transformation defined in (8) is a Borel transformation from \( X \) into \( Y \) which satisfies the conditions of the theorem.

If in the preceding theorem \( Y \) is the product of \( n \) metric spaces \( Y_1, \cdots, Y_n \), the Borel transformation (4) can be written in the form

\[
y_i = \Phi_i(x), \quad x \in X, \quad y_i \in Y_i, \quad i = 1, \cdots, n.
\]

By Lemma 2.5 we thus have the result:

**Corollary.** Each transformation \( y_i = \Phi_i(x), \; i = 1, \cdots, n \), given in (9) is a Borel transformation from \( X \) into \( Y_i \).

If \( X \) coincides with the unit cube \( 0 \leq x_j \leq 1, \; j = 1, \cdots, m \), of Euclidean \( m \)-space and \( Y \) coincides with the unit cube \( 0 \leq y_i \leq 1, \; i = 1, \cdots, n \), of Euclidean \( n \)-space, then we obtain as a special case the theorem of Cesari stated in the introduction.

**Bibliography**


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