SOME THEOREMS ON MEROMORPHIC FUNCTIONS

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Let \( f(z) \) be an integral function of finite order \( \rho \geq 0 \), and let \( n(r) \) denote the zeros of \( f(z) \) in \( |z| \leq r \). Set

\[ M(r, f) = \max_{|z| \leq r} |f(z)|. \]

Pólya\(^1\) proved

\[ \liminf_{r \to \infty} \frac{\log M(r, f)}{n(r)} < \infty \]

if \( \rho \) is not an integer, and Shah\(^2\) proved

\[ \liminf_{r \to \infty} \frac{\log M(r, f)}{n(r) \phi(r)} = 0 \]

if \( f(z) \) is a canonical product of genus \( p \) such as \( \rho = p \), and \( \phi(x) \) is any positive continuous nondecreasing function of a real variable \( x \) such that \( \int_a^\infty dx/x\phi(x) \) is convergent.

In this paper, some similar theorems will be obtained for meromorphic functions.

Let

\[ F(z) = \frac{f(z)}{g(z)} \]

be a meromorphic function of finite order, where

\[ f(z) = \prod_{n=1}^{\infty} E\left( \frac{z}{a_n}, p \right) \]

\[ = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right) \exp \left( \frac{z}{a_n} + \frac{1}{2} \left( \frac{z}{a_n} \right)^2 + \cdots + \frac{1}{p} \left( \frac{z}{a_n} \right)^p \right), \]

\[ g(z) = \prod_{n=1}^{\infty} E\left( \frac{z}{b_n}, q \right) \]

\[ = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{b_n} \right) \exp \left( \frac{z}{b_n} + \frac{1}{2} \left( \frac{z}{b_n} \right)^2 + \cdots + \frac{1}{q} \left( \frac{z}{b_n} \right)^q \right), \]

these being canonical products of genera \( p \), \( q \) and of orders \( \rho_1, \rho_2 \), respectively. Let \( s \) be the genus of \((1)\), that is \( s = \max (p, q) \), and de-

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note the number of the totality of \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \) in \( |z| \leq r \) by \( N(r) \). Moreover set
\[
M(r, F) = M(r) = \max_{|z| \leq r} |F(z)|.
\]
Then we have the following theorems:

**Theorem I.** If for any given meromorphic function (1) of finite order
\[
\max (\rho_1, \rho_2) = s \geq 0,
\]
then
\[
(2) \quad \lim_{r \to \infty} \inf \frac{1}{rN(r)\phi(r)} \int_0^r \log^+ M(t) dt = 0,
\]
where \( \phi(x) \) is any positive continuous nondecreasing function of a real variable \( x \) such that \( \int_0^x dx/x\phi(x) \) is convergent.

**Theorem II.** If for any given meromorphic function (1) of finite order \( \max (\rho_1, \rho_2) \) is not an integer, then
\[
(3) \quad \lim_{r \to \infty} \inf \frac{1}{rN(r)} \int_0^r \log^+ M(t) dt < \infty.
\]

To prove these theorems we establish the lemma:

**Lemma.** For any given meromorphic function (1) of genus \( s \), let \( c_1, c_2, \ldots \) denote the totality of \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \) in \( |z| \leq r \). Then
\[
(4) \quad \frac{1}{r} \int_0^r \log^+ M(t) dt \leq h_1 \sum_{n=1}^\infty \frac{r^{s+1}}{|c_n|^{s}(|c_n| + r)} \quad \text{when } s \geq 1; \text{ and for a number } k > 1,
\]
\[
(5) \quad \frac{1}{r} \int_0^r \log^+ M(t) dt \leq h_2 \sum_{n=1}^\infty \log \left( 1 + \frac{kr}{|c_n|} \right) \quad \text{when } s = 0.
\]

**Proof.** Assume \( s = \max (\rho, q) \geq 1 \). Since \( s \geq \rho \) and \( \rho \) is the genus of \( f(z) \), for a certain polynomial \( P(z) \) of degree \( s \leq s \) we get
\[
f(z) = e^{P(z)} \prod_{n=1}^\infty E \left( \frac{z}{a_n}, s \right),
\]
\[\text{Hereafter we assume that } h_1, h_2, \ldots \text{ are all some suitable fixed positive constants independent of } r.\]
whence

\[ \log^+ M(r, f) \leq h_3 r^s + \log^+ \prod_{n=1}^{\infty} \left| E \left( \frac{z}{a_n}, s \right) \right| \]

(6)

\[ \leq h_3 r^s + h_4 \sum_{n=1}^{\infty} \frac{r^{s+1}}{|a_n|^s (|a_n| + r)} \]

Similarly

(7) \[ \log^+ M(r, g) \leq h_5 \sum_{n=1}^{\infty} \frac{r^{s+1}}{|b_n|^s (|b_n| + r)} \]

Now, by the theory of meromorphic functions, we get

(8) \[ \frac{1}{r} \int_{0}^{r} \log^+ M(t)dt < C(k)T(kr, F), \]

where \( T(r, F) \) is the characteristic function of \( F(z) \), and \( C(k) \) depends on a number \( k > 1 \) only, and also we have

(9) \[ T(kr, F) = T(kr, f/g) \leq T(kr, f) + T(kr, g) + O(1) \]

\[ \leq \log^+ M(kr, f) + \log^+ M(kr, g) + O(1). \]

Hence from (6), (7), (8), and (9) follows (4).

Next, assume \( s = 0 \). Then by the same process we get (5).

**Proof of Theorem I.** By the lemma, when \( s \geq 0 \) we get

(10) \[ \frac{1}{r} \int_{0}^{r} \log^+ M(t)dt \leq h_7 r^{s+1} \int_{0}^{\infty} \frac{N(t)}{t^{s+1}(t + r)} dt; \]

for, when \( s \geq 1 \), from (4) this is obviously seen, and when \( s = 0 \), from (5) we have

\[ \frac{1}{r} \int_{0}^{\infty} \log^+ M(t)dt \leq h_2 \int_{0}^{\infty} \frac{krN(t)}{t(t + kr)} dt \]

\[ \leq kh_2 \int_{0}^{\infty} \frac{rN(t)}{t(t + r)} dt. \]

On the other hand the integral function

\[ G(z) = \prod_{n=1}^{\infty} E \left( \frac{z}{c_n}, s \right) \]
is of order $\rho = \max (\rho_1, \rho_2)$, since $c_1, c_2, \cdots$ are composed of $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$, whose exponents of convergence are $\rho_1, \rho_2$ respectively; and the genus of $G(z)$ is $s = \max (p, q)$, since $f(z)$ and $g(z)$ are of genera $p$ and $q$ respectively. Therefore by Shah\textsuperscript{4}

\[
\lim_{r \to \infty} \frac{1}{N(r)\phi(r)} \int_0^r \frac{N(t)}{t^{s+1}(t+r)} \, dt = 0
\]

for $\rho = s \geq 0$. Hence by (10) we get (2).

**Proof of Theorem II.** The order and genus of integral function $G(z)$ are $\rho = \max (\rho_1, \rho_2)$ and $s = \max (p, q)$, respectively, as already seen, so that the exponent of convergence of $|c_1|, |c_2|, \cdots$ is $\rho$, and $\rho > 0$, $s = [\rho]$, since $\rho$ is not an integer.

Now set $\Phi(x) = h_1 x^{(s+1)/1+x-1}$ for $1 < \rho$, and $\Phi(x) = h_2 \log (1+k/x)$ for $0 < \rho < 1$. Then $\Phi(x)$ is a positive and decreasing function for $x > 0$, and we have

\[
\Phi(x) < x^{-\rho+r} \quad (0 < \eta < \rho - s)
\]

for all values of $x > 0$ near $x = 0$, and

\[
\Phi(x) < x^{-s-r} \quad (0 < \eta < s + 1 - \rho)
\]

for all values of $x$ sufficiently great. Therefore by Pólya\textsuperscript{5}

\[
\lim_{r \to \infty} \frac{1}{N(r)} \sum_{n=1}^{\infty} \Phi \left( \frac{|c_n|}{r} \right) \leq \int_0^\infty \Phi(x^{1/r}) \, dx < \infty.
\]

Hence from the lemma

\[
\lim_{r \to \infty} \frac{1}{rN(r)} \int_0^r \log^+ M(t) \, dt \leq \lim_{r \to \infty} \frac{1}{N(r)} \sum_{n=1}^{\infty} \Phi \left( \frac{|c_n|}{r} \right) < \infty.
\]

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\textsuperscript{4} Shah, loc. cit. pp. 26–30.

\textsuperscript{5} Pólya, loc. cit., Theorem VI, pp. 173–174.