THE NON-EUCLIDEAN MIRROR

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We propose to show in this paper that some laws of geometrical optics hold true for hyperbolic geometry. Likewise optical properties of certain curves can be carried over. All the proofs given will be valid for elliptic geometry, as long as there are no parallel lines involved.

Fermat's principle may be stated as follows: The path of a light ray from a point A to a point B under known conditions is such that the time taken by the light to traverse it is a minimum. In order to find the law of reflection based on this principle, we shall use Cartesian coordinates [6, p. 138].[1] Let $(0, a)$ be the coordinates of A and (c, b) those of B, where $a>0$, $b>0$. A and B are joined to a point $P(x, 0)$ on the x-axis (surface of the mirror). Then the function $z=AP+PB$ has to be minimized (cf. [2, p. 164]). By trigonometry [3, p. 238]: $\cosh AP=\cosh a \cosh x$, whence

$$\sinh AP \frac{d}{dx}(AP) = \cosh a \sinh x.$$

Finding the derivative of PB in the same way, we have

$$\frac{dz}{dx} = \frac{\cosh a \sinh x}{\sinh AP} - \frac{\cosh b \sinh (c - x)}{\sinh PB} = 0.$$

If we now denote by α and β the acute angles formed by AP and PB with the x-axis, we obtain the trigonometric relations

$$\cos \alpha = \frac{\tanh x}{\tanh AP} = \frac{\cosh a \sinh x}{\sinh AP}, \qquad \cos \beta = \frac{\cosh b \sinh (c - x)}{\sinh PB}.$$

Thus,

$$\cos \alpha = \cos \beta \quad \text{or} \quad \alpha = \beta,$$

that is, the angle of incidence is equal to the angle of reflection for the required minimum. The second derivative of z is actually positive for any x. The same result could be found by a simple geometric method; but in its analytical form the above procedure can easily be generalized to yield the law of refraction.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Now suppose we want to solve the following problem: Find the shape of a mirror such that all light coming from one fixed point is reflected to another fixed point [5, p. 29]. A synthetic solution of this problem in Euclidean geometry, which applies to the hyperbolic case without change, can be found in R. Courant and H. Robbins, *What is mathematics?*, London, 1941, p. 333. Let the two fixed points be $F_1(-c, 0)$ and $F_2(c, 0)$. Take a characteristic point $P(x, y)$ of the curve, $Q(x, 0)$ being its projection on the x-axis. Let $PF_1=r_1$, $PF_2=r_2$, $\angle F_1PQ=\theta$, and $\angle F_2PQ=\phi$. Designate by α the angle through which PQ must be rotated in the counterclockwise sense in order to coincide with the tangent line at P. Then, according to the above law of reflection:

$$\alpha - \phi = \pi - \alpha - \theta, \quad \text{and} \quad \cos(\alpha - \phi) + \cos(\alpha + \theta) = 0,$$

or

$$\cos\theta \cot\alpha - \sin\theta + \cos\phi \cot\alpha + \sin\phi = 0. \tag{1}$$

It has been shown [4] that

$$\cot\alpha = -\operatorname{sech} y \frac{dy}{dx}.$$

On the other hand, from the right triangle F_1PQ:

$$\cos\theta = \frac{\tanh y}{\tanh r_1} = \frac{\cosh(x+c)\sinh y}{\sinh r_1}, \qquad \sin\theta = \frac{\sinh(x+c)}{\sinh r_1}.$$

Using the analogous relations for ϕ and substituting all this, we see that (1) becomes

$$\frac{\cosh(x+c)\tanh y dy + \sinh(x+c)dx}{\sinh r_1} + \frac{\cosh(x-c)\tanh y dy + \sinh(x-c)dx}{\sinh r_2} = 0. \tag{2}$$

Again, $\cosh r_1 = \cosh(x+c)\cosh y$, and by differentiation

$$\sinh r_1 dr_1 = \cosh(x+c)\sinh y dy + \sinh(x+c)\cosh y dx.$$

Thus, (2) reduces to

$$dr_1 + dr_2 = 0.$$

Hence

$$r_1 + r_2 = \text{const.}.$$

so that the only curves having the required property are ellipses [1, p. 146 and 148].

Let us consider one more problem: Find the surface of a mirror such that all light from a fixed point is reflected parallel to a fixed line. Let the fixed point coincide with the origin and take P and Q as before. Let $OP=r$ and $\angle OPQ=\theta$. The angle α has the same meaning as before and $\phi=\Pi(y)$ denotes the angle of parallelism for y. If we take the directed x-axis as the fixed line of our problem, we have from the law of reflection the above relation (1), but now

$$\cos\theta = \frac{\cosh x \sinh y}{\sinh r}, \qquad \sin\theta = \frac{\sinh x}{\sinh r}.$$

Also [6, p. 151],

$$\cos\phi = \tanh y, \qquad \sin\phi = \operatorname{sech} y.$$

Hence, (1) becomes upon substitution

$$\frac{\cosh x \tanh y dy + \sinh x dx}{\sinh r} + \tanh y \operatorname{sech} y dy - \operatorname{sech} y dx = 0. \tag{3}$$

As is seen in [6, p. 165], the distance from P to the limiting curve through O, which is perpendicular to all the parallels to the x-axis, is represented by

$$\xi = x - \log\cosh y.$$

Thus, $d\xi = dx - \tanh y dy$, and (3) is reduced to

$$dr - d\xi = 0.$$

Hence

$$r - \xi = \text{const.},$$

which represents a family of parabolas.

References

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