A REMARK ON A THEOREM OF MARSHALL HALL

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Marshall Hall [2] proved that the identity

(1) \( x(yz - zy)^2 = (zy - zy)^2 x \)

characterizes quaternion algebras among associative but not commutative division rings. Our remark is that (1) characterizes Cayley-Dickson algebras among alternative but not associative division rings. This follows from Hall’s proof and a result of A. A. Albert [1]. A theorem of R. D. Schäfer [4, Theorem 4] permits us to conclude that (1) and Hall’s Theorem L [2] (as universal) ensure that the coordinate ring of a projective plane is uniquely defined irrespective of the coordinate system.

It is easy to verify (1) in a Cayley-Dickson algebra. We should like to sketch a proof of the converse, which is independent of Albert’s result in [1]. Let \( A \) be an alternative division ring and let \( F \) be the set of all elements \( c \in A \) which satisfy \( cx = xc \) for every \( x \in A \). Then \( F \) is a field and, when 3 is nonzero in \( A \), \( F \) is the center of \( A \) [3; 5, Lemma 9]. If \( A \) satisfies (1), the proof of Hall’s Lemma 1 [2, p. 262] yields \( x^2 = t(x)x - n(x) \) for every \( x \in A \) not in \( F \), where \( t(x), n(x) \in F \).

Define \( t(c) = 2c, n(c) = c^2 \) for \( c \in F \).

When 3 is zero in \( A \) and \( A \) satisfies (1), \( F \) is still the center of \( A \) as we shall now show. Since 2 is a nonzero in \( A \), for each \( x \in A \) we have \( (x + c)^2 \in F \) for some \( c \in F \). If, then, \( c_1, c_2 \) are in \( F \), we have \( (c_1, c_2, (x + c)^2) = 0 \). By use of some identities of Zorn [6, (1.6) and (1.8); 5, (1)] we find that \( (c_1, c_2, (x + c)^2) = 2(x + c)(c_1, c_2, x + c) = 2(x + c) \cdot (c_1, c_2, x) \). We infer that \( (c_1, c_2, x) = 0 \) for every \( c_1, c_2 \) in \( F \) and every \( x \) in \( A \). Now let \( c \in F \) and \( x, y \in A \). As before, \( (x + c')^2 \in F \) for some \( c' \in F \) and

\[
0 = (c, (x + c')^2, y) = 2(x + c')(c, x + c', y) = 2(x + c')(c, x, y),
\]

so that \( (c, x, y) = 0 \) as desired.

To complete the argument, we first prove (i) \( t(x + y) = t(x) + t(y) \),

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1 Numbers in brackets refer to the references cited at the end of the paper.
2 To verify that \( c, c' \in F \) yields \( cc' \in F \) even if \( 3 = 0 \) in \( F \), we may use the equations \( (x, c, c') = (c', x, c) = (c, c', x) \), where \( (x, y, z) = x(yz) - (xy)z \) is the associator of \( x, y, \) and \( z \).
3 We are indebted to the referee for his observation that our original proof was incomplete in this case and also for simplifying our amended version.
(ii) \( t(cx) = ct(x) \), and (iii) \( n(xy) = n(x)n(y) \) for every \( x, y \in A \) and every \( c \in F \), and then we apply the arguments of R. D. Schäfer [4]. We indicate the proof of (i) and (iii) when \( xy \neq yx \). We compute
\[
(x+y)^2 = x^2 + xy + yx + y^2 = t(x+y)(x+y) - n(x+y),
\]
and we obtain (i) by taking the commutator with \( x \) and then (iii) by multiplying by \( xy \) on the left.

Addendum (October 25, 1949). When \( 1+1=0 \) in \( A \), we cannot obtain our result from Albert's Theorem 1 of [7] which employs \( 1+1 \neq 0 \). In this case we must complete the proof of (i)–(iii) by using Lemma 2 of [2, p. 262] and some straightforward but rather lengthy calculations. In fact, we may regard our proof as establishing Albert's Theorem 1 even when \( 1+1=0 \). First redefine quadratic algebra by deleting the coefficient 2 in \( f(\xi, x) \). Then assume that \( A \) is quadratic and alternative and observe that \( A \) is a division algebra. Unless \( A \) is commutative and hence a field over \( F \), one may use the arguments of Hall [2, pp. 262–263] to see that the center of \( A \) is \( F \). For, if \( xy \neq yx \), then \( Q = F[1, x, y, xy] \) is a quaternion algebra over \( F \), and the requirement that \( c \in A \) commute and associate with the elements of \( Q \) implies that \( c \in Q \) and hence that \( c \in F \), since the center of \( Q \) is \( F \). We are now in a position to apply the arguments of the present note.

References


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