TWO-ENDED TOPOLOGICAL GROUPS

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Introduction. Let $G$ be a locally compact, connected topological group (satisfying the second countability axiom). Let $G^*$ be a compact space which contains a dense subset $G'$ homeomorphic to the space $G$ and is such that $G^*-G'$ is totally disconnected. Then, Freudenthal has proved [1, Satz IX, p. 277] that the set $G^*-G'$ consists of at most two distinct points. Actually, Freudenthal's theorem even for topological groups is more general than here stated, and this theorem is an application to group spaces of a wider theory of "ends" of topological spaces. However, we shall quote only so much of Freudenthal's results as are necessary to this paper.

It will be convenient to regard $G'$ as identical with $G$ so that $G$ is topologically imbedded in $G^*$. We shall call a locally compact, connected group $G$ two-ended if a $G^*$ exists such that $G^*-G$ consists of two distinct points. The simplest example of such a group is the additive group of reals. Other examples are afforded by the direct product of this group and any compact connected topological group; it is likely that these are the only examples.

The principal objective of this note is the following theorem.

Theorem A. If a locally compact, connected topological group $G$ is two-ended, then $G$ contains a closed subgroup $T$ isomorphic to the group of reals such that the coset-space $G/T$ is compact; moreover, the space $G$ is the topological product of the axis of reals by a compact connected set homeomorphic to the space $G/T$.

1. Definitions. Now let $G$ be two-ended and $G^*$ compact, and necessarily connected, such that $G^*-G$ consists of a pair of points. We shall denote one of these by $e_L$ and the other by $e_R$. The space $G^*$ is not a group, but we may continue to speak of the group product $fg$ when $f$, $g\in G\subset G^*$. Moreover [1] each $f\in G$ may be regarded as a homeomorphism $f(G^*)=G^*$ by the definitions: $f(e_L)=e_L$, $f(e_R)=e_R$, $f(g)=fg\in G$ when $g\in G$. To the product of homeomorphisms $f$ and $g$ there corresponds the homeomorphism associated with $fg\in G$. We shall denote by $gK$ (resp. $Kg$), $g\in G$, $K\subset G$, the set of points $gk$ (resp. $kg$), where $k\in K$.

The two following properties [1] of $G^*$ are of great importance in this work.

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1 Numbers in brackets refer to the references cited at the end of the paper.

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(1.1) If \( g_n \in G, \ n = 1, 2, \ldots, \) and \( g_n \to e_R, \) then \( g_n^{-1} \to e_L; \) and if \( g_n \to e_L \) then \( g_n^{-1} \to e_R. \)

(1.2) If \( K \) is a compact subset of \( G \) and \( g_n \to e_R \) (resp. \( e_L \)), then \( g_nK \to e_R \) (resp. \( e_L \)).

In virtue of the known structure of locally compact abelian groups we may, in any locally compact group, distinguish between two classes of elements [2, Lemma 2, p. 96]. One of these is:

(1.3) The class which we shall denote by \( C \) consisting of those element of \( G \) which are contained in compact subgroups of \( G. \) For any elements \( c \in C, \) and any neighborhood \( V \) of the identity of \( G, \) there exists an integer \( n \) such that \( c^n \in V. \)

Each element of \( G \) not in \( C \) generates a subgroup of \( G \) isomorphic to the group of integers. For such elements no sequence of distinct powers has any limit in \( G. \) Now in a two-ended \( G \) we may distinguish further:

(1.4) The class \( R \) of elements of \( G \) such that if \( r \in R \) then \( r^n \to e_R \) (in \( G^* \)), and

(1.5) The class \( L \) such that \( l \in L \) if \( l^n \in e_L \) (in \( G^* \)).

It is clear from (1.1) that \( R \) and \( L \) are each the set of inverses (in \( G \)) of the other, are mutually exclusive, and are homeomorphic.

It is another matter, of course, to assert that none of these classes \( L, C, \) and \( R \) is empty, or that they comprise all elements of \( G. \) This will be shown in the succeeding paragraphs. The assumption that \( G \) is two-ended is to be understood throughout; many of the lemmas are not valid in the more general case.

2. The key lemma.

Principal Lemma. If \( g_n \to e_R, \ g_n \in G, \) then for all but a finite number, \( g_n \in R. \)

Proof. From the definition of ends [1] it follows at once that there exists a closed compact set \( K \subseteq G, \) and open sets \( A \) and \( B \) of \( G^* \) such that

(2.1) \[ G^* = A \cup K \cup B, \]

\[ e_L \subseteq A, \ e_R \subseteq B, \ A \cap K = K \cap B = B \cap \overline{A} = A \cap \overline{B} = 0. \]

By (1.1) for almost all \( n, \ g_nK \subseteq B. \) Let \( g \) denote any \( g_n \) for which \( gK \subseteq B. \) Then \( \overline{A} \cap gK = 0. \)

Now let \( A_0 \) denote the component of \( G^* - K \) which contains \( e_L. \) Then \( A_0 \) is a maximally connected subset of \( A, \) \( A_0 \) is closed in \( A, \) and \( \overline{A_0} \cap K \) is not empty since \( G^* \) is connected [3, p. 16]. Let \( k \) denote some element of the set \( \overline{A_0} \cap K. \)
(2.2) \[ G^* = g(G^*) = gA \cup gK \cup gB, \]
and it is easily seen that \( gA_0 \) is the component of \( G^* - gK \) which contains \( e_L = g(e_L) \). Then, since \( gK \subset B, \ A_0 \cap gK = 0 \) and \( A_0 \subset gA_0 \).
Moreover, since \( gK \subset B, gK \subset A_0 \) and there is some neighborhood \( V \) of \( k \) such that \( gV \cap A_0 = 0 \). Since \( k \in A_0, V \cap A_0 \) is not empty. Choose some \( a \in V \cap A_0 \). There is a neighborhood \( W \) of the identity such that \( Wa \subset V \). Then \( gWa \subset gV \), and \( (gWa) \cap A_0 = 0 \). Thus we have shown that
\[ (2.3) \quad A_0 \subset gA_0 - g(Wa \cap A_0), \]
and if we denote \( g^{-1} \) by \( f \), we may write this as
\[ (2.4) \quad fA_0 \subset A - Wa \cap A_0. \]

Now, for every integer \( n \),
\[ (2.5) \quad f^nA_0 = f^{n-1}fA_0 \subset f^{n-1}A_0 \subset \cdots \subset fA_0 \subset A - Wa \cap A_0. \]
In consequence of this, for every \( n, f^nA \subset Wa, \) and therefore \( f^nA \subset W. \)
But now it follows from (1.3) that no sequence of powers of \( f \) can converge to any element of \( G \). On the other hand, since \( f^nA \subset A, \) no subsequence of the points \( f^na, n = 1, 2, \cdots, \) can converge to \( e_R \in B. \)
Because of (1.2), no subsequence of the set \( f^n, n = 1, 2, \cdots, \) can converge to \( e_R. \) Finally, then, \( f^n \to e_L, \) and in consequence of this \( g^n \to e_R, \) since \( g = f^{-1}. \) This proves the lemma.

**Corollary.** There exists an open set \( O^* \subset G^*, e_R \in O^*, \) such that if \( g \in O^* \cap G, \) then \( g \in R. \)

This follows from the lemma by the observation that if no such \( O^* \) existed one could construct a sequence \( g_n \to e_R \) such that \( g_n \in R, \) and this would contradict the lemma.

**Lemma.** If \( g \in G \) and for some positive integer \( k, g^k \in R, \) then \( g \in R. \)

Let \( Q \) denote an arbitrary open subset of \( G^* \) containing \( e_R. \) Since \( g^k \in R, \) the sequence \( g^n \to e_R, \) \( k \) fixed. Therefore for each \( m = 0, 1, 2, \cdots, k - 1, \) the sequences \( g^m = g^{n+k-m} \to e_R \) \((k \text{ and } m \text{ fixed})\) and there exists an integer \( N \) such that for all \( n \geq N \) and all \( m = 0, 1, \cdots, k - 1, \) \( g^{n+m} \in Q. \) This means, of course, that in the sequence \( g, g^2, g^3, \cdots, \) all but a finite number belong to \( Q. \) Since \( Q \) is an arbitrary neighborhood of \( e_R, \) this implies that \( g^n \to e_R \) and, consequently, \( g \in R. \)

**Corollary.** \( R \) is open, and \( R \cup e_R \) is open.

Let \( g \in R \subset G. \) Then for some integer \( k, g^k \in O^* \cap G, \) for the \( O^* \) of the
preceding corollary. The set \( O^* \cap G \) is open and contained in \( R \).
Then there is a neighborhood \( V \) of \( g \) such that \( V^* \subseteq O^* \cap G \).
Now for any \( f \in V \), \( f^* \in O^* \cap G \cap R \) and consequently \( f \in R \).
This establishes that \( R \) is open and, since \( O^* \subseteq R \cup e_R \), it is clear that \( R \cup e_R \) is open.

3. **Another lemma.** We have proved now that \( R \) is open and not empty.
By symmetrical arguments, or by an application of (1.1), the set \( L \) is not empty and open.
Moreover, in view of the preceding corollary, \( e_L \cup L \cup R \cup e_R \) is an open set whose complement clearly is a closed and compact subset of \( G \).
Further, if \( c \) is an element of this complement, then \( c^* \in L \cup R \) for every integer \( n \), by a preceding corollary.
Then \( c \in C \) as defined in (1.3).
Since \( R \cap L = 0 \), and since \( G \) is connected, it follows that

\[
G = L \cup C \cup R,
\]
where \( C \) is closed, compact, not empty. We observe, from the form of (3.1), that \( C \) separates \( G^* \) between \( e_L \) and \( e_R \); that is, there is no connected subset of \( G^* - C \) which contains both \( e_L \) and \( e_R \).

**Lemma.** The identity \( e \) of \( G \) is a limit of element of \( R \).

We know that \( C \cap \overline{R} \) is not empty, since \( G \) is connected.
Let \( c \in C \cap \overline{R} \), and let \( V \) be an arbitrary neighborhood of the identity.
There exists an integer \( n \) such that \( c^n \in V \), by (1.3), and there must exist a neighborhood \( W \) of \( c \) such that \( W^n \subseteq V \).
This implies that there are in \( V \) elements of the form \( r^n, r \in W \cap R \).
For such \( r, r^n \in R \).
Therefore \( R \cap V \) is not empty, and since \( V \) is an arbitrary neighborhood of the identity we have established the lemma.

4. **One-parameter subgroups.** It is our next task to construct in \( G \) a one-parameter group \( T \) which is closed in \( G \).
Now if \( T \) denotes any one-parameter subgroup of any locally compact group \( G \), then by a result of which we have already taken partial advantage [2, p. 96] either \( T \) belongs to some compact subgroup of \( G \), and in that case \( T \subseteq C \), or \( T \) is a closed subgroup of \( G \).
Therefore, to construct in \( G \) a subgroup isomorphic to the additive group of reals it suffices to construct a one-parameter group which contains an element not in \( C \).

**Lemma.** \( G \) contains a connected abelian group \( A \) which contains an element of \( R \).

The group \( A \) will be constructed with the aid of the preceding lemma.
Let \( r_n \to e, r_n \in R \).
Denote by \( Q \) an open compact subset of \( G \) such that \( C \subseteq Q \), and denote by \( A_n \) the group generated by \( r_n \).
This
group is isomorphic to the additive group of integers. Then $A_n$ will have elements in $\mathbb{Q}$ and also in the complement of $\overline{\mathbb{Q}}$, and it is easy to see that there is a subsequence $A_{n_i}$ of the sequence $A_n$ such that some point of $\overline{\mathbb{Q}} - \mathbb{Q}$ is in the sequential limit set of the sequence of sets $A_{n_i}$. This sequential limit set is the desired group $A$. We have merely sketched the argument because it occurs in substantially the form needed here in [4, Lemma 5, p. 112].

**Corollary.** $G$ contains a subgroup $T$ isomorphic to the additive group of reals.

The closure of the group $A$ of the preceding lemma is a locally compact connected abelian group and is therefore the direct product of a compact subgroup by an $n$-dimensional vector group [5]. Since $\mathbb{A}$ cannot be compact, containing an element of $\mathbb{R}$, it follows that the vector group is not the identity and must contain a subgroup $T$ of the desired property. This is a closed subgroup of $G$, as well as of $\mathbb{A}$.

5. **Conclusion of the proof of Theorem A.** We are nearing the end of our task. We know that $G$ contains at least one subgroup isomorphic to the reals. Suppose that $T$ is any such closed one-parameter subgroup. Then the set $e_L \cup T \cup e_R$ is a closed subset of $G^*$ homeomorphic to a simple arc with end points $e_R$ and $e_L$. Moreover, this is also true of the set $e_L \cup gT \cup e_R$ for every $g \in G$. Then it follows from the remark following (3.1) that every coset $gT$ contains at least one point of $C$. Let us now consider the coset-space, for definiteness the right coset space $G/T$. Since $T$ is closed in $G$, $G/T$ may be topologized and is a locally compact topological space [5]. It may assist the reader to remark that an essential feature of this topology is the fact that if $g_n \to g$, where $g, g_n \in G$, and if the sequence $g_n t_n$, $t_n \in T$, has any limit point in $G$ then this limit point is of the form $gt$ for some $t \in T$.

**Lemma.** The coset-space $G/T$ is compact and homeomorphic to a space $C^*$ which is a continuous image of $C$.

Let us denote by $C_g$ the set $C \cap gT$, for $g \in C$. This is a closed set. Suppose now that $g_n \to g$, $g_n \in C$, and consider the associated sets $C_n = C \cap g_n T$. Let $k_n \in C_n$. Then $k_n = g_n t_n$, $t_n \in T$, $g_n \in C$. Since $C$ is compact, and $k_n \in C$, it is an easy consequence that the set of $t_n$, $n = 1, 2, \cdots$, is compact. From this it is clear that any limit point of the set $k_n$, $n = 1, 2, \cdots$, is of the form $gt \in C_g$, for some $t \in T$. Consequently the sets $C_g$, $g \in C$, give rise to an upper semi-continuous decomposition [3] of $C$, and there is a compact space $C^*$ which is
a continuous image of $C$, $\phi(C) = C^*$. For each $c^* \in C^*$, the set $\phi^{-1}(c^*)$ is some decomposition set $C_g = C \cap gT$.

Now it is clear that each point of $C^*$ is associated uniquely with a coset $gT$ and therefore with a point of the coset-space $G/T$. Since each coset intersects $C$, this correspondence extends over all of $G/T$ and all of $C^*$. From the definition, above, of $C^*$ and the topology of $G/T$ it follows easily that this correspondence is a homeomorphism. This establishes the major part of Theorem A.

The fact that the coset-space $G/T$ is homeomorphic to a compact set $C^*$ is not alone sufficient to insure that $G$ may be expressed as the topological product of $C^*$ and $T$. This would imply, and is implied by, the existence in $G$ of a closed set $\bar{C}$ meeting each coset $gT$ in one and only one point. But the fact that $T$ is a closed one-parameter group is enough to insure the existence of such a "cross-section." This was proved by Montgomery and the author [6] in a somewhat more general context, and depends on certain local "sections" of regular families of curves constructed by Whitney. We may conclude, then, that $G$ is representable as the direct product of two closed subsets, the set $T$ and a cross-sectioning set $\bar{C}$ which is of necessity homeomorphic to $C^*$. Since $G$ is connected, it is clear that $\bar{C}$ and $C^*$ are connected as well as compact. This concludes the proof of Theorem A.

6. An application. As an application of Theorem A, let us suppose that a two-ended group $G$ possesses a connected subgroup $H$ which is not compact. Then $H$ is locally compact and connected and also two-ended. Therefore, $H$ contains a subgroup $T$ isomorphic to the group of reals. Since $T$ is in $G$, it is evident from the preceding section that $G/T$ is compact. All the more, then, $G/H$ is compact.

It is an immediate consequence of this remark that if a locally compact connected group $G$ contains a non-compact connected subgroup $H$ such that the coset space $G/H$ is not compact, then $G$ cannot be a two-ended group. Therefore, by Freudenthal's theorem which forms the point of departure for this note, $G$ can have only a single "end." This sufficient condition for "one-endedness" generalizes a result due to Freudenthal, appearing in an appendix to his work [7] on the "ends" of discrete-spaces.

REFERENCES


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