ON THE NUMBER OF PRIMES LESS THAN OR EQUAL x

HAROLD N. SHAPIRO

In the beginnings of the theory of the distribution of prime numbers the unique factorization of $x!$ is found to be

$$x! = \prod_{p \leqq x} p^{\alpha_p} \tag{1}$$

where

$$\alpha_p = \left[\frac{x}{p}\right] + \left[\frac{x}{p^2}\right] + \cdots .$$

Taking the logarithm of both sides of (1) and using the very elementary result

$$\log x! = x \log x + O(x) \tag{2}$$

there results (see [1, pp. 72, 98])[1]

$$\sum_{p \leqq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x). \tag{3}$$

Since $[x/p] = [[x]/p]$, it is clear that (3) then holds for all real $x > 0$.

We propose to show in this note that the order of $\pi(x)$ = the number of primes less than or equal x (a result originally due to Tschebyschef, cf. [1]) may be derived very quickly from (3) as a consequence of a general theorem which has no particular relationship to prime numbers. This is of some further interest in that Landau, in [1, p. 82], says effectively that (3) does not suffice to give Tschebyschef's theorem. Our general theorem, which is in the nature of a weak Tauberian theorem, is as follows.

THEOREM. *Let $a_n \geqq 0$ be a sequence of real numbers such that*

$$A(x) = \sum_{n \leqq x} \left[\frac{x}{n}\right] a_n = x \log x + O(x). \tag{4}$$

Then for $S(x) = \sum_{n \leqq x} a_n$ there exist two positive constants α, β, such that for all sufficiently large x

$$\beta x \geqq S(x) \geqq \alpha x. \tag{5}$$

PROOF. Since $[z] - 2[z/2] \geqq 0$ for all positive z, we have from (4)

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[1] Numbers in brackets refer to the reference cited at the end of the paper.

$$O(x) = A(x) - 2A\left(\frac{x}{2}\right) = \sum_{n \leqq x} \left\{\left[\frac{x}{n}\right] - 2\left[\frac{x}{2n}\right]\right\} a_n$$

$$\geqq \sum_{x/2 < n \leqq x} \left[\frac{x}{n}\right] a_n = \sum_{x/2 < n \leqq x} a_n = S(x) - S\left(\frac{x}{2}\right).$$

Thus for some positive K_1 we have $0 \leqq S(x) - S(x/2) < K_1 x$. Then $0 \leqq S(x/2^i) - S(x/2^{i+1}) < K_1 x/2^i$, from which we obtain via addition $S(x) < 2K_1 x$. Taking $\beta = 2K_1$ yields the upper bound in (5).

Having proved that $S(x) = O(x)$ we return to (4) and obtain

$$\sum_{n \leqq x} \left[\frac{x}{n}\right] a_n = \sum_{n \leqq x} a_n \left\{\frac{x}{n} + O(1)\right\} = x \sum_{n \leqq x} \frac{a_n}{n} + O(S(x))$$

$$= x \log x + O(x)$$

whence

$$T(x) = \sum_{n \leqq x} \frac{a_n}{n} = \log x + O(1). \tag{6}$$

Thus we can write $T(x) = \log x + R(x)$ where $|R(x)| \leqq K_2$ for all sufficiently large x. Choosing $\alpha = e^{-2K_2 - 1}$, we have

$$T(x) - T(\alpha x) = \sum_{\alpha x < n \leqq x} \frac{a_n}{n} = \log \frac{1}{\alpha} + R(x) - R(\alpha x).$$

From this we obtain

$$\frac{1}{\alpha x} S(x) \geqq \frac{1}{\alpha x} \sum_{\alpha x < n \leqq x} a_n \geqq \log \frac{1}{\alpha} - 2K_2 = 1$$

or

$$S(x) \geqq \alpha x,$$

as desired. This completes the proof of the theorem.

In order to apply the above theorem to (3) we take

$$a_n = \begin{cases} \log n & \text{if } n \text{ is a prime,} \\ 0 & \text{otherwise,} \end{cases}$$

so that (4) reduces to (3). Then the theorem gives that for $\theta(x) = \sum_{p \leqq x} \log p$, we have two positive constants α, β, such that

$$\beta x \geqq \theta(x) \geqq \alpha x.$$

From the obvious inequalities

$$\{\pi(x) - \pi(x^{1/2})\} \log x^{1/2} \leqq \theta(x) \leqq \pi(x) \log x,$$

Tschebyscheff's theorem

$$\alpha \frac{x}{\log x} \leqq \pi(x) \leqq \gamma \frac{x}{\log x}$$

now follows at once.

Reference

1. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 1.

New York University

THE RADICAL OF A NON-ASSOCIATIVE RING

W. E. JENNER

In this paper a definition is proposed for the radical of a non-associative ring. Our results are somewhat similar to those given for algebras by Albert in [3],[1] but the difficulties that arose in the earlier theory from absolute divisors of zero have been overcome. With slight modifications, the present proofs are applicable to algebras.

A *non-associative ring* $\mathfrak{R}$ is an additive abelian group closed under a product operation with respect to which the two distributive laws hold. Multiplication on the right (left) by a fixed element $x \in \mathfrak{R}$ determines an endomorphism R_x (L_x) of $\mathfrak{R}$ as an additive group. For $x, y \in \mathfrak{R}$,

$$x \cdot y = xR_y = yL_x.$$

The R_x and L_y generate an associative ring $\mathfrak{A}$ called the *transformation ring* of $\mathfrak{R}$. Clearly $\mathfrak{R}$ can be construed as a representation space for $\mathfrak{A}$, and this representation is faithful. The two-sided ideals of $\mathfrak{R}$, which are defined as for associative rings, are exactly the $\mathfrak{A}$-subspaces of $\mathfrak{R}$. The theory of ring homomorphisms goes over intact to the non-associative case.

A nonzero element $a \in \mathfrak{R}$ is called an *absolute divisor of zero* if $a \cdot x = x \cdot a = 0$ for all $x \in \mathfrak{R}$. If $\mathfrak{A}$ has a unit element I and $\mathfrak{R}$ contains no absolute divisors of zero, then the unit element I is the identity

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[1] Numbers in brackets refer to the references cited at the end of the paper.