NOTE ON A RESULT OF LEVINE AND LIFSCITZ

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In a paper some years ago, [1], Levine and Lifschitz considered, among other questions, the relationship between the gaps of a Fourier series and its admissible integral orders of zeros, a problem first treated by S. Mandelbrojt [2]. Let \( f(t) \) be a function summable over the interval \((-\pi, \pi)\) and let

\[
(1) \quad f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt
\]

be its associated Fourier series. Let \( \psi(\alpha) \) be a continuous non-negative nondecreasing function defined for \( 0 \leq \alpha \leq 2\pi \). We shall say that \( f(t) \) has a left-hand zero of integral order \( \psi(\alpha) \) at \( t = \pi \) if

\[
(0 \leq \alpha \leq 2\pi).
\]

(One could equally well consider such a zero at any other point.) Natural choices for \( \psi(\alpha) \) are:

a. \( \psi(\alpha) = \alpha^n \) \( \quad (n = 1, 2, \ldots) \),

b. \( \psi(\alpha) = e^{-\frac{1}{\alpha^p}} \) \( \quad (0 < p < \infty) \),

c. \[
\psi(\alpha) \begin{cases} 
0 & (0 \leq \alpha \leq \alpha_0), \\
> 0 & (\alpha_0 < \alpha \leq 2\pi).
\end{cases}
\]

If \( \psi(\alpha) \neq 0 \), then there exists a number \( \alpha_0 \) such that \( \psi(\alpha) = 0 \) \( (0 \leq \alpha \leq \alpha_0) \) and \( \psi(\alpha) > 0 \) \( (\alpha_0 < \alpha \leq 2\pi) \). Let

\[
r = -\log \frac{\psi(\alpha)}{\alpha} \quad (\alpha_0 < \alpha \leq 2\pi).
\]

It is easily seen that for all \( r \) sufficiently large this equation may be inverted to give \( \alpha \) as a function \( \alpha = \eta(r) \) of \( r \). It is \( \eta(r) \) which we shall use as the measure of the zero of \( f(t) \). It should be noted that such extreme behavior as a. and c. may be permitted and that our theorems are significant throughout this entire range.

Let \( N(t) \) be the number of indices \( n < t \) whose coefficients \( a_n \) and \( b_n \) are not both equal to zero. The density of these indices may be characterized by the function

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1 Numbers in brackets refer to the references cited at the end of the paper.
\[ f(r) = \log r + 2r^2 \int_0^\infty \frac{N(t)}{t(t^2 + r^2)} \, dt. \]

This slight correction to the original formula of Levine and Lifschitz was pointed out by S. Mandelbrojt in his review of their paper, see [3].

Levine and Lifschitz proved that if
\[ \lim \inf_{r \to \infty} [3f(2r) - r\eta(r)] = -\infty, \]
then \( f(t) = 0 \) almost everywhere in \(( -\pi, \pi )\). In particular
\[ \lim \inf_{r \to \infty} [cf(r) - \eta(r)] = -\infty \]
implies \( f(x) = 0 \) almost everywhere if \( c = 12 \), since \( \xi(\alpha r) \leq \alpha^2 \xi(r), \alpha > 1, \) \( r \) sufficiently large. In the other direction Levine and Lifschitz proved that this does not hold for \( c < 1/2 \). It is the object of this note to show that it does hold for \( c = \pi \).

The method of proof is to construct the function
\[ Q(z) = \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{n_k^2} \right) \]
where \( n_k \) are the values of the indices of the nonzero coefficients in the series (1). \( Q(z) \) assumes its maximum modulus for a circle with center the origin on the imaginary axis and
\[ \log |Q(iy)| = \xi(y) \quad (y > 0). \]

The integral function
\[ F(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)e^{-it\alpha} \, dt \]
has zeros at those integers where \( Q(z) \) does not. Thus it is easily proved that the function
\[ \Phi(z) = \frac{F(z)Q(z)}{\sin \pi z} \]
is integral and satisfies the bound
\[ |\Phi(z)| < D\varepsilon^{(1+1)} \]
for all \( z \), and for \( y > 0 \) the bound
\[ |\Phi(z)| < D\varepsilon^{(1+1)-y\varepsilon(y)} \quad (z = x + iy = re^{\theta}). \]
For these results we refer to [1].
Suppose that \( \Phi(z) \neq 0 \); then \( z = 0 \) is a zero of some finite order \( m, 0 \leq m < \infty \). Let us regard the circle \( C: |z| = r \). On this circle

\[
\log | \Phi(z)/Dz^m | \leq \zeta(r) - m \log r \quad (\pi \leq \arg z \leq 2\pi)
\]

\[
\leq \zeta(r) - m \log r - r \sin \theta \eta(r \sin \theta) \quad (0 \leq \arg z \leq \pi)
\]

\[
\leq \zeta(r) - m \log r - \sin \theta \eta(r),
\]

since \( \eta(r) \) is a nonincreasing function of \( r \). Because the harmonic measure at the origin of an infinitesimal arc of length \( rd\theta \) on the circle \( C \) is \( (1/2\pi)d\theta \), we have, by an integrated form of the two constants theorem,

\[
\log | \Phi(z)/Dz^m |_{z=0} \leq \frac{1}{2\pi} \int_0^{2\pi} [\zeta(r) - m \log r] d\theta - \frac{1}{2\pi} \int_0^r r \eta(r) \sin \theta d\theta
\]

\[
\leq \frac{1}{\pi} \left[ \pi \zeta(r) - r \eta(r) - \pi m \log r \right].
\]

Because we have assumed

\[
\lim_{r \to \infty} \inf [\pi \zeta(r) - r \eta(r)] = -\infty,
\]

it follows that \( [\Phi(z)/z^m]_{z=0} = 0 \), a contradiction. Thus we must have \( \Phi(z) = 0 \). Consequently \( F(z) = 0 \) and \( f(t) = 0 \) almost everywhere in the interval \((-\pi, \pi)\).

\( f(t) \) is said to have a two-sided zero of integral order \( \psi(\alpha) \) at \( t = \pi \) if

\[
\phi(\alpha) = \int_{-\pi}^{\pi} |f(t)| dt + \int_{-\pi-\alpha}^{-\pi} |f(t)| dt \leq \psi(\alpha) \quad (0 \leq \alpha \leq \pi).
\]

In this case a simple adaptation of the above argument shows that the condition

\[
\lim_{r \to \infty} \inf \left[ \frac{\pi}{2} \zeta(r) - r \eta(r) \right] = -\infty
\]

is sufficient in order to have \( f(t) = 0 \) almost everywhere in \((-\pi, \pi)\).

If in the above conditions we replace \( \lim \inf \) by \( \lim \), more precise results may be obtained. Indeed if \( f(t) \) has a two-sided zero at \( t = \pi \), the condition

\[
\lim_{r \to \infty} [(1 + \epsilon) \zeta(r) - r \eta(r)] = -\infty \quad (\epsilon > 0)
\]

implies that \( f(t) = 0 \) almost everywhere in \((-\pi, \pi)\). This follows.
from the fact that on the lines $\text{arg } z = \pi/2 + \delta$, $\text{arg } z = -\pi/2 - \delta$, we have the bound

$$|\Phi(z)| \leq D\delta^{\prime}(|z|) - |z| \cos \delta \eta(|z| \cos \delta) \leq D\delta^{\prime}(|z|) - \cos \delta |z| \eta(|z|)$$

so that if $\cos \delta \geq (1 + \epsilon)^{-1}$, $\Phi(z)$ tends to 0 as we go to infinity on these lines. Then, since $\Phi(z)$ is at most of order one, by the Phragmén-Lindelöf Principle it is bounded in the enclosed angle and this follows similarly for the angle $-\pi/2 + \delta \leq \text{arg } z \leq \pi/2 - \delta$ and the two complementary angles. Thus $\Phi(z) = 0$ and the result is proved.

If $f(t)$ has a one-sided zero at $t = \pi$, the same argument applied to $\Phi(z)\Phi(-z)$ shows that the condition

$$\lim_{r \to \infty} [(2 + \epsilon)\xi(r) - r\eta(r)] = -\infty \quad (\epsilon > 0)$$

implies that $f(t) = 0$ almost everywhere in $(-\pi, \pi)$.

It is easy to give examples which provide lower bounds for the constant $c$ which may occur in the above result. Indeed, let $f_{i}(t)$ be identically zero from $-\pi + \delta$ to $\pi$ and arbitrary (but not zero) in the rest of $(-\pi, \pi)$. We see at once that $\xi(r)$ is asymptotically not greater than $\pi r$ while for the choice $\psi(\alpha) = \Phi(\alpha)$, $r\eta(r)$ is asymptotically equal to $(2\pi - \delta)r$. Thus the constant $c$ cannot be taken less than 2. Similarly the function $f_{2}(t)$ identically zero outside $(-\delta, \delta)$, arbitrary in this interval, provides an example showing that for a two-sided zero the constant $c$ cannot be taken as less than 1.

It should be mentioned that this does not, of course, supersede the result of Levine and Lifschitz since they showed that $c$ cannot be taken less than $1/2$ even when only very restricted classes of lacunary series are admitted.

REFERENCES


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