LINEAR FUNCTIONALS ON CERTAIN BANACH SPACES

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1. Introduction. The purpose of this note is to present what is believed to be a simple proof of a theorem on the representation of linear functionals on spaces L_p ($p > 1$) and some generalizations of such spaces. Instead of making use of the Radon-Nikodym theorem, the proof utilizes some simple consequences of uniform convexity, and applies at once to spaces L_p over arbitrary sets X on a family of whose subsets a measure function is defined, no decomposition of the space being involved. The proofs are stated for complex Banach spaces; the real case allows some obvious simplifications.

2. Linear functionals and derivatives of norms.

LEMMA 1. *Let g be an element of a complex Banach space B and L a linear continuous functional such that $L(g) = |L| \cdot \|g\|$. Then for each f in B we have*

$$(1) \qquad |L| \, D^-(\|g + tf\|)_{t=0} \leqq \mathcal{R}L(f) \leqq |L| \, D_+(\|g + tf\|)_{t=0}.$$

If $|L|$ or $\|g\|$ is 0, this is trivial. If neither of these is 0, we find that there is no loss of generality in assuming that $|L| = \|g\| = 1$; in fact, we shall apply the lemma only to cases in which $\|g\| = 1$. For all complex z and all f in B we have $L(g+z[f-L(f)g]) = L(g) = 1 = |L|$, so

$$(2) \qquad \|g + z[f - L(f)g]\| \geqq 1 \qquad \text{for all complex } z.$$

The equation

$$g + tf = (1 + tL(f))\{g + t[1 + tL(f)]^{-1}(f - L(f)g)\}$$

is an identity for $t \neq -1/L(f)$, and by (2) the norm of the vector in braces is never less than 1, which is its value at $t = 0$. Hence for all real t near 0 we have

$$\|g + tf\| - \|g\| \geqq |1 + tL(f)| - 1 \geqq t(\mathcal{R}L(f)).$$

Dividing by negative t and letting t approach zero yields the first of inequalities (1); with positive t we obtain the other inequality.

LEMMA 2. *Let g be a nonzero element of a complex Banach space B, and let L be a linear continuous functional on B such that $L(g) = |L| \cdot \|g\|$. Assume that for each f in B the function defined for all real t by*

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the expression $\|g+tf\|$ is differentiable at $t=0$. Then for all f the equation

$$(3) \qquad L(f) = |L| \frac{d}{dt} [\|g + tf\| + i\|g - itf\|]_{t=0}$$

is satisfied.

Under the present hypotheses, the conclusion of Lemma 1 becomes an equation. Writing this for f and for $-if$, we obtain

$$\mathfrak{R}L(f) = |L| \frac{d}{dt} \|g + tf\|_{t=0},$$

$$\mathfrak{I}L(f) = \mathfrak{R}L(-if) = |L| \frac{d}{dt} \|g - itf\|_{t=0},$$

establishing (3).

Next we recall [1, p. 396][1] that the space B is *uniformly convex* if to each positive ϵ there corresponds a positive $\delta(\epsilon)$ such that if $\|f\| = \|g\| = 1$ and $\|f-g\| \geq \epsilon$, then

$$(4) \qquad \|(f + g)/2\| \leq 1 - \delta(\epsilon).$$

LEMMA 3. *If B is uniformly convex, to each linear continuous functional L on B there corresponds a unit vector g_L such that*

$$|L| D^-\|g_L + tf\| \leq \mathfrak{R}L(f) \leq |L| D_+\|g_L + tf\|$$

for all f in B. In particular, if $\|g_L+tf\|$ is a differentiable function of t at $t=0$ for all f in B, the equation

$$(5) \qquad L(f) = |L| \frac{d}{dt} (\|g_L + tf\| + i\|g_L - itf\|)_{t=0}$$

holds for all f in B.

Choose a sequence of unit vectors $f_1, f_2, \cdots$ such that $L(f_n)$ tends to $|L|$. Let ϵ be positive; then so is $\delta(\epsilon)$, and for all large m and n we have

$$L([f_m + f_n]/2) = [L(f_m) + L(f_n)]/2 > |L| [1 - \delta(\epsilon)],$$

so that $\|(f_m+f_n)/2\| > 1 - \delta(\epsilon)$. By the definition of uniform convexity, this implies that $\|f_m - f_n\| < \epsilon$, and the sequence converges. Let g_L be its limit; this is a unit vector. Also

$$L(g_L) = |L| = |L| \cdot \|g_L\|,$$

[1] Numbers in brackets refer to the bibliography at the end of the paper.

so the hypotheses of Lemma 1 are satisfied, and the conclusion
follows from Lemmas 1 and 2.

3. **Spaces $L_p(B_1)$.** Now let m be a non-negative measure function
on a σ-field of subsets of a space X. A function f on X to a Banach
space B_1 is said to belong to the space $L_p(B_1)$ if $\|f(\cdot)\|_1^{p-1}f(\cdot)$ is
summable in the sense of Bochner over X, where $\|\cdot\|_1$ denotes the
norm in the space B_1. (If the reader is interested only in the usual
space L_p, he should interpret B_1 as the real or complex number sys-
tem, $\|\cdot\|_1$ as absolute value, and the integral as the Lebesgue integral.)
Assume $p>1$ and define as usual $p'=p/(p-1)$. It is then known that
if B_1 is uniformly convex (in particular, if it is the real or complex
number system), $L_p(B_1)$ is also uniformly convex [2, p. 504]; we
append another proof of this fact to the present note.

THEOREM I. *Let B_1 be a uniformly convex Banach space such that
for each nonzero g in B_1 and each f in B_1, the function $=\|g+tf\|_1$ is
differentiable at $t=0$. Let p be greater than 1, and let L be a functional
linear and continuous on $L_p(B_1)$. Then there exists a unit vector g_L
in $L_p(B_1)$ such that for every f in $L_p(B_1)$ we have*

$$
(6)\quad L(f) = |L| \int_X \|g_L(x)\|_1^{p-1} \frac{d}{dt}\,[\|g_L(x) + tf(x)\|_1
$$
$$
+ i\|g_L(x) - itf(x)\|_1]_{t=0}m(dx),
$$

*wherein the derivative in the integrand is to be assigned any arbitrary
finite value when the factor multiplying it has the value 0.*

Let g_L be the unit vector in $L_p(B_1)$ whose existence is guaranteed by
Lemma 3, and let f belong to $L_p(B_1)$. The equation

$$
\frac{d}{dt}\|g_L(x) + tf(x)\|_1^p = p\|g_L(x) + tf(x)\|_1^{p-1}\frac{d}{dt}\|g_L(x) + tf(x)\|_1
$$

holds whenever $g_L(x)+tf(x)\neq0$ by elementary differentiation, and
when that vector is 0 it continues to hold if the derivative in the
right member is assigned any finite value. For $|t|<1$ the right mem-
ber cannot exceed the function $p[\|g_L(x)\|_1+\|f(x)\|_1]^p$, which is sum-
mable. This permits us to differentiate under the integral sign in the
integral which defines $\|g_L+tf\|$; recalling that $\|g_L\|=1$, we obtain

$$
(7)\quad \frac{d}{dt}\|g_L + tf\|_{t=0} = \int_X \|g_L(x)\|_1^{p-1}\left[\frac{d}{dt}\|g_L(x) + tf(x)\|_1\right]_{t=0} m(dx).
$$

Thus the hypotheses of Lemma 2 are satisfied, and the conclusion of

Lemma 2 is equation (6).

As a special case, we obtain the representation of the general linear continuous functional on complex L_p.

THEOREM II. *Let L be a continuous linear functional on the complex Banach space L_p of functions on X, wherein $p > 1$. Then there exists an element g of $L_{p'}$, where $p' = p/(p-1)$, such that the norm of g in $L_{p'}$ is $|L|$ and*

$$(8) \qquad L(f) = \int_X g(x)f(x)m(dx)$$

for all f in L_p.

If a and b are complex numbers such that $a \neq 0$, from

$$|a + tb| = |a|\left[\,|1 + t(b/a)|\,\right]$$
$$= |a|\,[(1 + \Re tb/a)^2 + (\Im tb/a)^2]^{1/2}$$

we obtain

$$[d\,|a + tb|\,/dt]_{t=0} = |a|\,\Re(b/a) = \Re \bar{a}b/|a|,$$

whence

$$(9) \quad \begin{aligned} [d\,|a + tb|\,/dt + id\,|a - itb|\,/dt]_{t=0} \\ = \Re[\bar{a}b/|a|] + i\Im[\bar{a}b/|a|] = \bar{a}b/|a|. \end{aligned}$$

Thus the integrand in (6) takes the form

$$\|g_L(x)\|^{p-2}\overline{g_L(x)}f(x)$$

whenever $g_L(x) \neq 0$. If we now define

$$g(x) = |L| \cdot \|g_L(x)\|^{p-2}\overline{g_L(x)} \quad \text{where} \quad g_L(x) \neq 0,$$
$$g(x) = 0 \quad \text{wherever } g_L(x) = 0,$$

we see that (8) holds. Now $g(x)$ is everywhere a non-negative multiple of the conjugate of $g_L(x)$, and by the definition of p' we have

$$|g(x)|^{p'} = |L|^{p'}|g_L(x)|^p.$$

Hence g is in $L_{p'}$, and its norm in $L_{p'}$ is $|L|$, completing the proof.

THEOREM III. *Let μ be a measure function on a σ-field of subsets of a set Y, and let m be a measure function on a σ-field of subsets of a set X. Let L_q be the space of complex functions on Y such that $|f(y)|^{q-1}f(y)$ is summable over Y, and $L_p(L_q)$ the space of functions on X to L_q such that $\|f(x)\|_q^{p-1}f(x)$ is Bochner summable over X, the symbol $\|\cdot\|_q$ denoting*

the norm in L_q. Assume p and q greater than 1, and define $p' = p/(p-1)$, $q' = q/(q-1)$. Let L be a linear continuous functional on the complex Banach space $L_p(L_q)$. Then there exists an element g of $L_{p'}(L_{q'})$, with norm $|L|$ in that space, such that for every element

$$f = (f(x) \mid x \in X) = ((f(y, x) \mid y \in Y) \mid x \in X)$$

of the space $L_p(L_q)$ we have

(10) $$L(f) = \int_X \left\{ \int_Y g(y, x) f(y, x) \mu(dy) \right\} m(dx).$$

For L we first use equation (6), recalling that for each x the value of $g_L(x)$ is the function $(g_L(y, x) \mid y \in Y)$. For the derivative in the integrand of (6) we substitute from (7), applied to the space Y instead of X. The inner integrand will contain a derivative, which we replace by its value as computed in (9). The result is

$$L(f) = |L| \int_X \left\{ \|g_L(x)\|^{p-1} \int_Y |g_L(y, x)|^{q-2} \overline{g_L(y, x)} f(y, x) \mu dy \right\} m(dx),$$

where the inner integrand is to be understood to mean 0 whenever $g_L(y, x)$ vanishes. We define

$$g(y, x) = |L| \, \|g_L(x)\|^{p-1} |g_L(y, x)|^{q-2} \overline{g_L(y, x)}$$

wherever $g_L(y, x) \neq 0$, and set $g(y, x) = 0$ where $g_L(y, x) = 0$. Then (10) holds. The verification of the other statements about g is much the same as in the proof of Theorem II.

In Theorem II we extended the representation theorem for linear functionals on L_p to maximum generality as regards the space X. It is not possible to extend the theorem on the conjugate space of L_1 to a corresponding generality, as is shown by the following example, which is the result of a conversation with T. A. Botts and V. L. Klee. Let X consist of the points of the Euclidean plane. Let the exterior measure of a countable set E be 1 or 0 according as the origin is or is not in E; the exterior measure of every uncountable set is $+\infty$. This exterior measure generates a Carathéodory measure which coincides with itself, all sets being measurable. A function f is summable if and only if it is zero except on a countable set and $f(0)$ is finite, and in this case its integral is $f(0)$. All spaces L_p ($p \geq 1$) coincide, and the norm is $\|f(\cdot)\| = |f(0)|$, and all are conjugate to each other. But the space conjugate to L_1 is L_1, which is distinct from the space of essentially measurable functions on X.

4. Proof of uniform convexity of spaces $L_p(B)$. We now prove that

when $L_p(B)$ is defined as above, and B is uniformly convex and $p > 1$, the space $L_p(B)$ is also uniformly convex. For the norms in B and $L_p(B)$ we shall use the symbols $\|\cdot\|$, $\|\cdot\|_p$ respectively. As a first step, we show that to each positive ϵ corresponds a positive $\delta(\epsilon)$ such that if f and g are in B, and $1 = \|f\| \geq \|g\|$, and $\|f - g\| \geq \epsilon$, then

$$(11) \qquad \|(f + g)/2\|^p \leq [1 - \delta(\epsilon)][\|f\|^p + \|g\|^p]/2.$$

Assume this false; then there exist $\epsilon > 0$ and sequences f_n, g_n of elements of B such that $1 = \|f_n\| \geq \|g_n\|$, $\|f_n - g_n\| \geq \epsilon$, and

$$(12) \qquad \lim_{n \to \infty} \frac{\|(f_n + g_n)/2\|^p}{[\|f_n\|^p + \|g_n\|^p]/2} = 1.$$

From the convexity of t^p $(0 \leq t \leq 1)$ we obtain

$$(13) \qquad ([1 + t]/2)^p < (1 + t^p)/2 \qquad \text{for } 0 \leq t < 1$$

so by (12) and the triangle inequality

$$\lim_{n \to \infty} \frac{(1 + \|g_n\|)^p 2^{p-1}}{1 + \|g_n\|^p} = 1.$$

But by (13) this implies $\lim \|g_n\| = 1$. Define $u_n = g_n/\|g_n\|$; then $\lim \|u_n - g_n\| = 0$, so $\lim \inf \|u_n - f_n\| = \lim \inf \|g_n - f_n\| \geq \epsilon$, and by (12)

$$\lim \|(f_n + u_n)/2\| = 1.$$

This contradicts the uniform convexity of B.

As a corollary, if f and g are not both 0 we have

$$(14) \quad \begin{aligned} \|(f + g)/2\|^p \\ \leq [1 - \delta(\|f - g\|/\sup(\|f\|, \|g\|))][\|f\|^p + \|g\|^p]/2. \end{aligned}$$

Now assume that $f(\cdot)$ and $g(\cdot)$ are in $L_p(B)$, and that ϵ is positive, and that $\|f\| = \|g\| = 1$ and $\|f - g\|_p \geq \epsilon$. Let E be the subset of X on which

$$(15) \quad \begin{aligned} \|f(x) - g(x)\|^p &\geq (\epsilon^p/4)(\|f(x)\|^p + \|g(x)\|^p) \\ &\geq (\epsilon^p/4) \sup(\|f(x)\|^p, \|g^p(x)\|). \end{aligned}$$

On this set we have, by (14)

$$(16) \quad \|[f(x) + g(x)]/2\|^p \leq [1 - \delta(\epsilon/4^{1/p})][\|f(x)\|^p + \|g(x)\|^p]/2.$$

The integral of $\|f - g\|^p$ over $X - E$ is at most $\epsilon^p/2$, so

$$(17) \qquad \int_E \|f(x) - g(x)\|^p m(dx) \geq \epsilon^p/2.$$

Thus the functions which coincide with $f(\cdot)$ and $g(\cdot)$ on E and vanish on $X - E$ have distance at least $\epsilon/2^{1/p}$. So at least one of them has distance not less than $\epsilon/2 \cdot 2^{1/p}$ from the origin, and

$$(18) \quad \sup\left(\int_E \|f(x)\|^p m(dx), \quad \int_E \|g(x)\|^p m(dx) \right) \geq \epsilon^p/2^{p+1}.$$

Now by (16) and (18),

$$\int_X [(\|f(x)\|^p + \|g(x)\|^p)/2]m(dx) - \int_X [\|(f(x) + g(x)\|/2]^p m(dx)$$

$$\geq \int_E [(\|f(x)\|^p + \|g(x)\|^p)/2]m(dx) - \int_E [\|f(x) + g(x)\|/2]^p m(dx)$$

$$\geq \int_E \delta(\epsilon/4^{1/p})\{[\|f(x)\|^p + \|g(x)\|^p]/2\}m(dx)$$

$$\geq \delta(\epsilon/4^{1/p})(\epsilon^p/2^{p+2}),$$

whence

$$(19) \quad \|f(\cdot) + g(\cdot)\|_p \leq [1 - \delta(\epsilon/4^{1/p})(\epsilon^p/2^{p+2})]^{1/p}.$$

If we define $\delta_1(\epsilon)$ to be the difference between 1 and the right member of (19), we see that $\delta_1(\epsilon) > 0$, and $L_p(B)$ is uniformly convex.

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