Thus, with respect to the coordinate system \( w(\xi) \), the product function of \( G \) is of class \( C^* \) and hence \( G \) is a local Lie group.

**Proof of the Theorem.** When \( G \) has the discrete center, Lemma 5 shows that \( G \) is a local Lie group.

When \( G \) has the non-discrete center \( N \), by Lemma 4, \( N \) is an abelian Lie group, and by Lemma 5 again, \( G/N \) is a local Lie group. Then we can introduce a canonical coordinate of the second kind by \( x_1^{\lambda_1} \cdots x_m^{\lambda_m} \), where \( x_i^{\lambda_i} (i = 1, 2, \cdots, m) \) are one-parameter subgroups of \( G/N \). Take \( x_i \) of \( G \) from the coset \( x_i^{\lambda_i} \) for each \( i \), we can easily show that the set \( M = \{ y; y = x_1^{\lambda_1} \cdots x_m^{\lambda_m}, |\lambda_i| \leq 1 \} \) satisfies the condition (3) of Lemma 6. Consequently, by Lemma 6, \( G \) is a local Lie group. This completes our proof.

**References**


**Tokyo Institute of Technology**

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**NOTE ON A THEOREM OF KOKSMA**

WM. J. LEVEQUE

In 1935 Koksma [2] showed, among other things, that the sequence \( x, x^2, x^3, \cdots \) is uniformly distributed (mod 1) for almost all \( x > 1 \); that is, that if \( N(n, \alpha, \beta, x) \) denotes the number of elements \( x^i \) of the sequence \( x, x^2, \cdots, x^n \) for which

\[
0 \leq \alpha \leq x^i - [x^i] < \beta \leq 1,
\]

then

\[
\lim_{n \to \infty} \frac{N(n, \alpha, \beta, x)}{n} = \beta - \alpha
\]

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1 The author is indebted to Professor Mark Kac for his help in connection with this paper.

* Numbers in brackets refer to the bibliography at the end of the paper.
for almost all $x>1$. The purpose of this note is to provide another proof of this theorem, based on a lemma used in a recent paper [1] of Kac, Salem, and Zygmund on quasi-orthogonal functions. The assertion is contained in the more general theorem:

**Theorem 1.** Let $g(x, n)$ be any function of the real variable $x$ and the positive integral variable $n$ with the following properties in the interval $a<x<b$:

(i) $dg/dx, d^2g/dx^2$ exist,

(ii) $g'(x, j) - g'(x, k)$ is monotonic, and is different from zero for $j \neq k$,

(iii) for $x = a$ and $x = b$, the inequality

$$|g'(x, j) - g'(x, k)| \leq C |j - k|^\epsilon$$

is fulfilled for some $C > 0$, $0 < \epsilon \leq 1$.

Then the sequence $g(x, 1), g(x, 2), \cdots$ is uniformly distributed (mod 1) for almost all $x \in (a, b)$.

This theorem is weaker than Theorem 3 of Koksma's paper, but it is easily verified that the conditions of the theorem are satisfied for

$$g(x, n) = x^n, \quad 1 \leq a < b,$$

$$g(x, n) = n^tx, \quad \text{every } a, b \quad (t \text{ a positive integer}),$$

$$g(x, n) = n^x, \quad 1 \leq a < b,$$

$$g(x, n) = x^{M(n)}, \quad 1 \leq a < b,$$

where $M(n)$ is positive and such that $|M(j) - M(k)| \geq N$ for some $N > 0$ and all $j \neq k$. This last case is Theorem 2 of Koksma's paper.

We use the following specialization of Lemma 1 of [1]: Let $g(x, n)$ $(n = 1, 2, \cdots)$ be any sequence of real continuous functions in $(a, b)$, and let $m$ be an integer different from zero. Suppose that for all positive integers $j, k$ with $j \neq k$, we have

$$\left| \int_a^b e^{2\pi i g(x, j) - g(x, k)} dx \right| \leq \frac{C_1}{|j - k|^\epsilon},$$

for some $\epsilon$ with $0 < \epsilon \leq 1$. Then the series

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i g(x, n)}}{n^{1-\delta}}$$

converges almost everywhere in $(a, b)$ for every $\delta < \epsilon/2$.

(Lemma 1 of [1] is stated for real-valued functions; in the case of complex-valued functions the integrand should be replaced by $f_i(x)\overline{f_k(x)}$, where $\overline{f}$ is the complex conjugate of $f$.)
We shall show that for a function $g$ satisfying (i), (ii), and (iii) of the theorem, the conditions of the lemma are also satisfied for each $m$. The validity of the theorem then follows from Weyl's criterion [3, p. 91] for uniform distribution (mod 1) upon noting that the convergence of $\sum a_n/n$ implies that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n = 0.$$ 

Assume (i), (ii), and (iii). Then

$$\left| \int_{a}^{b} \exp \left\{ 2\pi im(g(x, j) - g(x, k)) \right\} dx \right|$$

$$= \left| \int_{a}^{b} \exp \left\{ 2\pi im(g(x, j) - g(x, k)) \right\} \cdot \frac{g'(x, j) - g'(x, k)}{g'(x, j) - g'(x, k)} \ dx \right|$$

$$\leq \frac{1}{2\pi} \left| \frac{1}{m} \left[ \exp \left\{ 2\pi im(g(x, j) - g(x, k)) \right\} \right]_{a}^{b} \right|$$

$$+ \left| \int_{a}^{b} \exp \left\{ 2\pi im(g(x, j)ight.$$ 

$$\left. - g(x, k)) \right\} \frac{d}{dx} (g'(x, j) - g'(x, k))^{-1} dx \right|$$

$$\leq \frac{1}{2\pi} \left| \frac{1}{m} \left[ \frac{1}{|g'(a, j) - (g'(a, k)|} + \frac{1}{|g'(b, j) - g'(b, k)|} \right.$$

$$\left. + \int_{a}^{b} \frac{d}{dx} (g'(x, j) - g'(x, k))^{-1} dx \right| \right.$$ 

By (ii), this is

$$\leq \frac{1}{2\pi} \left| \frac{1}{m} \left[ \frac{1}{|g'(a, j) - (g'(a, k)|} + \frac{1}{|g'(b, j) - g'(b, k)|} \right.$$

$$\left. + \int_{a}^{b} \frac{d}{dx} (g'(x, j) - g'(x, k))^{-1} dx \right| \right.$$ 

$$\leq \frac{1}{\pi} \left| \frac{1}{m} \left[ \frac{1}{|g'(a, j) - (g'(a, k)|} + \frac{1}{|g'(b, j) - g'(b, k)|} \right.$$

and this, by (iii), is

$$\leq \frac{2}{C\pi |m| \cdot |j - k|},$$
so that the assumptions of the lemma hold with $C_1 = 2/C \pi \vert m \vert$.

Actually, we have proved considerably more than is stated in Theorem 1, at least for the special functions $g(x, n)$ cited above. In each of these cases, the hypothesis of Theorem 1 holds with $\epsilon = 1$. It follows that for every integer $m \neq 0$, the series

$$\sum_{n=1}^\infty \frac{\epsilon^{2 \pi i m g(x, n)}}{n^{1/2+\theta}}, \quad \theta > 0,$$

converges for almost all $x \in (a, b)$. Using Abel's partial summation formula we deduce the following theorem.

**Theorem 2.** Under the assumptions of Theorem 1,

$$\sum_{n=1}^N \epsilon^{2 \pi i m g(x, n)} = o(N^{1/2+\theta}), \quad \theta > 0,$$

for every integer $m \neq 0$ and for almost all $x \in (a, b)$.

This is a much stronger statement than an assertion about uniform distribution. In view of its generality it is remarkably close to best possible, since it is known [4] that for $g(x, n) = xn^2$, this sum is not $o(N^{1/2})$ for any irrational $x$.

**Bibliography**


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Harvard University