Thus, with respect to the coordinate system \( w(t) \), the product function of \( G \) is of class \( C^1 \) and hence \( G \) is a local Lie group.

**Proof of the Theorem.** When \( G \) has the discrete center, Lemma 5 shows that \( G \) is a local Lie group.

When \( G \) has the non-discrete center \( N \), by Lemma 4, \( N \) is an abelian Lie group, and by Lemma 5 again, \( G/N \) is a local Lie group. Then we can introduce a canonical coordinate of the second kind by \( x_1^{s_1} \cdots x_m^{s_m} \), where \( x_i^{s_i} \) (\( i = 1, 2, \cdots, m \)) are one-parameter subgroups of \( G/N \). Take \( x_i \) of \( G \) from the coset \( x_i^* \) for each \( i \), we can easily show that the set \( M = \{ y; y = x_1^{\lambda_1} \cdots x_m^{\lambda_m}, |\lambda_i| \leq 1 \} \) satisfies the condition (3) of Lemma 6. Consequently, by Lemma 6, \( G \) is a local Lie group. This completes our proof.

**References**


**Tokyo Institute of Technology**

**NOTE ON A THEOREM OF KOKSMA**

WM. J. LEVEQUE

In 1935 Koksma [2]\(^2\) showed, among other things, that the sequence \( x, x^2, x^3, \cdots \) is uniformly distributed \((mod 1)\) for almost all \( x > 1 \); that is, that if \( N(n, \alpha, \beta, x) \) denotes the number of elements \( x^i \) of the sequence \( x, x^2, \cdots, x^n \) for which

\[
0 \leq \alpha \leq x^i - \lfloor x^i \rfloor < \beta \leq 1,
\]

then

\[
\lim_{n \to \infty} \frac{N(n, \alpha, \beta, x)}{n} = \beta - \alpha
\]

Presented to the Society, September 10, 1948, under the title *A metric theorem on uniform distribution\((mod 1)\)*; received by the editors January 24, 1949 and, in revised form, February 10, 1949.

\(^1\) The author is indebted to Professor Mark Kac for his help in connection with this paper.

\(^2\) Numbers in brackets refer to the bibliography at the end of the paper.
for almost all \( x > 1 \). The purpose of this note is to provide another proof of this theorem, based on a lemma used in a recent paper [1] of Kac, Salem, and Zygmund on quasi-orthogonal functions. The assertion is contained in the more general theorem:

**Theorem 1.** Let \( g(x, n) \) be any function of the real variable \( x \) and the positive integral variable \( n \) with the following properties in the interval \( a < x < b \):

(i) \( \frac{dg}{dx}, \frac{d^2g}{dx^2} \) exist,
(ii) \( g'(x, j) - g'(x, k) \) is monotonic, and is different from zero for \( j \neq k \),
(iii) for \( x = a \) and \( x = b \), the inequality

\[
|g'(x, j) - g'(x, k)| \leq C \cdot |j - k|^{\epsilon}
\]

is fulfilled for some \( C > 0 \), \( 0 < \epsilon \leq 1 \).

Then the sequence \( g(x, 1), g(x, 2), \ldots \) is uniformly distributed (mod 1) for almost all \( x \in (a, b) \).

This theorem is weaker than Theorem 3 of Koksma's paper, but it is easily verified that the conditions of the theorem are satisfied for

\[
g(x, n) = x^n, \quad 1 \leq a < b,
g(x, n) = n^t x, \quad \text{every } a, b \quad (t \text{ a positive integer}),
g(x, n) = n^x, \quad 1 \leq a < b,
g(x, n) = x^{M(n)}, \quad 1 \leq a < b,
\]

where \( M(n) \) is positive and such that \( |M(j) - M(k)| \geq N \) for some \( N > 0 \) and all \( j \neq k \). This last case is Theorem 2 of Koksma's paper.

We use the following specialization of Lemma 1 of [1]: Let \( g(x, n) \ (n = 1, 2, \ldots) \) be any sequence of real continuous functions in \( (a, b) \), and let \( m \) be an integer different from zero. Suppose that for all positive integers \( j, k \) with \( j \neq k \), we have

\[
\left| \int_a^b e^{2\pi i m(g(x,j) - g(x,k))} \, dx \right| \leq \frac{C_1}{|j - k|^{\epsilon}}
\]

for some \( \epsilon \) with \( 0 < \epsilon \leq 1 \). Then the series

\[
\sum_{n=1}^{\infty} \frac{e^{2\pi i m g(x,n)}}{n^{1-\delta}}
\]

converges almost everywhere in \( (a, b) \) for every \( \delta < \epsilon/2 \).

(Lemma 1 of [1] is stated for real-valued functions; in the case of complex-valued functions the integrand should be replaced by \( f_j(x) \bar{f}_k(x) \), where \( \bar{f} \) is the complex conjugate of \( f \).)
We shall show that for a function $g$ satisfying (i), (ii), and (iii) of the theorem, the conditions of the lemma are also satisfied for each $m$. The validity of the theorem then follows from Weyl’s criterion [3, p. 91] for uniform distribution (mod 1) upon noting that the convergence of \( \sum \frac{a_n}{n} \) implies that

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n = 0.
$$

Assume (i), (ii), and (iii). Then

$$
\left| \int_{a}^{b} \exp \{ 2\pi im(g(x, j) - g(x, k)) \} \, dx \right|
\leq \frac{1}{2\pi} \left( \frac{1}{m} \left| g'(a, j) - g'(a, k) \right| + \frac{1}{m} \left| g'(b, j) - g'(b, k) \right| \right)
$$

By (ii), this is

$$
\leq \frac{1}{2\pi} \left( \frac{1}{m} \left| g'(a, j) - g'(a, k) \right| + \frac{1}{m} \left| g'(b, j) - g'(b, k) \right| \right)
$$

and this, by (iii), is

$$
\leq \frac{2}{C\pi \, m \cdot |j - k|},
$$
so that the assumptions of the lemma hold with $C_1 = 2/C \pi \lvert m \rvert$.

Actually, we have proved considerably more than is stated in Theorem 1, at least for the special functions $g(x, n)$ cited above. In each of these cases, the hypothesis of Theorem 1 holds with $\epsilon = 1$. It follows that for every integer $m \neq 0$, the series

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i m g(x, n)}}{n^{1/2 + \theta}}, \quad \theta > 0,$$

converges for almost all $x \in (a, b)$. Using Abel's partial summation formula we deduce the following theorem.

**Theorem 2.** Under the assumptions of Theorem 1,

$$\sum_{n=1}^{N} e^{2\pi i m g(x, n)} = o(N^{1/2 + \theta}), \quad \theta > 0,$$

for every integer $m \neq 0$ and for almost all $x \in (a, b)$.

This is a much stronger statement than an assertion about uniform distribution. In view of its generality it is remarkably close to best possible, since it is known [4] that for $g(x, n) = xn^2$, this sum is not $o(N^{1/2})$ for any irrational $x$.

**Bibliography**

3. ———, *Diophantische Approximationen*, Ergebnisse der Mathematik vol. 4, no. 4, 1936.

**Harvard University**