ON MATRICES WHOSE CHARACTERISTIC EQUATIONS ARE IDENTICAL

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In a previous paper [1][1] it was shown that if A and C are matrices such that $ACA=0$ and B is an arbitrary matrix, then AB and $A(B+C)$ have the same characteristic equation. It is the purpose of this note to prove two theorems, each of which gives the above result as a special case.

THEOREM 1. *Let A be an $n\times m$ matrix of rank $r<n$ and let C be an $m\times n$ matrix such that $ACA=kA$ (k a scalar). If B is an $m\times n$ matrix, the characteristic equation of AB is $x^{n-r}\phi(x)=0$ and the characteristic equation of $A(B+C)$ is $x^{n-r}\phi(x-k)=0$.*

Let P and Q be nonsingular matrices such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad Q^{-1}BP^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \text{ and } Q^{-1}CP^{-1} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

where B_1 and C_1 are square matrices of order r. Then since $ACA=kA$,

$$PACAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} kI_r & 0 \\ 0 & 0 \end{pmatrix}$$

and hence $C_1=kI_r$.

Also

$$PACP^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} kI_r & C_2 \\ 0 & 0 \end{pmatrix}$$

and

$$PABP^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}$$

and hence

$$P(AB+AC)P^{-1} = \begin{pmatrix} B_1 + kI_r & B_2 + C_2 \\ 0 & 0 \end{pmatrix}.$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

It follows that the characteristic equation of AB is $x^{n-r}\phi(x)=0$ and the characteristic equation of $A(B+C)$ is $x^{n-r}\phi(x-k)=0$ where $\phi(x)=0$ is the characteristic equation of B_1. The result of the previous paper follows if $k=0$.

In case $k\neq 0$ it may as well be assumed that $k=1$ so that $ACA=A$ Set $U=AB$, $E=AC$ and $V=U+E$. Then $E^2=E$, $EU=U$, and $EV=V$. Hence $V^\bullet U=(U+1)^\bullet U$ and $U^\bullet V=(V-1)^\bullet V$ so that $f(V)U=f(U+1)U$ and $f(U)V=f(V-1)V$ for all polynomials $f(x)$. Let $g(x)$ be the minimum function of U and $h(x)$ the minimum function of V. Then $g(V-1)V=g(U)V=0$ and $h(U+1)U=h(V)U=0$ so that $h(x)\,|\,xg(x-1)$ and $g(x)\,|\,xh(x+1)$. Hence the minimum functions of U and V satisfy one of the four relations:

(1) $h(x) = g(x-1)$; (2) $(x-1)h(x) = xg(x-1)$;

(3) $(x-1)h(x) = g(x-1)$; (4) $h(x) = xg(x-1)$.

That all four relations are actually possible may be shown by examples.

THEOREM 2. *If M and N are square matrices such that NM (or MN) $=N^2=0$, then M and $M+N$ have the same characteristic equation.*

Since $N^2=0$, N is similar to

$$\begin{pmatrix} 0_r & I_r & 0 \\ 0_r & 0_r & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where r is the rank of N. There is no loss of generality in assuming that N is in this form. Then since $NM=0$

$$M = \begin{pmatrix} M_1 & M_2 & M_3 \\ 0_r & 0_r & 0 \\ M_4 & M_5 & M_6 \end{pmatrix}$$

where M_1 and M_2 are square matrices of order r. Any principal minor of M which contains elements of M_2 also contains a row of zeros from rows $r+1$ to $2r$. Hence the characteristic equation of M is independent of the elements of M_2. But M and $M+N$ differ only in the elements of M_2 and therefore their characteristic equations are identical. If $MN=N^2=0$, then $N'M'=N'^2=0$, so that M' and $M'+N'$ have the same characteristic equation. But the characteristic equation of any matrix is the same as that of its transposed matrix and the proof of the theorem is complete.

If $ACA=0$ then $(AC)(AB)=(AC)^2=0$ and hence AB and $A(B+C)$ have the same characteristic equation.

It is well known that if A and B are square matrices, AB and BA have the same characteristic equation and if either A or B is non-singular, the two products are similar. Roth [2] has pointed out that if both A and B are singular, the products may be similar or not.

With P and Q as defined in Theorem 1

$$PABP^{-1} = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} = M + N_1$$

where $N_1M=N_1^2=0$ and

$$Q^{-1}BAQ = \begin{pmatrix} B_1 & 0 \\ B_3 & 0 \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B_3 & 0 \end{pmatrix} = M + N_2$$

where $MN_2=N_2^2=0$.

Hence

$$AB = P^{-1}MP + P^{-1}N_1P = K + S$$

where $SA=S^2=0$ and

$$BA = QMQ^{-1} + QN_2Q^{-1} = L + T$$

where $AT=T^2=0$. Since $K=P^{-1}MP=AB-S$ and $L=QMQ^{-1}=BA-T$ the following theorem is established.

THEOREM 3. *If A and B are square matrices, there exist matrices S and T such that $SA=S^2=AT=T^2=0$ and such that $AB-S$ is similar to $BA-T$.*

BIBLIOGRAPHY

1. W. V. Parker, *On the characteristic equations of certain matrices*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 115–116.

2. W. E. Roth, *A theorem on matrices*, Amer. Math. Monthly vol. 44 (1937) p. 95.

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