

$+F_2^p(X, A)$. The reverse inclusion is clear and the equality shows that τ is onto.

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A NOTE ON A COMBINATORIAL PROBLEM

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The purpose of this note is to prove the following theorem.

THEOREM. *Let x_1, x_2, \dots, x_v denote v distinct elements and let T_1, T_2, \dots, T_v denote v sets formed from these elements. Suppose further that each T_i consists of exactly k distinct elements and that every pair of distinct sets T_i and T_j have exactly λ elements in common ($0 < \lambda < k < v$). Then*

$$\lambda = \frac{k(k-1)}{v-1}.$$

The combinatorial problem under consideration makes its appearance in the study of projective planes, Hadamard matrices, and block designs. In the combinatorial problem of Todd arising in the study of Hadamard matrices it is usually assumed that $\lambda = k(k-1)/(v-1)$ [3; 4].¹ For the symmetrical block designs each element of the arrangement is required to occur exactly k times, and it is then easy to verify that $\lambda = k(k-1)/(v-1)$ [1]. Further theorems concerning the possibility of this combinatorial problem for a given v , k , and λ may be found in [3].

To prove the theorem let the elements x_1, \dots, x_v be listed in a row, and let the sets T_1, \dots, T_v be listed in a column. Form the incidence matrix A of the arrangement in the usual way by inserting a one in column i and row j if x_i belongs to set T_j , and a zero in the contrary case [2; 3]. Now let s_i denote the sum of column i of the

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¹ Numbers in brackets refer to the references cited at the end of the paper.

matrix A . Let row r of A have ones in the k columns $r_1 < r_2 < \dots < r_k$, and let $t = \lambda(v-1) + k$. Then by the hypothesis of the theorem it follows that

$$s_{r_1} + s_{r_2} + \dots + s_{r_k} - t = 0.$$

The matrix of the above system of v equations in v unknowns is simply A . Moreover, it is clear that

$$s_1 + s_2 + \dots + s_v - kv = 0.$$

Now $t \neq 0$, and consequently the matrix

$$C = \begin{bmatrix} & & & 1 \\ & A & & \vdots \\ & & & 1 \\ 1 \dots 1 & & kv/t & \end{bmatrix}$$

has a zero determinant. Multiplying C by its transpose implies that

$$\begin{vmatrix} k+1 & \lambda+1 & \dots & \lambda+1 & r \\ \lambda+1 & k+1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \lambda+1 & & & k+1 & r \\ r & \dots & & r & s \end{vmatrix} = 0,$$

where $r = k + kv/s$ and $s = v + k^2v^2/t^2$. Multiplying row $v+1$ by $-r/s$ and adding to each of the other rows gives $s \cdot \det D = 0$, where the matrix D has $k+1 - r^2/s$ in the main diagonal and $\lambda+1 - r^2/s$ in all other positions. An evaluation of the determinant of the matrix D implies that

$$(k - \lambda)^{v-1} \{ k + 1 - r^2/s + (v-1)(\lambda + 1 - r^2/s) \} = 0,$$

and consequently

$$t + v - r^2v/s = 0.$$

Since

$$\frac{r^2v}{s} = \frac{k^2(t+v)^2}{t^2 + k^2v},$$

it is clear that

$$(t+v)(t^2 + k^2v) = k^2(t+v)^2.$$

Thus $t^2 + k^2v = k^2(t+v)$ and $t = k^2$.

COROLLARY. *Every element in the arrangement described in the preceding theorem occurs exactly k times and every pair of elements occurs exactly λ times.*

The corollary is an immediate consequence of the preceding theorem and Theorem 1 derived in [3].

A simple and direct alternative proof of the corollary may be obtained by noting that for $\lambda = k(k-1)/(v-1)$, the equations $s_1 + \dots + s_v = kv$ and $s_1(s_1-1) + \dots + s_v(s_v-1) = \lambda v(v-1)$ imply that $s_i = k$ (see [1]). The writer is indebted to Professor Marshall Hall, Jr., for this suggestion.

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