ON LACUNARY DIRICHLET SERIES

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The following theorem is suggested by a result of S. Mandelbrojt [2, p. 101] concerning lacunary Fourier series.

**Theorem 1.** Let \( f(s) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k s}, \) where \( 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \) \( \lim_{k \to \infty} \lambda_k = \infty. \) We denote by \( \gamma_c \) the abscissa of convergence of this series and by \( \gamma_a \) the abscissa of analyticity. It is assumed that \( \gamma_c < \infty, \gamma_a > -\infty. \) Let \( \nu < 1 \) be the exponent of convergence of the \( \lambda_k \)'s and suppose that
\[
|f(s_a + \sigma)| = O[e^{-(1/\nu)s}] \quad (\sigma \to 0+)
\]
where \( s_a = \gamma_a + i\tau_a. \) Then if \( \mu > \nu/(1-\nu), \) \( f(s) \equiv 0. \)

We shall in what follows prove a more general theorem including Theorem 1 as a special case. The methods of the present paper are closely related to and in part derived from the work of L. Schwartz [4]. However no appeal is made to other than standard theorems of analysis.

Let \( m(0) = 0 \) and let \( m(\sigma) \) be an increasing function defined for \( 0 \leq \sigma \leq a, a > 0. \) A function \( f(s) \) which we may suppose analytic in the half-plane \( \sigma > \gamma_1 \) is said to have a zero of modular order \( m(\sigma) \) at \( s_1 = \gamma_1 + i\tau_1 \) if
\[
|f(s_1 + \sigma)| \leq m(\sigma) \quad (0 < \sigma \leq b)
\]
for some \( b > 0. \) Let us define \( \sigma \) as a function of \( \rho, \sigma = \eta(\rho), \) by the equation \( e^{-\rho\sigma} = m(\sigma). \) Since \( m(\sigma) \) decreases as \( \sigma \) decreases to 0 this definition is effective. It is \( \eta(\rho) \) which we shall use as the measure of the zero of \( f(s). \) As an example, if \( m(\sigma) = \exp [-(\sigma^{-\nu})] \) then \( \eta(\rho) = \rho^{-1/(\mu+1)}. \)

As the measure of the degree of lacunarity of our Dirichlet series we introduce
\[
\xi(\rho) = \sum_{k=1}^{\infty} \log \left(1 + \frac{2\rho}{\lambda_k}\right).
\]
If the sequence \( \{\lambda_k\}_{1}^{\infty} \) has exponent of convergence \( \nu, \) then by a standard theorem on integral functions [5, p. 251],
\[
\xi(\rho) = O(\rho^{1+\epsilon}) \quad (\rho \to +\infty).
\]
We may now state our principal theorem.

**Theorem 2.** Let $0 < \lambda_1 < \lambda_2 < \cdots, \sum_i \lambda_i^{-1} < \infty$, and let $\xi(\rho)$ be defined as above. Let

$$f(s) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k s}$$

have abscissa of convergence $\gamma_c < \infty$ and abscissa of analyticity $\gamma_a > -\infty$, and let $\gamma_a + i\tau_a$ be a zero of $f(s)$ of modular order $m(\sigma)$. Then if $\eta(\rho)$ is defined as above and if

$$(1) \quad \lim_{\rho \to \infty} f(s) - \eta(\rho) = -\infty,$$

we have $f(s) \equiv 0$.

In the case considered in Theorem 1 we have, as we have seen, $\xi(\rho) = O(\rho^{1+\epsilon})$, $\eta(\rho) = \rho^{-1/(\mu+1)}$. The assumption $\mu > \nu/(1-\nu)$ shows that condition (1) is satisfied. Thus Theorem 2 does include Theorem 1.

It is clearly sufficient to prove our theorem for $\gamma_a = \tau_a = 0$.

It is immediately verifiable that the function

$$\frac{1}{(1 + z)} \prod_{k=1, k \neq n}^{\infty} \left( \frac{\lambda_k + \rho - z}{\lambda_k + \rho + z} \right) = k_n(z, \rho) \quad (\rho > 0)$$

is analytic for $x \geq 0$, and that

$$\int_{-\infty}^{\infty} \left| k_n(x + iy, \rho) \right|^2 dy \leq \pi \quad (x \geq 0).$$

By a simple extension of Plancherel’s theorem, see [3, p. 8], we see that if

$$\phi(n, \rho, \sigma) = \lim_{\sigma \to -\infty} \frac{1}{2\pi i} \int_{-\sigma}^{\sigma} \frac{k_n(z, \rho)}{k_n(\lambda_n + \rho, \rho)} e^{\gamma z} dz \quad (0 \leq \sigma < \infty),$$

then

$$(2) \quad \int_{0}^{\infty} \phi(n, \rho, \sigma) e^{-\gamma z} d\sigma = \frac{k_n(z, \rho)}{k_n(\lambda_n + \rho, \rho)} \quad (x > 0).$$

Further

$$\|\phi(n, \rho, \sigma)\|_2 \equiv \left[ \int_{0}^{\infty} \left| \phi(n, \rho, \sigma) \right|^2 d\sigma \right]^{1/2} \leq \left[ 2^{1/2} k_n(\lambda_n + \rho, \rho) \right]^{-1}.$$
We have

\[
\frac{1}{2^{1/2}} \prod_{k=1, k \neq n}^{\infty} \left( 1 + \frac{\lambda_n}{\lambda_k + 2\rho} \right) \left[ \frac{1 + \lambda_n + \rho}{\lambda_n + 2\rho} \right] \prod_{k=1}^{\infty} \left( 1 - \frac{\lambda_n}{\lambda_k} \right)^{-1} \sim \frac{\lambda_n}{2^{1/2}} \prod_{k=1, k \neq n}^{\infty} \left( 1 - \frac{\lambda_n}{\lambda_k} \right)^{-1} (\rho \to \infty),
\]

which proves our assertion.

We define

\[
F(\rho, s) = e^{-\rho s} f(s).
\]

We assert that if

\[
\limsup_{\rho \to \infty} \|F(\rho, \sigma)\|_{2e^{-\rho}} < \infty.
\]

Let \( M \) be a bound for \( f(\sigma) \) for \( 0 < \sigma < \infty \). We have

\[
\|F(\rho, \sigma)\| \leq \int_{0}^{\infty} e^{-2\rho \sigma} |f(\sigma)|^2 d\sigma \left[ \frac{1}{2^{1/2}} \prod_{k=1, k \neq n}^{\infty} \left( 1 + \frac{\lambda_n}{\lambda_k + 2\rho} \right) \left[ \frac{1 + \lambda_n + \rho}{\lambda_n + 2\rho} \right] \prod_{k=1}^{\infty} \left( 1 - \frac{\lambda_n}{\lambda_k} \right)^{-1} \right]^{1/2} \leq M e^{-\rho \lambda_n} + \int_{0}^{\infty} e^{-2\rho \sigma} |f(\sigma)|^2 d\sigma \leq (M + 1) e^{-\rho \lambda_n},
\]

which proves relation (4).

It follows from a well known theorem concerning restricted over-convergence of Dirichlet series, see [1, p. 141], that there exists an increasing sequence of integers \( l_k \) for which

\[
\lim_{k \to \infty} \sum_{j=0}^{l_k} a_j e^{-\lambda_j^\sigma} = f(\sigma)
\]
uniformly for $\epsilon \leq \sigma < \infty$ for any $\epsilon > 0$. Thus if $\epsilon > 0$,

$$\int_{0}^{\infty} \phi(n, \rho, \sigma) F(\sigma + \epsilon, \rho) d\sigma = \lim_{k \to \infty} \int_{0}^{\infty} \phi(n, \rho, \sigma) e^{-\sigma(\epsilon + 1)} \sum_{j=1}^{l_k} a_j e^{-\lambda_j(\sigma + 1)} d\sigma.$$ 

By equation (2)

$$\int_{0}^{\infty} \phi(n, \rho, \sigma) e^{-\sigma(\epsilon + 1)} a_j e^{-\lambda_j(\sigma + 1)} d\sigma = \begin{cases} 0, & j \neq n, \\ e^{-\sigma - \lambda_n} a_n, & j = n. \end{cases}$$

Hence

$$\int_{0}^{\infty} \phi(n, \rho, \sigma) F(\rho, \sigma + \epsilon) d\sigma = e^{-\sigma - \lambda_n} a_n.$$ 

By Schwarz's inequality

$$|a_n| \leq e^{\epsilon \rho + \lambda_n} \|\phi(n, \rho, \sigma)\|_2 \|F(\sigma + \epsilon, \rho)\|_2.$$ 

If $\epsilon$ is allowed to approach zero through positive values, we obtain

$$|a_n| \leq \|\phi(n, \rho, \sigma)\|_2 \|F(\sigma, \rho)\|_2.$$ 

Using equations (3) and (4) we see that

$$\log |a_n| \leq \zeta(\rho) - \rho \gamma(\rho) + C$$

where $C$ is a constant which depends on $n$ but not upon $\rho$. If we allow $\rho$ to increase without limit, assumption (1) of Theorem 2 implies that

$$\log |a_n| = -\infty,$$

that is, $a_n = 0$.

Since this holds for each $n = 1, 2, \ldots, f(s) = 0$ and our theorem is proved.

We shall now prove that if assumption (1) is replaced by the weaker assumption

$$\liminf_{\rho \to \infty} c\zeta(\rho) - \rho \gamma(\rho) = -\infty$$

where $c < 2^{-3/2}$, then Theorem 2 is false.

Let us take for the constants $\lambda_k, 0 < a < b < 1 < 4 < 9 < \cdots$. As in Titchmarsh [5, p. 271] we find that

$$\log \left[ \prod_{1}^{\infty} \left(1 + \frac{x}{k^2}\right) \right] \sim \pi x^{1/2} \quad (x \to \infty).$$
It follows that

\begin{equation}
\zeta(\rho) = \log \left( 1 + \frac{2\rho}{a} \right) + \log \left( 1 + \frac{2\rho}{b} \right) + \sum_{k=1}^{\infty} \log \left( 1 + \frac{2\rho}{k^2} \right) \sim \pi(2\rho)^{1/2} \quad (\rho \to \infty).
\end{equation}

We define

\begin{equation}
f(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{zs}}{(1 + \frac{z}{a})(1 + \frac{z}{b}) \prod_{k=1}^{\infty} \left( 1 + \frac{z}{k^2} \right)} \, ds.
\end{equation}

Deforming the line of integration of this integral to \( \text{Re} \, z = -(n+1/2)^2 \) and allowing \( n \) to approach infinity we obtain for \( s \) real and positive an expansion of \( f(s) \) into a Dirichlet series with exponents \( a, b, 1, 4, 9, \ldots \) which then converges in the half-plane \( \text{Re} \, s > 0 \). The details of this are left to the reader. Again deforming the line of integration, this time to \( \text{Re} \, z = (\pi/2\sigma)^2 \), and using equation (5), we see that if \( \epsilon > 0 \) is fixed, then for \( \sigma > 0 \) sufficiently small we have

\[
|f(s)| \leq \exp \left[ -\frac{1 - 2\epsilon}{4} \frac{\pi^2}{\sigma} \right] M(\sigma)
\]

where

\[
M(\sigma) = \frac{1}{2\pi} \int_{R1 z = (\pi/2\sigma)^2} \left| \frac{ds}{\left( 1 + \frac{z}{a} \right) \left( 1 + \frac{z}{b} \right)} \right| = o(1) \quad (\sigma \to 0 +).
\]

Thus we may associate with \( f(s) \) and the origin the modular order \( m(\sigma) = \exp \left[ -((1-2\epsilon)/4)(\pi^2/\sigma) \right] \). We immediately find that

\begin{equation}
\eta(\rho) = \frac{\pi}{2} (1 - 2\epsilon)^{1/2} \rho^{-1/2}.
\end{equation}

Comparing equations (6) and (8), we deduce that Theorem 2 is false if \( c < 2^{-1/2} \), which is what we wished to show.

References


2. S. Mandelbrojt, "Analytic functions and classes of infinitely differentiable func-
ON THE ABSOLUTE CONVERGENCE OF TRIGONOMETRICAL SERIES

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1. Theorem 1. Suppose that a trigonometrical series
\[ \sum \rho_n \cos (nx - \alpha_n) \quad (\rho_n \geq 0, \, n = 1, 2, \ldots) \]
and its conjugate series
\[ \sum \rho_n \sin (nx - \alpha_n) \]
are convergent absolutely at \( x = x_0 \) and \( x = x_1 \), respectively. If
\[ x_1 - x_0 = \frac{p\pi}{q} \quad (p/q \text{ irreducible}) \]
where \( p \) is an integer positive, negative, or zero, and \( q \) is an odd integer, then
\[ \sum \rho_n < \infty. \]

We shall see in Theorem 2 that the above theorem is no longer true if \( p/q \) is replaced by \( p'/q' \) with an even \( q' \) and \( p' \neq 0 \), or by an irrational number.

Proof of Theorem 1. If we put \( x_1 - x_0 = \frac{p\pi}{q} = h \), then by Fatou's theorem\(^1\)
\[ \sum \rho_n \left| \cos (n(x_1 - sh) - \alpha_n) \right| < \infty \]
for every odd \( s \). Hence from the identity
\[ \sin (nx_1 - \alpha_n) = \cos nsh \sin (n(x_1 - sh) - \alpha_n) \]
\[ + \sin nsh \cos (n(x_1 - sh) - \alpha_n) \]
we deduce immediately that
\[ \sum \rho_n \left| \cos nsh \sin (n(x_1 - sh)) - \alpha_n \right| < \infty. \]

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\(^1\) See, for example, A. Zygmund, Trigonometrical series, p. 134.