ON CHERN'S INVARIANT FOR RIEMANNIAN 4-MANIFOLDS

HANS SAMELSON

1. Introduction. S. S. Chern has defined, guided by geometric considerations, a differential geometric invariant for closed, orientable, 4-dimensional Riemannian manifolds [1]; the invariant is closely connected with the fact that the orthogonal group in 4 variables is essentially, that is, up to a finite covering, the direct product of two orthogonal groups in 3 variables. It is the purpose of this note to exhibit a space for which this invariant is not zero; at the same time we shall verify the well known generalized Gauss-Bonnet formula for the curvatura integra. The space is the complex projective plane $K^2$, in its elliptic metric; this metric is discussed in detail in [2], and we shall freely refer to this paper for the necessary formulae of Riemannian geometry in Cartan’s formulation; for the basic definitions of differential forms we refer to [3]. We make use of the well known fact that $K^2$ is the base space of a fiber decomposition of $S^5$ with $S^1$ as fiber (see [4] and [5]).

2. The space $K^2$. We now consider the complex plane $K^2$. As is well known, it is obtained from the 5-sphere $S^5$ by a fiber decomposition [4]: We represent $S^5$ as unit sphere $\sum |z_i|^2 = 1$ in a complex-euclidean 3-dimensional space $C^3$ with coordinates $(z_1, z_2, z_3)$. Two points $y$ and $z$ of $S^5$ are called equivalent, under the relation $0$, if there exists a real $\theta$ such that $y^* = z^* e^{i\theta}$; the equivalence classes, or fibers, are great circles, with $\theta$ as arc length.

One finds that the decomposition space $S^5/\Theta$ with the induced topology is a 4-dimensional manifold; it is called $K^2$, and with the canonical map $\phi: S^5 \to K^2$ the system $(S^5, S^1, K^2, \phi)$ is a fiber bundle [5] (with the rotations of $S^1$ as structure group). In addition there is a natural way of making $K^2$ into a differentiable (even analytic) manifold, such that $\phi$ is a differentiable (analytic) map: if $z^0 = (z_1^0, z_2^0, z_3^0)$ with, say, $z_1^0 \neq 0$, is a point of $S^5$, then the real and imaginary part of $z_2/z_1$ and $z_3/z_1$ serve as analytic coordinates in the neighborhood of $\phi(z^0)$. Now $S^5$ is a Riemannian manifold, as subspace of $C^3$, with fundamental form $\sum |dz_i|^2$. One verifies after some computation that for a vector $\xi = (dz_1, dz_2, dz_3)$, tangent to $S^5$ at $(z_1, z_2, z_3)$, the expression $\sum |dz_i|^2 - \sum \xi d\bar{z}_i|^2$ depends only on the image vector of $\xi$ under the map $\phi$, and that it represents a positive definite
(analytic) quadratic form on the tangent space to $K^2$; the Riemannian geometry determined by it is called the elliptic hermitian geometry of $K^2$; from now on we consider $K^2$ with this metric (see [2]). Any vector of $S^6$, which is perpendicular to the fiber through its point of support, is mapped by $\phi$ into a vector of the same length. Using this fact, and considering the volume forms $(g^{\mu
u}dx_1 \cdots dx_n)$ of the Riemannian manifolds $S^6$, $K^2$, and $S^1$, one can show that the volumes $v(S^6)$, $v(K^2)$, and $v(S^1)$ satisfy the relation $v(S^6) = v(K^2) \cdot v(S^1)$; since $v(S^1) = 2\pi$, and $v(S^6) = \pi^3$ [6, p. 303], we have

\[ (*) \quad v(K^2) = \pi^2/2. \]

3. The differential forms. Chern's invariant and the total curvature form are expressed in terms of Cartan's differential forms [1]. We first consider the complex curvature forms $\Omega_{ij}$ for $K^2$; we take the necessary formulae from [2]: there are two complex 1-forms $\omega_1$ and $\omega_2$, and the $\Omega_{ij}$'s are given by

\[ \Omega_{ij} = \omega_i \omega_j + \sum \omega_k \omega_k; \quad \Omega_{ij} = \omega_j \omega_i, \quad i \neq j. \]

We have to set $\omega_1 = \theta_1 + i\psi_1$, and $\omega_2 = \theta_2 + i\psi_2$.

The volume form $dv$ is then given by $\theta_1 \theta_2 \psi_1 \psi_2$, and we note, in consequence of (*), \S 2, that

\[ (**) \quad \int_{K^2} \theta_1 \theta_2 \psi_1 \psi_2 = \int_{K^2} dv = \frac{\pi^2}{2}. \]

Substituting into the $\Omega_{ij}$'s we have

\[ \Omega_{ij} = (\theta_i + i\psi_i)(\theta_i - i\psi_i) + \sum (\theta_k + i\psi_k)(\theta_k - i\psi_k) \]
\[ = 2i\psi_i \theta_i + 2i \sum \psi_i \theta_k, \]
\[ \Omega_{ij} = (\theta_j + i\psi_j)(\theta_i - i\psi_i) = \theta_j \theta_i + \psi_j \psi_i + i(\psi_j \theta_i - \theta_j \psi_i), \quad i \neq j. \]

Setting $\Omega_{ij} = \theta_{ij} + i\Psi_{ij}$, and denoting the real curvature forms by $\Omega_{ab}, a, b = 0, 1, 2, 3$, we obtain for the matrix $\Psi = (\Omega_{ab})$ by [2, (44)] the following:

\[ \Omega^r = \begin{pmatrix} \Theta & -\Psi \\ \Psi & \Theta \end{pmatrix} \quad \text{with} \quad \Theta = \begin{pmatrix} 0 & \theta_2 \theta_1 + \psi_2 \psi_1 \\ \theta_1 \theta_2 + \psi_1 \psi_2 & 0 \end{pmatrix} \]

and

\[ \Psi = \begin{pmatrix} 4\psi_1 \theta_1 + 2\psi_2 \theta_2 & \psi_1 \theta_2 - \theta_1 \psi_2 \\ \psi_1 \theta_2 - \theta_1 \psi_2 & 4\psi_2 \theta_2 + 2\psi_1 \theta_1 \end{pmatrix}. \]

With the quantities $\Omega_{ab}$ we now compute the Chern form $\Gamma$, and
the total curvature form $K$; they are given by $\Gamma = \sum_{a < b} [\Omega_{ab}]^2$ and

$$K = \Omega_{01} \Omega_{23} + \Omega_{02} \Omega_{31} + \Omega_{03} \Omega_{12}$$

(see [1, p. 970]). By a straightforward multiplication we obtain

$$\Gamma = -24 \theta_1 \theta_2 \psi_1 \psi_2 = -24 \psi_1$$

and

$$K = 24 \theta_1 \theta_2 \psi_1 \psi_2 = 24 \psi_1.$$

Incidentally this shows that $\Gamma$ really is a form of $K^2$ and not only of a fiber bundle over $K^2$.

We obtain now for the characteristic, using (**),

$$\chi(K) = -\frac{1}{4\pi^2} \int_{K^2} K = -\frac{6}{\pi^2} \int_{K^2} d\psi = -\frac{6}{\pi^2} \frac{\pi^2}{2} = 3,$$

which checks with well known facts, and for the Chern invariant

$$\eta = \frac{1}{2\pi^2} \int_{K^2} \Gamma = -\frac{12}{\pi^2} \int_{K^2} d\psi = -6,$$

which is different from 0; as predicted by the theory, the value of $\eta$ is integral. The two invariants $I_r$ and $I_1$ of [1] have the values 6 and $-18$ for $K^2$.

References


