ON DOUBLE FOURIER SERIES
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1. In the present note, the author proves some inequality theorems on double Fourier series. We begin with some preliminary lemmas. Let \( f(x, y) \) be an integrable function and let its Fourier series be

\[
f(x, y) \sim \sum_{m,n} A_{m,n}(x, y)
\]

and let

\[
\Delta_{m,n}(x, y) = \sum_{m=2^{r}}^{2^{r+1}-1} \sum_{n=2^{s}}^{2^{s+1}-1} A_{m,n}(x, y).
\]

**Lemma 1.** Let \( s_{r,m,n}(x, y) \) be the \( m, n \)th partial sum of Fourier series of \( f_{r}(x, y) \), then

\[
\int_{0}^{2\pi} \int_{0}^{2\pi} \left( \sum_{r}^{2} s_{r,m,n}(x, y) \right)^{\gamma/2} \, dx \, dy 
\]

\[
\leq A_{\gamma} \int_{0}^{2\pi} \int_{0}^{2\pi} \left( \sum_{r}^{2} f_{r}^{2} \right)^{\gamma/2} \, dx \, dy, \quad \gamma > 1.
\]

**Lemma 2.** If \( \gamma > 1 \),

\[
\int_{0}^{2\pi} \int_{0}^{2\pi} \left( \sum_{r}^{2} | \Delta_{m,n}(x, y) |^{2} \right)^{\gamma/2} \, dx \, dy 
\]

\[
\leq B_{\gamma} \int_{0}^{2\pi} \int_{0}^{2\pi} | f(x, y) |^{\gamma} \, dx \, dy 
\]

\[
\leq B'_{\gamma} \int_{0}^{2\pi} \int_{0}^{2\pi} \left( \sum_{r}^{2} | \Delta_{m,n}(x, y) |^{2} \right)^{\gamma/2} \, dx \, dy.
\]

These lemmas are due to J. Marcinkiewicz [2].

2. **Theorem 1.** If \( \gamma > 1 \),

\[
\int_{0}^{2\pi} \int_{0}^{2\pi} \left( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} | s_{m,n}(x, y) - \sigma_{m,n}(x, y) |^{2} \right)^{\gamma/2} \, dx \, dy 
\]

\[
\leq C_{\gamma} \int_{0}^{2\pi} \int_{0}^{2\pi} | f(x, y) |^{\gamma} \, dx \, dy,
\]

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1 Numbers in brackets refer to the references cited at the end of the paper.

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and
\[
\int_0^{2\pi} \int_0^{2\pi} \left| \max_{m,n} s_{2^m,2^n}(x, y) \right|^\gamma dx dy \leq D\gamma \int_0^{2\pi} \int_0^{2\pi} |f(x, y)|^\gamma dx dy.
\]
Accordingly \( s_{2^m,2^n}(x, y) \) converges to \( f(x, y) \) almost everywhere, when \( f(x, y) \in L^\gamma (\gamma > 1) \).

By Schwarz's inequality
\[
\sum_{m} \sum_{n} \left| s_{2^m,2^n} - \sigma_{2^m,2^n} \right|^2 \\
\leq \sum_{m} \sum_{n} \frac{1}{2^{2m}.2^{2n}} \left( \sum_{m=0}^{m} \sum_{n=0}^{n} \left| s_{2^m,2^n} - s_{\mu,\nu} \right|^2 \right) \\
\leq \sum_{m} \sum_{n} \sum_{i=0}^{i} \frac{1}{2^{2m}.2^{2n}} \left( \sum_{m=0}^{m} \sum_{n=0}^{n} \left| s_{2^m,2^n} - s_{\mu,\nu} \right|^2 \right) \\
\leq \sum_{m} \sum_{n} \sum_{i=0}^{i} \frac{1}{2^{2m}.2^{2n}} \left\{ \sum_{m=0}^{m} \sum_{n=0}^{n} \left( \sum_{m=0}^{m} \sum_{n=0}^{n} \left| s_{2^m,2^n} - s_{\mu,\nu} \right|^2 \right) \right\}.
\]

In view of Lemma 1, we get
\[
\int_0^{2\pi} \int_0^{2\pi} \left( \sum_{m} \sum_{n} \left| s_{2^m,2^n} - \sigma_{2^m,2^n} \right|^2 \right)^{\gamma/2} \, dx \, dy \\
\leq E\gamma \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{m} \sum_{n} \frac{1}{2^{2m}.2^{2n}} \left\{ \sum_{i=0}^{i} \sum_{j=0}^{j} \left( 2^{i+1} \left| s_{2^m,2^n} - s_{2^{i+1},2^{j+1}} \right|^2 \right) \right\} \right]^{\gamma/2} \, dx \, dy.
\]
Since \( |s_{2^m,2^n} - s_{2^{i+1},2^{j+1}}| \leq \sum_{k=1}^{m} \sum_{i=1}^{n} |\Delta_{k,i}| \), we have
\[
\sum_{m} \sum_{n} \frac{1}{2^{2m}.2^{2n}} \sum_{i=0}^{i} \sum_{j=0}^{j} 2^{i+1} \left| s_{2^m,2^n} - s_{2^{i+1},2^{j+1}} \right|^2 \\
\leq \sum_{m} \sum_{n} \sum_{i=0}^{i} \sum_{j=0}^{j} 2^{i+1} \left[ \sum_{k=1}^{m} \sum_{l=1}^{n} \left| \Delta_{k,l} \right| (2^{k+2l})^{1/4} (2^{k+2l})^{-1/4} \right] \\
\leq \sum_{m} \sum_{n} \sum_{i=0}^{i} \sum_{j=0}^{j} 2^{i+1} \left\{ \sum_{k=1}^{m} \sum_{l=1}^{n} \left| \Delta_{k,l} \right| (2^{k+2l})^{1/4} (2^{k+2l})^{-1/4} \right\} \\
\leq \left\{ \sum_{k=1}^{m} \sum_{l=1}^{n} (2^{k+2l})^{-1/2} \right\} \\
\cdot \left\{ \sum_{k=1}^{m} \sum_{l=1}^{n} (2^{k+2l})^{-1/2} \right\}.
\]
By using Lemma 2, we get the first half of the theorem. The remaining part is an easy consequence of the former.

3. Lemma 3. If $k > 1$ and $\gamma > 1$, we get

$$
\int_0^{2\pi} \int_0^{2\pi} \left( \sum_{r} |s_{r, m, n}|^k \right)^{\gamma/k} \, dx \, dy
\leq E_\gamma \int_0^{2\pi} \int_0^{2\pi} \left( \sum_{r} |f_r|^k \right)^{\gamma/k} \, dx \, dy.
$$

The proof is carried out analogously to Marcinkiewicz's proof of Lemma 1, using the result of Boas and Bochner [1].

Theorem 2. If $k \geq 2$ and $r > 1$,

$$
\int_0^{2\pi} \int_0^{2\pi} \left( \sum_{m} \sum_{n} \left| \frac{s_{m, n} - \sigma_{m, n}}{mn} \right|^k \right)^{\gamma/k} \, dx \, dy
\leq F_\gamma \int_0^{2\pi} \int_0^{2\pi} \left| f(x, y) \right|^\gamma \, dx \, dy.
$$

Accordingly if $f(x, y) \in L^\gamma$ ($\gamma > 1$),

$$
\left( \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \left| s_{\mu, \nu} - f |^k \right) = o(mn) \quad (k \geq 1)
$$

almost everywhere.

The proof is analogous to my previous paper [4], using Lemma 3. The last part of the theorem was obtained by Marcinkiewicz [3] in more generalized form.
LEMMA 4. If \( r > 1 \), then
\[
\int_0^{2\pi} \int_0^{2\pi} \left( \sum_{\mu} \sum_{\nu} |\Delta_{\mu, \nu}|^2 \right)^{\gamma/2} dx dy \\
\leq \int_0^{2\pi} \int_0^{2\pi} \left( \sum_{m} \sum_{n} \frac{|\sigma_{m,n} - \sigma_{m,n}|^2}{mn} \right)^{\gamma/2} dx dy.
\]
The proof is easy (cf. the author's note [5]).

THEOREM 3. If \( \gamma > 1 \),
\[
\int_0^{2\pi} \int_0^{2\pi} |f(x, y)|^\gamma dx dy \\
\leq G_{\gamma} \int_0^{2\pi} \int_0^{2\pi} \left( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |s_{2m,2n} - \sigma_{2m,2n}|^2 \right)^{\gamma/2} dx dy
\]
and
\[
\int_0^{2\pi} \int_0^{2\pi} |f(x, y)|^\gamma dx dy \\
\leq H_{\gamma} \int_0^{2\pi} \int_0^{2\pi} \left( \sum_{m} \sum_{n} \frac{|\sigma_{m,n} - \sigma_{m,n}|^2}{mn} \right)^{\gamma/2} dx dy.
\]
The proof is immediate by Lemma 4 and Lemma 2.

BIBLIOGRAPHY


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