Several proofs have been given of the results of M. Riesz and Zygmund.

(a) The conjugate of the Fourier series of a function \( f(x) \) of \( L^p, p > 1 \), is Fourier series of a function \( \tilde{f}(x) \) of the same class, and

\[
\int_0^{2\pi} |\tilde{f}(x)|^p dx \leq A_p \int_0^{2\pi} |f(x)|^p dx
\]

holds, \( A_p \) is a constant depending only on \( p \).

(b) If the function \( |f(x)| \log^+ |f(x)| \) is integrable, the conjugate of the Fourier series of \( f(x) \) is the Fourier series of a function \( \tilde{f}(x) \) of the class \( L \). Moreover, there exist two constants \( A \) and \( B \) such that

\[
\int_0^{2\pi} |\tilde{f}(x)| \, dx \leq A \int_0^{2\pi} |f(x)| \log^+ |f(x)| \, dx + B.
\]

In view of the importance of these theorems it may be of interest to give another proof of them based on a different idea. Actually it is
easy to see (and very well known) that they are equivalent to the following proposition about analytic functions.

Let \( F(z) = u(z) + iv(z) \), \( u(z) > 0 \), \( v(0) = 0 \) be a function regular inside the circle \( |z| < 1 \). Then the inequalities

\[
\begin{align*}
(1) & \quad \int_0^{2\pi} |v(re^{iz})|^p dx \leq A_p \int_0^{2\pi} |u(re^{iz})|^p dx; \quad 1 < p \leq 2; \quad r < 1, \\
(2) & \quad \int_0^{2\pi} |v(re^{iz})| dx \leq C \int_0^{2\pi} |u(re^{iz})| dx \\
& \quad \quad + D \int_0^{2\pi} u \log u dx - 2\pi Du(0) \log u(0)
\end{align*}
\]

hold, \( C \) and \( D \) being positive constants and \( A_p \) the same as above.

Let us prove (1) first. The argument is based on the inequality

\[
|\sin \phi|^p \leq A_p |\cos \phi|^p - B_p \cos p\phi
\]

valid for \(-\pi/2 \leq \phi \leq \pi/2\), \( 1 < p \leq 2 \), \( A_p \) and \( B_p \) being two positive constants depending on \( p \).

The proof of it is simple. Since \( \cos p\phi < 0 \) for \( \phi = \pm \pi/2 \), taking \( B_p \) large enough we get \(-B_p \cos p\phi > 1\) in a small neighborhood of \( \phi = \pm \pi/2 \). Now \(|\cos \phi|^p\) is greater than a positive constant in every closed interval interior to \((-\pi/2, \pi/2)\), so that choosing \( A_p \) large enough we shall have \( A_p |\cos \phi|^p - B_p \cos p\phi > 1\) in \((-\pi/2, \pi/2)\) and a fortiori the inequality (3).

To show (1) now, let us set

\[
F(z) = Re^{i\phi}; \quad u = R \cos \phi; \quad v = R \sin \phi.
\]

Owing to the fact that \( u > 0 \), we may choose \(-\pi/2 < \phi < \pi/2\). For the same reason, \( F(z) \neq 0 \) and so \( F(z)^p \) is regular in \( |z| < 1 \). Integrating now along the circle \( |z| = r \) we get

\[
\Re \left[ \frac{1}{2\pi i} \int_C \frac{F(z)^p}{z} \, dz \right] = \frac{1}{2\pi} \int_0^{2\pi} R^p \cos p\phi dx = \Re [F(0)^p] = u(0)^p > 0.
\]

Multiplying now (3) by \( R^p \) and integrating, we obtain

\[
\int_0^{2\pi} |v(re^{iz})|^p dx \leq A_p \int_0^{2\pi} |u(re^{iz})|^p dx - B_p \int_0^{2\pi} R^p \cos p\phi dx,
\]

but since the last term is positive, we may drop it, and the desired result follows. A similar argument works in proving (2). Here instead of (3) we use
\[ |\sin \phi| \leq C \cos \phi + D(\cos \phi \log \cos \phi + \phi \sin \phi), \]

which can be proved in exactly the same way.

Again, since \( F(z) \) does not vanish in \(|z| < 1\), \( F \log F \) is regular there, and integrating along \(|z| = r\) we get

\[
\Re \left[ \frac{1}{2\pi i} \int_{C} F(z) \log F(z) \frac{dz}{z} \right] = \frac{1}{2\pi} \int_{0}^{2\pi} \Re [F \log F] dx = u(0) \log u(0),
\]

and replacing

\[
\Re [F \log F] = \Re [R(\cos \phi + i \sin \phi)(\log R \cos \phi - \log \cos \phi + i\phi)]
\]

\[
= R \cos \phi \log R \cos \phi - R(\cos \phi \log \cos \phi + \phi \sin \phi),
\]

we obtain

\[
\frac{1}{2\pi} \int_{0}^{2\pi} R(\cos \phi \log \cos \phi + \phi \sin \phi) dx
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} u \log u dx - u(0) \log u(0).
\]

Finally, multiplying (4) by \( R \) and integrating, we obtain

\[
\int_{0}^{2\pi} |v(re^{ix})| dx \leq C \int_{0}^{2\pi} |u(re^{ix})| dx
\]

\[
+ D \int_{0}^{2\pi} u \log u dx - 2\pi Du(0) \log u(0),
\]

as stated in (2).

\section*{Bibliography}


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