TRANSITIVE SYSTEMS OF LINEAR OPERATORS
ON A BANACH SPACE

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Let $\mathcal{X}$ be an infinite-dimensional Banach space. Mackey has shown, under more general conditions, that if $x_1, \ldots, x_n, y_1, \ldots, y_n$ are any two sets of $n$ linearly independent elements in $\mathcal{X}$, then there exists an isomorphism $T$ of $\mathcal{X}$ with itself such that $T(x_i) = y_i$, $i = 1, \ldots, n$. In other words the collection of isomorphisms is transitive for linearly independent sets of elements of $\mathcal{X}$. This property is shared by many other sets of linear operators, for example by the set of all operators with finite-dimensional range in $\mathcal{X}$. These two sets of linear operators are semi-groups. In this note we give a condition for the transitivity property above which is necessary for all sets of linear operators and which while not in general sufficient is so for semi-groups. In this result the operators need not be assumed to be bounded or defined everywhere in $\mathcal{X}$. $\mathcal{C}(\mathcal{X})$ will be used to designate the collection of all bounded linear operators with domain $\mathcal{X}$ and range in $\mathcal{X}$.

**Theorem 1.** Let $\mathcal{G}$ be a collection of linear operators with domain and range in an infinite-dimensional Banach space $\mathcal{X}$. For $\mathcal{G}$ to be transitive for linearly independent sets of elements of $\mathcal{X}$ it is necessary that for each finite-dimensional subspace $F$ of $\mathcal{X}$ there exists a $T \in \mathcal{G}$ which is the identity on $F$ and a number $\epsilon > 0$ such that if $U \in \mathcal{C}(\mathcal{X})$ takes $F$ into $F$ and $\|U\| < \epsilon$, then there exists an operator $V \in \mathcal{G}$ which agrees with $T + U$ on $F$. If $\mathcal{G}$ is a semi-group, this condition is sufficient.

It is readily seen that the condition is necessary. That it is not sufficient in general can be seen by taking $\mathcal{G}$ to be an $\epsilon$-neighborhood of the identity in the Banach space $\mathcal{C}(\mathcal{X})$.

Let $\mathcal{G}$ be a semi-group. Let $F$ be a finite-dimensional subspace of $\mathcal{X}$ and $T_0$ be its associated operator in $\mathcal{G}$, and let $x_1, \ldots, x_n$ be linearly independent elements of $F$. There exist $x_i^* \in \mathcal{X}^*$ (the conjugate space of $\mathcal{X}$) such that $x_i^*(x_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. For elements

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1 This paper is based on a portion of the author's dissertation, Yale University, 1947. The author wishes to thank the referee for his suggestions which made possible an appreciable shortening of proof of Theorem 1 and clarified the nature of that proof.


3 By a semi-group of linear operators is meant a collection which contains $TU$ whenever it contains $T$ and $U$. 

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\( v_i \in F \) for which \( \|v_i - x_i\| < \delta = \min \{1, \epsilon\} / 2n \max \|x_i^*\| \), let the transformation \( W \) be determined by

\[
W(x) = \sum_{i=1}^{n} x_i^*(x)(v_i - x_i).
\]

Then \( \|W\| < \min \{1, \epsilon\} / 2 \). Thus \( T_0 + W \) is an isomorphism on \( F \). The operator \( U = \sum_{i=1}^{n} (-1)^i W_i \) has the properties \( \|U\| < \epsilon \) and \( (T_0 + U)(T_0 + W)x = x \) for all \( x \) in \( F \). By hypothesis there exist transformations \( U_i \) and \( W_i \) in \( \mathcal{G} \) which agree with \( T_0 + U \) and \( T_0 + W \) respectively in \( F \). Then by the above \( W_i(x_i) = v_i \) and \( U_i(v_i) = x_i \), \( i = 1, \ldots, n \). If also for each \( i \), \( \|v_i - x_i\| < \delta \), \( v_i \in F \), we may define \( W_i' \in \mathcal{G} \) as above so that \( W_i'(x_i) = v_i' \). Then \( W_i' U_i(v_i) = v_i' \), \( i = 1, \ldots, n \). The semi-group property of \( \mathcal{G} \) shows that (a) for each \( F \) and \( x_1, \ldots, x_n \) linearly independent in \( F \) there exists a \( \delta > 0 \) such that for every \( v_i, w_i \) in \( F \) with \( \|v_i - x_i\| < \delta \) and \( \|w_i - x_i\| < \delta \), there exists \( V \) in \( \mathcal{G} \) with \( V(v_i) = w_i, i = 1, \ldots, n \).

Next for a fixed \( F \) and \( n \leq \dim F \) consider the Cartesian product space \( \overline{F} = F \times \cdots \times F \) (\( n \) factors) made up of elements of the form \( \bar{x} = (x_1, \ldots, x_n) \) with \( \|\bar{x}\| = \max_{i=1}^{n} \|x_i\| \). \( \overline{F} \) is a finite-dimensional Banach space.\(^4\) Let \( \overline{L} \) be the set of all \( \bar{x} \in \overline{F} \) such that \( x_1, \ldots, x_n \) are linearly independent in \( F \). By (a), \( \overline{L} \) is open in \( \overline{F} \) and to each point \( \bar{x} \) of \( \overline{L} \) there exists a \( \delta > 0 \) such that \( \|\bar{x} - \bar{v}\| < \delta \) and \( \|\bar{x} - \bar{w}\| < \delta \) imply the existence of a transformation \( V \in \mathcal{G} \) such that \( \overline{V}(\bar{v}) = \bar{w} \) where \( \overline{V}(\bar{v}) = (V(v_1), \ldots, V(v_n)) \). Let \( \overline{K} \) be the set of \( \bar{y} \) in \( \overline{L} \) which can be reached from \( \bar{x} \) along a finite chain of such spheres each overlapping its predecessor. Then by (a) for such a \( \bar{y} \) there exists a sequence \( V_1, \ldots, V_r \) in \( \mathcal{G} \) such that for the product \( W = V_r \cdots V_1, \overline{W}(\bar{x}) = \bar{y} \) and by the semi-group property, \( W \in \mathcal{G} \).

From this it follows by a standard topological argument, used in connection with analytic continuation and elsewhere,\(^5\) that the set \( \overline{K} \) is open and closed in \( \overline{L} \). Therefore (b) if \( \bar{x} \) and \( \bar{y} \) are in the same component of \( \overline{L} \), there is a \( V \in \mathcal{G} \) such that \( \overline{V}(\bar{x}) = \bar{y} \).

Call \( \bar{x} \) and \( \bar{y} \) fully independent if \( x_1, \ldots, x_n, y_1, \ldots, y_n \) form a linearly independent set of \( 2n \) elements and suppose that \( \bar{x}, \bar{y} \in \overline{L} \). For such a pair \( \bar{x} \) and \( \bar{y} \), and for any real \( \alpha \), we have \( \alpha \bar{x} + (1 - \alpha) \bar{y} \in \overline{L} \), for \( \sum_{i=1}^{n} \beta_i \alpha x_i + (1 - \alpha) y_i = 0 \) implies \( \beta_i = 0 \), \( i = 1, \ldots, n \). Thus the line segment joining \( \bar{x} \) and \( \bar{y} \) is in \( \overline{L} \). Therefore (c) if \( \bar{x} \) and \( \bar{y} \) are fully independent in \( \overline{L} \), they must belong to the same component of \( \overline{L} \).

\(^4\) S. Banach, Théorie des opérations linéaires, Warsaw, 1932, see p. 182.
\(^5\) The use of the set \( \overline{K} \) and of this argument was suggested by the referee to replace a more lengthy Heine-Borel argument.
If $\tilde{x}$ and $\tilde{y}$ are two linearly independent sets of $n$ elements in $X$ which are not fully independent, consider a set $\tilde{v}$ of $n$ elements which is fully independent of $\tilde{x}$ and also of $\tilde{y}$. Such a set $\tilde{v}$ exists because $\mathcal{X}$ is assumed to be infinite-dimensional. The subspace $F$ can be taken to contain all the vectors $x_i, y_i,$ and $v_i$. For the $L$ which corresponds to this $F$, $\tilde{x}$, $\tilde{y}$, and $\tilde{v}$ must lie in the same component by (c). Then by (b) there exists a $U \in \mathcal{G}$ with $U(\tilde{x}) = \tilde{y}$. This completes the proof.

Next we consider all our transformations as being in the space $\mathcal{E}(\mathcal{X})$. For $\mathcal{E}(\mathcal{X})$ the strong topology is defined, following the ideas of von Neumann, as that where the neighborhoods of a transformation $U_0$ are those of the form

$$N(U_0; x_1, \cdots, x_k, \epsilon) = \{ T \in \mathcal{E}(\mathcal{X}) \mid \| T(x_i) - U_0(x_i) \| < \epsilon, i = 1, \cdots, k \}$$

where $\epsilon > 0$ and $x_i \in \mathcal{X}, i = 1, \cdots, k$.

**Theorem 2.** A subset $\mathcal{A}$ of $\mathcal{E}(\mathcal{X})$ satisfying the conclusion of Theorem 1 is dense in the strong topology of $\mathcal{E}(\mathcal{X})$.

It suffices to show that the neighborhood $N(U_0; x_1, \cdots, x_k, \epsilon)$ contains an operator in $\mathcal{A}$. By renumbering, if necessary, we let $x_1, \cdots, x_r$ be a linearly independent subset of the $x$'s which generates the same linear manifold as is generated by all of them. We put $x_i = \sum_{j=1}^r a_{ij}x_j, i = r+1, \cdots, k$ and $A = \max |a_{ij}|$ for these values of $i$ and for $j = 1, \cdots, r$. Also we set $\eta = \min \{ \epsilon, \epsilon/(rA) \}$. We choose $w_1, \cdots, w_r$ in $\mathcal{X}$ where each $w_i$ is linearly independent of the elements $U_0(x_i), i = 1, \cdots, r,$ and of the previously selected $w$'s and each $\| w_i \| < \eta$. The collection $\{ U_0(x_i) + w_i \}$ is a linearly independent set. By the assumption on $\mathcal{A}$, there exists $T \in \mathcal{A}$ such that $T(x_i) = U_0(x_i) + w_i, i = 1, \cdots, r$. Thus for these values of $i$, $\| T(x_i) - U_0(x_i) \| < \epsilon$. For $i = r+1, \cdots, k$ we have

$$\| T(x_i) - U_0(x_i) \| = \| (T - U_0) \left( \sum_{j=1}^r a_{ij}x_j \right) \| \leq \sum_{j=1}^r |a_{ij}| \| T(x_j) - U_0(x_j) \| < \epsilon.$$ 

Then $T \in N(U_0; x_1, \cdots, x_k; \epsilon)$.

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