which is a nonalgebraic integral of the system (1; 1, 0).

This conclusion contradicts the fact of F_0 being an algebraic function, therefore the initial hypothesis that assumes F to be dependent upon x_3 and y_3 is absurd.

F being independent of x_3 and y_3, it follows, as initially said, that any algebraic integral of the system (1; 1, 1) is an algebraic combination of the four integrals (12; 1, 1), that is, such a system does not admit any other algebraic integral distinct from the integrals (12; 1, 1).

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NOTE ON THE FUNCTIONAL EQUATIONS $f(xy) = f(x) + f(y), f(x^n) = nf(x)$

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Introduction. M. Augustin-Louis Cauchy in his famous Cours d'Analyse de l'école Royale Polytechnique (Part I, *Analyse algêbrique*, chap. 5, p. 109), published in 1821, gives a beautiful proof that the only continuous solution of the functional equation $f(x)+f(y)=f(xy)$, where $f(x)$ is defined for all real numbers x, is the function $f(x)=a \ln x$. Cauchy's proof reduces the equation to the Cauchy equation $f(x)+f(y)=f(x+y)$. In 1905 G. Hamel in the Mathematische Annalen proved that the discontinuous solutions of Cauchy's equation are totally discontinuous. In 1919 B. Blumberg proved that the only measurable function satisfying Cauchy's equation is the function Ax, hence the only measurable solution of

$$f(xy) = f(x) + f(y), \qquad f(x^n) = nf(x) \tag{1}$$

is $a \ln x$. In the past thirty years there have been a number of papers on these functional equations where the domain of the variable has usually been the real number system. In 1946 Erdös proved that if

Presented to the Society, October 29, 1949; received by the editors May 23, 1949.

$f(m)$ is additive and $f(m+1)\geqq f(m)$ then $f(m)=c \ln m$ (Ann. of Math. vol. 47 (1946)). This implies Theorem I of this paper. Professor H. Shapiro of New York University suggested that a more elementary proof of Theorem I than Erdös's should be possible and this led me to the proof of Theorem I, given below, in which I show that any solution $f(m)$ can be imbedded in a continuous solution $f(x)$, thus rendering the theorem a corollary of Cauchy's. I wish to express my thanks to Professor Freidman of New York University for some suggestions he made regarding this paper.

THEOREM I. *If $f(m+1)>f(m)$ and $f(mn)=f(m)+f(n)$ hold for all positive integers m and n, then $f(m)=c \ln m$.*

PROOF. If m/n, m and n are positive integers, $f(m/n)=f(m)-f(n)$ since $m=(m/n)\cdot n$. If m and n are relatively prime positive integers, we define $f(m/n)=f(m)-f(n)$. If λ is an integer and m and n are relatively prime, $f(m/n)=f(m)+f(\lambda)-f(n)-f(\lambda)=f(\lambda m)-f(\lambda n)$, so that if we define $f(m/n)=f(m)-f(n)$ for any two integers m and n, $f(m/n)$ is a function of the rational number m/n.

If a, b, c, and d are positive integers such that $a/b>c/d$, then $f(a/b)>f(c/d)$, since $f(a/b)=f(ad)-f(bd)$. $f(c/d)=f(bc)-f(bd)$ and $f(ad)>f(bc)$.

If a, b, c, and d are positive integers,

$$\begin{aligned} f(ac/bd) &= f(ac) - f(bd) \\ &= f(a) + f(c) - f(b) - f(d) \\ &= f(a/b) + f(c/d). \end{aligned}$$

$f(m)=f(m)+f(1)$, therefore $f(1)=0$.

If $b_i>1$ and $\{b_i\}$ is a sequence having 1 as a limit where b_i is a rational number, for i sufficiently large $f(b_1)>f(b_i)>f(1)=0$. Therefore there exists a monotonically decreasing subsequence $\{b_i'\}$ of $\{b_i\}$ having 1 as a limit and a corresponding monotonically decreasing sequence $\{f(b_i)\}$ having a real number f as a limit. Suppose $f\neq 0$. Let N be an integer such that $Nf>f(2)$; let b_λ' be a b_i' such that $1<b_\lambda'<2^{1/2N}$, then $1<b_\lambda'^N<2$,

$$f(b_\lambda'^N) = Nf(b_\lambda')$$

or

$$f(b_\lambda') = \frac{f(b_\lambda'^N)}{N} < \frac{f(2)}{N} < f$$

contrary to the choice of $f(b_i')$ as a monotonically decreasing sequence approaching f as a limit. Therefore $f=0$. Since f was any limit point of the sequence $\{f(b_i)\}$, we have the result that for any sequence of rationals $b_i>1$ such that $b_i\to 1$, $f(b_i)\to 0$. If r is rational and positive

$$f(1/r) = f(1) - f(r) = -f(r).$$

Therefore if $b_i\to 1$ and $b_i<1$, $f(b_i)\to 0$. From the above we conclude that if $b_i\to 1$, $f(b_i)\to 0$.

If $\{a_i\}$ is a convergent sequence of positive rationals $f(a_m)-f(a_n) = f(a_m/a_n)\to 0$ as $|a_m-a_n|\to 0$, for as $|a_m-a_n|\to 0$, $a_m/a_n\to 1$, therefore $\{f(a_i)\}$ converges. Let $\lim_{i\to\infty} a_i=a$. If a is rational, then $a_i/a\to 1$ and $f(a_i/a)=f(a_i)-f(a)\to 0$ and hence $f(a_i)\to f(a)$. If a is irrational, define $f(a)$ as $\lim_{i\to\infty} f(a_i)$. $f(a)$ is unique for if $a_i\to a$, $f(a_i)\to f$ and if $a_i'\to a$, $f(a_i')\to f'$, then

$$f(a_i) - f(a_i') = f\left(\frac{a_i}{a_i'}\right)\to f - f',$$

but $a_i/a_i'\to 1$, hence $f-f'=0$. We have now defined $f(x)$ for all positive real numbers x so that when $x_i\to x$, $f(x_i)\to f(x)$, that is $f(x)$ is continuous. If x and y are any real numbers, let $\{x_i\}$ and $\{y_i\}$ be sequences of rational numbers such that $x_i\to x$ and $y_i\to y$, then $f(x_iy_i) =f(x_i)+f(y_i)$ and hence $f(xy)=f(x)+f(y)$ for all positive real numbers x and y. The only continuous solution to the functional equation

$$f(x) + f(y) = f(xy)$$

is

$$f(x) = c \ln x.$$

Therefore

$$f(n) = c \ln n.$$

THEOREM II. *If $f(x)$ is a monotonically increasing function defined for all positive real numbers x, such that $f(x^n)=nf(x)$ for all positive integers n, then $f(x)=k \ln (x)$ where $k=f(e)$.*

PROOF. $f(x)=nf(x^{1/n})$ or $f(x^{1/n})=1/nf(x)$. $f(x^{m/n})=(m/n)f(x)$, or if r is a rational number, $f(x^r)=rf(x)$. In particular $f(e^r)=rf(e)=f(e) \cdot \ln (e^r)$. Let x be any positive real number and let $y=\ln x$, that is $x=e^y$. Let the sequence $\{r_i\}$ be a monotonically increasing sequence of rational numbers having y as a limit, then $e^{r_i}\to x$ and $\{e^{r_i}\}$ is monotonically increasing and $\{f(e^{r_i})\}$ is monotonically increasing and has the limit of $f(e)\ln x$. If $f(x)<f(e)\ln x$, then there is an r_n such

that $f(x) < f(e^{r_n})$ with $x > e^{r_n}$ contrary to the monotonicity of $f(x)$. If $f(x) > f(e) \ln x$, let $\{r_i\}$ be a decreasing sequence of rational numbers having y as a limit, then $\{e^{r_i}\}$ is monotonically decreasing to the limit x and $\{f(e^{r_i})\}$ is a monotonically decreasing sequence having $f(e) \ln x$ as a limit. We can now find an r_m such that $f(x) > f(e^{r_m})$ with $x < e^{r_m}$, contrary to the monotonicity of $f(x)$. Therefore $f(x) = k \ln x$.

COROLLARY. *If $f(x^p) = pf(x)$ and $f(x)$ is monotonically increasing for all real numbers $x > 0$ and all prime numbers p, then $f(x) = k \ln x$.*

COROLLARY. *If $f(x^p) = pf(x)$ and $f(x)$ is monotonically decreasing for all real numbers $x > 0$, and all prime numbers p, then $f(x) = -k \ln x$.*

COROLLARY. *If $f(r)$ is a monotonically increasing function on a set S which is everywhere dense among the positive real numbers and contains all positive integral powers of all members of the set, for example, the rational numbers, and $f(r^p) = pf(r)$ for any prime p, then $f(r) = k \ln r$.*

PROOF. $f(r^n) = nf(r)$ for all r in S and positive integers n. Let R be a real number and let $\{r_i\}$ be an increasing sequence of elements of S having R as a limit, then if $\{r_i'\}$ is another such sequence, the limit L of $\{f(r_i)\}$ = the limit L' of $\{f(r_i')\}$. For suppose $L > L'$; then there exists an r_n such that $f(r_n) > L'$, and an $r_m' > r_n$, hence $L' > f(r_m') > f(r_n)$, a contradiction. Similarly, we arrive at a contradiction if we assume $L' > L$. Therefore $L = L'$. We may thus extend the domain of $f(r)$ unambiguously to all real numbers greater than 0, by defining $f(R)$ to be the limit of $\{f(r_n)\}$ for any irrational positive number R. Under this definition monotonicity of the function holds in the new extended domain of all positive real numbers. Since R^n is the limit of $\{r_i^n\}$, r_i defined above, $f(R^n) = nf(R)$ for any real number R and the corollary follows from the theorem.

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