DEFORMATION THEORY OF SUBSPACES IN A RIEMANN SPACE

V. HLAVATÝ

Summary. This paper deals with the infinitesimal transformation (2,1) of a family \( (V_m) \) of subspaces \( V_m \) in a Riemann space. In §§1–4 the transformation (2,1) is applied on internal objects of a \( V_m \), while in §§5 and 6 the characteristic mixed tensors \( K^s_{a_1...a_l} \) (cf. the equation (1,2; 4)) of a \( V_m \) are investigated with respect to (2,1). Finally, some applications of the theory are given in §§7 and 8. In particular the statement expressed by the equation (8,10)b is the generalization of the well known Levi-Civita result for \( m=1 \), while the statement expressed by the equation (8,10)c generalizes the classical result (for \( m=1 \), \( n=2 \)) by Jacobi.

1. Preliminary.

(1) Let \( V_n \) be a \( n \)-dimensional Riemann space (\( n \geq 2 \)), referred to the real coördinate system \( \xi^1, \ldots, \xi^n \), its metric tensor, \( \Gamma^r_{\lambda\mu} = \Gamma^r_{\mu\lambda} \) the corresponding Christoffel symbols and \( \nabla \) the covariant derivative operator in \( V_n \) with respect to \( \Gamma^r_{\lambda\mu} \). The curvature tensor of \( \Gamma^r_{\lambda\mu} \) will be denoted by

\[
R^s_{\alpha\beta\lambda\mu} = \frac{\partial}{\partial \xi^\mu} \Gamma^r_{\alpha\lambda} - \frac{\partial}{\partial \xi^\mu} \Gamma^r_{\alpha\beta} + \Gamma^r_{\alpha\beta} \Gamma^\lambda_{\mu\lambda} - \Gamma^r_{\alpha\lambda} \Gamma^\lambda_{\beta\mu} \quad \left( \partial_a = \frac{\partial}{\partial \xi^a} \right).
\]

(2) Let \( V_m \) be a \( m \)-dimensional subspace of \( V_n \) (\( 1 \leq m < n \)) referred to the real parameters\( ^1 \eta^a \) and let

\[
\xi^r = \xi^r(\eta^1, \ldots, \eta^m)
\]

be its parametric equations.\( ^2 \) Throughout this paper we consider only the case where the matrix of the mixed tensor

\[
T^r_a = \partial_a \xi^r \quad \left( \partial_a = \frac{\partial}{\partial \eta^a} \right)
\]

is of rank \( m \). Hence the metric tensor\( ^3 \)

\[
g_{ab} = g_{ba} = g_{\lambda\mu} T^{\lambda\mu}_{ab}
\]

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\(^1\) Greek (Latin) indices run from 1 to \( n \) (from 1 to \( m \)).

\(^2\) Any function considered in this paper is understood to be a real and continuous one, as well as its derivatives which appear in the discussion.

\(^3\) Here and later on we put \( T_{a_1...a_v}^{\lambda...s} = T_{a_1...a_v}^{\lambda...s}, \quad v = 2, 3, \ldots \).
SUBSPACES IN A RIEMANN SPACE

is of rank \( m \) and consequently the corresponding Christoffel symbols \( \Gamma^\epsilon_{ab} = \Gamma^\epsilon_{ba} \) exist. The curvature tensor of \( V_m \) will be denoted by

\[
P^a_{de} = \partial_d \Gamma^a_{bc} - \partial_c \Gamma^a_{bd} + \Gamma^h_{dc} \Gamma^a_{bh} - \Gamma^h_{db} \Gamma^a_{hc}.
\]

Considering the connections \( \Gamma^\epsilon_{\mu} \) and \( \Gamma^\epsilon_{\nu} \) we may introduce three different kinds of covariant derivatives, namely

\[
(1,2;3a) \quad \nabla_{\mu} V^\lambda_{\cdots} = \partial_{\mu} V^\lambda_{\cdots} + \Gamma^\alpha_{\mu} V^\alpha_{\cdots} + \cdots - \Gamma^\alpha_{\lambda} V^\alpha_{\cdots} - \cdots,
\]

for a tensor field defined over \( V_n \),

\[
(1,2;3b) \quad \nabla_{\mu} V^\lambda_{\cdots} = \partial_{\mu} V^\lambda_{\cdots} + T^\mu_{a} \{ \Gamma^\alpha_{\mu} V^\alpha_{\cdots} + \cdots - \Gamma^\alpha_{\lambda} V^\alpha_{\cdots} - \cdots \},
\]

for a tensor field defined over \( V_m \) and

\[
(1,2;3c) \quad D_b V^\lambda_{\cdots} = \partial_b V^\lambda_{\cdots} + \Gamma^\alpha_{\lambda} T^b_{\cdots} + \cdots - \Gamma^\alpha_{\lambda} V^\alpha_{\cdots} - \cdots,
\]

for a mixed tensor field defined over \( V_m \). The embedding theory of a \( V_m \) in a \( V_n \) may be described by means of the mixed tensors \( T^\alpha_{a} \) and

\[
(1,2;4) \quad K^\lambda_{\cdots} = D_{\alpha} \cdots D_{\alpha} T^\lambda_{\cdots}, \quad r = 2, \ldots, N,
\]

where \( N \) is the number of osculating spaces of \( V_m \).

Incidentally \( K^\alpha_{\beta} = K^\alpha_{\beta} \) lies with its index \( \nu \) in the first normal space of \( V_m \).

(3) A family \( (V_m) \) of a set of \( V_m \)'s is defined as a set of \( V_m \)'s such that through any generic point of the \( V_n \) there is only one element \( V_m \) of \( (V_m) \). Each element \( V_m \) of the family \( (V_m) \) is referred to the parameter system \( \eta^a \) and we keep the parameter transformation \( \eta^a \leftrightarrow \eta^{a^*} \) independent of the coordinate system \( \xi^a \). Hence

\[
(1,3;1) \quad \frac{\partial \eta^{a^*}}{\partial \eta^a} = 0 \quad \frac{\partial \eta^a}{\partial \eta^{a^*}} = 0.
\]

On the other hand, an object \( \Omega \) defined over \( (V_m) \) is defined also over \( V_n \) and consequently may be thought of as expressed either by means of the \( \eta^a \) or by means of the \( \xi^a \). This is in particular true for \( T^a_{\alpha}, g_{ab}, \Gamma^a_{ab} \). Whenever we apply on such an object the operator \( \partial_{\mu} \), it is understood that we consider it expressed by means of the \( \xi^a \). Taking in account (1,3;1) we see that\(^5 \)

\[
\nabla_{\mu} T^a_{\alpha} = \partial_{\mu} T^a_{\alpha} + \Gamma^a_{\alpha b} T^b_{\imath},
\]

according to (1,2;3a), while (cf. (1,2;3b)) \( \nabla_{b} T^\alpha_{\imath} = \partial_{b} T^\alpha_{\imath} + \Gamma^\alpha_{\lambda b} T^\lambda_{\imath} \). ---

\(^4\) Cf. V. Hlavaty: Embedding theory of a \( W_m \) in a \( W_n \) (to be published in Actualités Scientifiques et Industrielles) where the case of a Riemann space is included. Cf. also J. A. Schouten and E. R. van Kampen, Über die Krümmung einer \( V_m \) in \( V_n \); eine Revision der Krümmungstheorie (Math. Ann. vol. 105, p. 144-159).

\(^5\) \( \nabla_{\mu} T^a_{\alpha} = \partial_{\mu} T^a_{\alpha} + \Gamma^a_{\alpha b} T^b_{\imath} \Gamma^\lambda_{ab} T^\lambda_{\imath} \) according to (1,2;3a), while (cf. (1,2;3b)) \( \nabla_{b} T^\alpha_{\imath} = \partial_{b} T^\alpha_{\imath} + \Gamma^\alpha_{\lambda b} T^\lambda_{\imath} \).
tensors (resp. vectors) not only with respect to the coördinate trans-
formation but also with respect to the parameter transformation. Let
\( P \) be a generic point of \( V^n \) and let \( V_m \) be the element of \((V_m)\) contain-
ing \( P \). If a tensor field \( T \) defined over \( V^n \) has the property that \( T(P) \)
is in the tangential space (or in some of the normal spaces) of the
\( V_m \) already mentioned, then we say that the field \( T \) is tangent
(normal) to \( (V_m) \).

2. **Fundamental definitions.** Let \( V^r = V^r(\xi) \) be the components of a
contravariant vector field given over \( V^n \) and let

\[
*V^r = V^r + \varepsilon V^r \quad (\varepsilon \to 0 \text{ is a constant})
\]

be the infinitesimal transformation of a Lie group with the generator
\( V^r \partial_r \). If a generic point \( P(\xi) \) describes a subspace \( V_m \) of the family
\((V_m)\) then the point \( *P = P(*\xi) \) describes a subspace \( *V_m \). We shall
assume throughout this paper that the set of all \(*V_m\) constructed in
this way is a family, which we denote by \((*V_m)\). In the following defi-
nition \( X = X(P) \) denotes a tensor field (a set of tensor fields) defined
either over \((V_m)\) or over \( V^n \) and \( F[X] = F(\xi) \) is a function of \( X \).

**Definition (2,1).** \( X' \) is the value of \( X \) in \(*P\), \( X' = X(*P) \), \( \circ X \) is the
tensor at \(*P\) which one gets by parallel displacement in \( V^n \) of \( X(P) \)
from \( P \) to \(*P\). If \( X \) is defined over \((V_m)\), then \(*X \) is the tensor field (a set
of tensor fields) defined over \((*V_m)\) in the same way as \( X \) is defined over
\((V_m)\). If \( X \) is defined over \( V^n \), then \(*X \equiv X' \). Furthermore

\[
\begin{align*}
*F = F[*X], & \quad \circ F = F[\circ X], & \quad F' = F(*\xi) \\
\end{align*}
\]

in the first case and

\[
\begin{align*}
*F = F' \equiv F(*\xi), & \quad \circ F = F[\circ X] \\
\end{align*}
\]

in the second case.

We use these symbols in the following definition of the symbols \( \tau \),
\( \omega \), and \( \Delta \):

**Definition (2,2).** The operators \( \tau \), \( \omega \), \( \Delta \) are defined by the following
equations

\[
\begin{align*}
(2,3) & \quad \begin{align*}
(\text{a}) \quad \tau F &= \lim_{\varepsilon \to 0} \frac{*F - \circ F}{\varepsilon} \quad \text{(translation of } F), \\
(\text{b}) \quad \omega F &= \lim_{\varepsilon \to 0} \frac{F' - \circ F}{\varepsilon} \quad \text{(variation of } F), \\
(\text{c}) \quad \Delta F &= (\tau - \omega)F \quad \text{(deformation of } F). \\
\end{align*}
\end{align*}
\]

In this paper we shall investigate the application of these operators
on the objects of \((V_m)\).

3. The fields \(T_a^r\) and \(g_{ab}\).

**Theorem (3,1).** We have

\[
(3.1) \quad (a) \quad \tau T_a^r = \nabla_a V^r, \quad (b) \quad \omega T_a^r = V^\mu \nabla_a T_a^r, \\
(c) \quad \Delta T_a^r = L_a^r,
\]

where

\[
(3.2) \quad L_a^r = \nabla_a V^r - V^\mu \nabla_a T_a^r = \partial_a V^r - V^\mu \partial_a T_a^r.
\]

**Proof.** Let \(X^F[X] = T_a^r\). Then

\[
*F = \frac{\partial *F}{\partial \eta^a} = T_a^r + \epsilon \partial_a V^r, \quad F' = T_a^r(*F) = T_a^r + \epsilon V^\mu \partial_a T_a^r + \cdots,^4
\]

\[
\circ F = \circ T_a^r = T_a^r - \epsilon V^\mu \Gamma^r_{\mu a} T_a^r + \cdots
\]

and consequently

\[
(3.3) \quad (a) \quad *F - \circ F = T_a^r + \epsilon (\partial_a V^r + \Gamma^r_{\mu a} T_a^r) - T_a^r + \cdots,^4 \\
(b) \quad F' - \circ F = T_a^r + \epsilon V^\mu (\partial_a T_a^r + \Gamma^r_{\mu a} T_a^r) - T_a^r + \cdots.\]

From (3.3)ab we have (3,1)ab and these equations together with (3,2) lead to (3,1)c.

**Theorem (3,2).** We have

\[
(3.4) \quad (a) \quad \tau g_{ab} = 2W_{ab},^7 \quad (b) \quad \omega g_{ab} = G_{ab}, \\
(c) \quad \Delta g_{ab} = 2g_{ab}L_a^\alpha (\epsilon T_b^\alpha)
\]

where

\[
(3.5) \quad W_{ab} = T_{(a\beta)}^\lambda V^\lambda V_{\mu}, \quad G_{ab} = V^\mu \partial_\mu g_{ab}.
\]

**Proof.** Put \(X = g_{\lambda \mu}\). Then

\[
(3.6) \quad (a) \quad *X = X' = g_{\lambda \mu} + \epsilon V^\omega \partial_\omega g_{\lambda \mu} + \cdots, \\
(b) \quad \circ X = g_{\lambda \mu} + \epsilon V^\omega (\Gamma^r_{\lambda \mu} g_{r\mu} + \Gamma^r_{\mu \rho} g_{r\lambda}) + \cdots = g_{\lambda \mu}.
\]

On the other hand, if we denote by \(X\) the set of tensor fields \(g_{\lambda \mu}\), \(T_a^r\) we have for \(F[X] = g_{\lambda \mu} T_a^\lambda T_b^\mu = g_{ab}\) by virtue of (3,6)

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^4 Throughout this paper the dots denote the coefficient of \(\epsilon^2\).

^7 Substantially equivalent formulas to (3,1)a and (3,4)a may be found also in the papers in the bibliography, which deal with the theory of deformation of subspaces from a different point of view.
(a) \[ *F = g_{ab}^{\mu} \cdot T_{a}^{\lambda} \cdot T_{b}^{\mu} = (g_{\lambda \mu} + \epsilon V_{\lambda} \nabla \mu g_{\lambda \mu} + \cdots) \]
\[ \cdot \left( T_{a}^{\lambda} + \epsilon \partial_{a} V_{\lambda} + \cdots \right) \left( T_{b}^{\mu} + \epsilon \partial_{b} V_{\mu} + \cdots \right) \]
\[ = g_{ab} + \epsilon \left[ V_{\lambda} T_{a}^{\lambda} \partial_{a} g_{\lambda \mu} + 2 g_{\lambda \mu} T_{\left( b \partial_{a} \right)} V_{\lambda} \right] + \cdots \]
\[ = g_{ab} + 2 \epsilon T_{\left( a \right) T_{b}} \nabla_{a} V_{\mu} + \cdots , \]

(3.7) (b) \[ F' = g_{ab}^{\mu} \cdot T_{a}^{\lambda} \cdot T_{b}^{\mu} = (g_{\lambda \mu} + \epsilon V_{\lambda} \partial_{a} g_{\lambda \mu} + \cdots) \]
\[ \cdot \left( T_{a}^{\lambda} + \epsilon V_{\lambda} \nabla_{a} T_{b}^{\mu} + \cdots \right) \left( T_{b}^{\mu} + \epsilon V_{\mu} \nabla_{b} T_{a}^{\lambda} + \cdots \right) \]
\[ = g_{ab} + \epsilon \left[ V_{\lambda} T_{a}^{\lambda} \partial_{a} g_{\lambda \mu} + 2 g_{\lambda \mu} V_{\mu} \left( \partial_{a} T_{\left( a \right) b} \right) \right] + \cdots \]
\[ = g_{ab} + \epsilon V_{\lambda} \partial_{a} g_{ab} + \cdots , \]

(c) \[ \circ F = g_{ab}^{\mu} \cdot T_{a}^{\lambda} \cdot T_{b}^{\mu} = \left[ g_{\lambda \mu} + \epsilon V_{\lambda} \left( \Gamma_{\lambda \rho}^{\nu} g_{\rho \mu} + \Gamma_{\mu \rho}^{\nu} g_{\lambda \rho} \right) + \cdots \right] \]
\[ \cdot \left( T_{a}^{\lambda} + \epsilon V_{\lambda} \nabla_{a} T_{b}^{\mu} + \cdots \right) \left( T_{b}^{\mu} + \epsilon V_{\mu} \nabla_{b} T_{a}^{\lambda} + \cdots \right) \]
\[ = g_{ab} + \epsilon \left[ T_{\lambda}^{\mu} T_{a}^{\lambda} \left( \Gamma_{\lambda \rho}^{\nu} g_{\rho \mu} + \Gamma_{\mu \rho}^{\nu} g_{\lambda \rho} \right) - 2 g_{\lambda \mu} T_{\left( a \right) b} V_{\lambda} T_{\left( a \right) b} \right] + \cdots \]
\[ = g_{ab} + \cdots . \]

Hence

(3.7) (d) \[ *F - \circ F = g_{ab}^{\mu} - g_{ab}^{\mu} = 2 \epsilon T_{\left( a \right) b} \nabla_{a} V_{\mu} + \cdots , \]

(e) \[ F' - \circ F = \epsilon \partial_{a} g_{ab} = \epsilon V_{\lambda} \partial_{a} g_{ab} + \cdots . \]

These equations lead to (3.4)ab and from these we have (3.4)c.

**Definition (3.1).** The transformation (2,1) is called a rigid translation if \( T_{g_{ab}} = 0. \)

**Theorem (3.3a).** Let \( V' \) be a tangential vector field to \( (V_{m}) \). A necessary and sufficient condition that (2,1) be a rigid translation is the (Killing) equation

\[ D_{a} V_{b} = 0. \]

**Proof.** In our case we have\(^8\) \( V_{a} = V_{\lambda} T_{a}^{\lambda} \), \( V_{\lambda} = V_{\lambda} T_{a}^{\epsilon} \) and consequently (by virtue of \( \Gamma_{\lambda}^{\mu} T_{a}^{\lambda} = T_{a}^{\lambda} \nabla_{a} T_{a}^{\lambda} - \Gamma_{ab}^{\lambda} T_{a}^{\lambda} \))

\[ D_{b} V_{a} = \partial b V_{a} T_{a}^{\lambda} - T_{\lambda}^{\epsilon} \left( \nabla_{b} T_{a}^{\epsilon} \right) V_{c} = \left( \partial b V_{a} \right) T_{a}^{\lambda} + V_{\lambda} \partial b T_{a}^{\lambda} - \left( \nabla_{b} T_{a}^{\lambda} \right) V_{\lambda} \]

(3.9)
\[ = T_{\lambda}^{\epsilon} \left( \partial b V_{a} - \Gamma_{\mu}^{\lambda} T_{a}^{\mu} V_{\lambda} \right) = T_{b} T_{a}^{\lambda} \nabla_{a} V_{\lambda}. \]

The proof follows from (3.9), (3.5), and (3.4)a.

**Theorem (3.3b).** Let \( V' \) be a normal vector field to \( (V_{m}) \). A necessary and sufficient condition that (2,1) be a rigid translation is: \( V' \) is in the \( x \)th normal space of \( V_{m} \), \( x = 2, 3, \cdots. \)

\(^8 T_{a}^{\lambda} = T_{a}^{\mu} g_{\lambda}^{\epsilon} g_{\epsilon \mu}. \)
Proof. In our case we have $V_\lambda T^\lambda_a = 0$ and consequently

$$T^\lambda_a T^\mu_b \nabla_V V_\mu = - T^\lambda_a V_\mu \nabla_T T^\mu_b = - (D_a T^\mu_b) V_\mu = - K^\mu_a V_\mu.$$  

The tensor $K^\mu_a$ lies with $\mu$ in the first normal space of $(V_m)$. By virtue of this fact, we get the theorem (3,3b) from (3,4)a and (3,5).

Remark. The theorem (3,3b) is the generalization of the well known fact that the “deformation” of the arc of a curve in a three-dimensional Euclidean space along its binormal is equal to zero.

4. The field $\Gamma^e_{ab}$.

Theorem (4,1). We have

$$\tau \Gamma^e_{ab} = W^e_{ab} = W^{ed}_{ab} g^d, \quad \omega \Gamma^e_{ab} = G^e_{ab},$$

$$\Delta \Gamma^e_{ab} = Q^e_{ab}$$

where

$$W^{ad}_{ab} = D_a W_{db} + D_b W_{da} - D_d W_{ba},$$

$$G^{ab} = V^\omega \partial_\omega \Gamma_{ab},$$

$$Q^e_{ab} = W^{ed}_{ab} g^d - G^e_{ab}.$$

Proof. First of all we have

$$\partial_a W_{db} + \partial_b W_{da} - \partial_d W_{ab} = W_{ab} + 2 \Gamma^e_{ba} W_{ae}.$$

Furthermore, if $X$ is the set of tensors $g_{ab}$, $g^{cd}$, we get for $F[X] = \Gamma^e_{ab}$ by virtue of (3,7)d and (4,3)

$$F = * \Gamma^e_{ab} = 1/2 \left( \partial_a * g_{bd} + \partial_b * g_{ad} - \partial_d * g_{ab} \right)$$

$$= 1/2\left[ \partial_a \left( g_{bd} + 2 \epsilon W_{bd} + \cdots \right) \right] \left[ \partial_d \left( g_{ad} + 2 \epsilon W_{ad} + \cdots \right) \right]$$

$$+ \partial_b \left( g_{ab} + 2 \epsilon W_{ab} + \cdots \right)$$

$$= \Gamma^e_{ba} + \epsilon \left( g_{ed} \left[ W_{ab} + 2 \Gamma^e_{ba} W_{ae} \right] - 2 W^{cd} \Gamma^e_{ba} g_{ed} \right) + \cdots.$$

On the other hand, we see from (3,7)c that

$$\Gamma^e_{ba} = \Gamma^e_{ba} + \cdots.$$

From (4,5), (4,4) we have (4,1)a. The equation (4,1)b follows from

$$F' = \Gamma^e_{ab} = \Gamma^e_{ab} + \epsilon V^\omega \partial_\omega \Gamma^e_{ab} + \cdots$$

and (4,5). The equation (4,1)c is obvious.
Later on we shall need also the coefficients

\begin{equation}
\Gamma^c_{ab} = \frac{1}{2} \varepsilon e^{cd} (\partial_d g'_{ab} + \partial_a g'_{bd} - \partial_b g'_{ad})
\end{equation}

which are related to \( \Gamma^{e'}_{ab} \) by a relation introduced in the following lemma:

**Lemma (4,1).** We have

\begin{equation}
\Gamma^c_{ab} = \frac{\varepsilon}{2} e^{cd} \left( L^e_\alpha \partial_\alpha g_{db} + L^e_\beta \partial_\beta g_{da} - L^e_\gamma \partial_\gamma g_{ab} \right) + \cdots.
\end{equation}

**Proof.** We get from (4,7) and (3,7)b

\begin{equation}
\begin{aligned}
\Gamma^c_{ab} &= \frac{1}{2} [g^{cd} + \varepsilon V'^\alpha \partial_a g^{cd} + \cdots] \left[ \partial_d (g_{bd} + \varepsilon V^\rho \partial_\rho g_{bd} + \cdots) \\
&\quad + \partial_b (g_{ad} + \varepsilon V^\rho \partial_\rho g_{ad} + \cdots) \\
&\quad - \partial_a (g_{ab} + \varepsilon V^\rho \partial_\rho g_{ab} + \cdots) \right] \\
&= \Gamma^c_{ab} + \frac{\varepsilon}{2} \left\{ 2 \Gamma^l_\alpha g_{dl} V^\alpha \partial_\rho g^{cd} + g^{cd} \left[ (\partial_\alpha V^\rho) \partial_\rho g_{bd} \\
&\quad + (\partial_\beta V^\rho) \partial_\rho g_{ad} - (\partial_\gamma V^\rho) \partial_\rho g_{ab} \\
&\quad + g^{cd} V^\rho (\partial_\beta \partial_\rho g_{bd} + \partial_a \partial_\rho g_{ad} - \partial_a \partial_\rho g_{ab}) \right] \right\} + \cdots.
\end{aligned}
\end{equation}

On the other hand

\begin{equation}
\begin{aligned}
\Gamma^{e'}_{ab} &= \Gamma^c_{ab} + \varepsilon V^\rho \partial_\rho \Gamma^c_{ab} + \cdots = \Gamma^c_{ab} + \varepsilon [V^\rho (\partial_\rho g^{cd}) \Gamma^d_\alpha g_{bd} \\
&\quad + \frac{1}{2} g^{cd} V^\rho (\partial_\beta \partial_\rho g_{bd} + \partial_\beta \partial_\rho g_{ad} - \partial_\beta \partial_\rho g_{ab})] + \cdots,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
(\partial_\alpha \partial_\rho - \partial_\rho \partial_\alpha) g_{bd} &= (T^\lambda_\alpha \partial_\lambda \partial_\rho - (T^\lambda_\rho \partial_\lambda) \partial_\alpha - (T^\lambda_\rho \partial_\lambda) \partial_\beta g_{bd} \\
&\quad = - (\partial_\rho T^\rho_\alpha) (\partial_\lambda g_{bd}).
\end{aligned}
\end{equation}

Comparing (4,9), (4,10), and (4,11), we obtain

\begin{equation}
\begin{aligned}
\Gamma^c_{ab} - \Gamma^{e'}_{ab} &= \frac{\varepsilon}{2} g^{cd} \left[ (\partial_\alpha V^\rho - V^\lambda \partial_\lambda T^\rho_\alpha) \partial_\rho g_{bd} + (\partial_\beta V^\rho - V^\lambda \partial_\lambda T^\rho_\beta) \partial_\rho g_{ad} \\
&\quad - (\partial_\gamma V^\rho - V^\lambda \partial_\lambda T^\rho_\gamma) \partial_\rho g_{ab} + \cdots \right]
\end{aligned}
\end{equation}

and this equation gives us at once (4,8).

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5. The field \( K_{\alpha \beta} \). The first theorem concerns the translation of the
tensor

\[ K'_{ba} = K'_{ab} = D_b T'_a. \]

**Theorem (5,1).** The following equation holds:

\[ \tau K'_{ba} = D_b D_a V^\nu - V^\nu T^\lambda_{ba} R^\mu_{\omega \lambda} - W^\nu_{ba} T'_{\omega}. \]

**Proof.** First of all we have according to (2,2)b for \( X = g_{\lambda \nu}, g'^{\lambda}, F[X] = F(\xi) = \Gamma_{\mu}^\nu \)

\[ *\Gamma'_{\lambda \mu} = \Gamma'_{\lambda \mu} = \varepsilon V^\nu \partial_a \Gamma_{\lambda \mu} + \cdots \]

and moreover by virtue of (4,4) and (4,5)

\[ *\Gamma'_{\sigma a} = \varepsilon W^\sigma_{ba} + \cdots. \]

Hence

\[ *K'_{ba} = *D_b *T'_{a} = \partial_b (T'_{a} + \varepsilon \partial_a V^\nu + \cdots) \]

\[ + \varepsilon V^\sigma (\partial_a \Gamma_{\lambda \mu} ) (T^\nu_{b} + \varepsilon \partial_b V^\nu + \cdots) (T^\lambda_{a} + \varepsilon \partial_a V^\lambda + \cdots) \]

\[ - \varepsilon W^\sigma_{ba} (T'_{c} + \varepsilon \partial_c V^\nu + \cdots) \]

and consequently

\[ *K'_{ba} = \partial_b T'_{a} + \varepsilon \{ \partial_b \partial_a V^\nu + V^\nu (\partial_a \Gamma'_{\lambda \mu}) T^\lambda_{ba} - W^\nu_{ba} T'_{c} \} + \cdots \]

\[ = D_b T'_{a} + \varepsilon \{ D_b D_a V^\nu - V^\nu [\partial_{a} \Gamma'_{\omega \lambda} - \partial_{a} \Gamma_{\mu \lambda}] T^\lambda_{ba} - W^\nu_{ba} T'_{c} \} + \cdots \]

\[ = D_b T'_{a} + \varepsilon \{ D_b D_a V^\nu - V^\nu T^\mu_{ba} R^\omega_{\mu \lambda} - W^\omega_{ba} T'_{c} \} + \cdots. \]

On the other hand

\[ \circ K'_{ba} = D_b T'_{a} + \cdots = K'_{ba} + \cdots. \]

The equations (5,4)b and (5,4)c lead to (5,2) with the \( \doteq \) sign. This equation, being a tensor equation valid for a special coordinate (and parameter) system at a generic point \( P \), holds for all coordinate (and parameter) systems at a generic point.

The second theorem concerns the variation of \( K'_{ba} \):

**Theorem (5,2).** The following equation holds:

\[ \omega K'_{ba} = V^\nu [D_b V_{a} T'_{a} - T^\mu_{ba} R^\nu_{\omega \lambda} + (\nabla_{\mu} T'_{a}) (\nabla_{a} T'_{b})] - G^\nu_{ba} T'_{c}. \]

**Proof.** First of all we have
\[
\partial_\omega \partial_b T^\omega_a - \partial_b \partial_\omega T^\omega_a = \partial_a T^\omega_b \partial_b T^\omega_a - T^\omega_b \partial_\omega T^\omega_a = (\partial_a T^\omega_b)(\partial_b T^\omega_a)
\]
\[
\pm (\nabla_\omega T^\omega_b)(\nabla_\mu T^\omega_c)
\]
\[
D_\omega \nabla_\omega T^\omega_a \pm \partial_b \partial_\omega T^\omega_a + T^\mu_\omega (\partial_\mu T^\omega_\omega).
\]

Consequently
\[
K^\omega_{ba} = (D_\omega T^\omega_a)^\prime = K^\omega_{ba} + \epsilon V^\omega \partial_\omega K^\omega_{ba} + \cdots
\]
\[
\pm K^\omega_{ba} + \epsilon V^\omega \{\partial_\omega \partial_b T^\omega_a + (\partial_\omega \Gamma^\omega_\mu) T^\mu_\omega - (\partial_\omega \Gamma^\omega_\mu) T^\omega_\omega \} + \cdots
\]
\[
= K^\omega_{ba} + \epsilon V^\omega \{ [D_\omega \nabla_\omega T^\omega_a + T^\mu_\omega (\partial_\mu T^\omega_\omega - \partial_\omega T^\omega_\mu)] + (\nabla_\omega T^\omega_b)(\nabla_\mu T^\omega_c) \} - G^\omega_{ba} T^\omega_c
\]

The proof follows from (5,6) and (5,4)c.

**Theorem (5.3).** Put

\[
(5,7)a \quad \bar{L}^\omega_{ba} = D_\omega L^\omega_a + L^\mu_\omega T^\mu_a = Q^\omega_{ba} T^\omega_c.
\]

Then

\[
(5,7)b \quad \bar{L}^\omega_a = D_\omega L^\omega_a - V^\omega T^\mu_\omega R^\mu_\omega - Q^\omega_{ba} T^\omega_c
\]

\[
+ (T^\mu_\omega V^\omega \nabla_\omega - V^\omega \nabla_\omega T^\mu_\omega) T^\omega_a
\]

and

\[
(5,8) \quad \Delta K^\omega_{ba} = \bar{L}^\omega_{ba}.
\]

**Proof.** We have from (5,2) and (5,5)

\[
(5,9) \quad \Delta K^\omega_{ba} = (\tau - \omega) K^\omega_{ba} = D_\omega D_\omega V^\omega - V^\omega D_\omega \nabla_\omega T^\omega_a
\]

\[
- V^\omega (\nabla_\omega T^\mu_\omega) \nabla_\mu T^\omega_a - Q^\omega_{ba} T^\omega_c.
\]

Since

\[
D_\omega L^\omega_a = D_\omega [D_\omega V^\omega - V^\omega \nabla_\omega T^\omega_a]
\]

\[
= D_\omega D_\omega V^\omega - (D_\omega V^\omega) \nabla_\omega T^\omega_a - V^\omega D_\omega \nabla_\omega T^\omega_a,
\]

the equation (5,9) reduces to

\[
\Delta K^\omega_{ba} = D_\omega L^\omega_a + (\nabla_\mu T^\mu_\omega)(\nabla_\omega V^\omega - V^\omega \nabla_\mu T^\mu_\omega) - Q^\omega_{ba} T^\omega_c
\]

which proves (5,7)a and (5,8). The equation (5,7)b follows from (5,7)a and

\[
(5,10) \quad (T^\mu_\omega V^\omega \nabla_\omega - V^\omega \nabla_\omega T^\mu_\omega) T^\omega_a = [L^\mu_\omega + V^\omega T^\mu_\omega(\nabla_\mu \nabla_\omega - \nabla_\omega \nabla_\mu)] T^\omega_a
\]

\[
= L^\mu_\omega T^\omega_a - V^\omega T^\mu_\omega R^\mu_\omega.
\]
6. The field $K^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha}$. The last theorem may easily be generalized:

**Theorem (6,1).** Put

$$L^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} = D_{\alpha}L^{'\alpha-1\ldots\alpha}_{\alpha-1\ldots\alpha} + L^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha}$$  

(6,1a)

$$- \sum_{1}^{r-1} Q_{a_{\alpha}}K^{'\alpha}_{\alpha-1\ldots\alpha+1|c_{a_{\alpha}-1\ldots\alpha}}.$$  

Then

$$L^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} = D_{\alpha}L^{'\alpha-1\ldots\alpha}_{\alpha-1\ldots\alpha} + (T^{'\alpha}_{\alpha}T^{'\alpha}_{\alpha}V_{\alpha} - V^{'\alpha}_{\alpha}T^{'\alpha}_{\alpha}V_{\alpha})K^{'\alpha-1\ldots\alpha}_{\alpha-1\ldots\alpha}$$  

(6,1b)

$$- V^{'\alpha}_{\alpha}T^{'\alpha}_{\alpha}K^{'\alpha}_{\alpha-1\ldots\alpha} - \sum_{1}^{r-1} Q_{a_{\alpha}}K^{'\alpha}_{\alpha-1\ldots\alpha+1|c_{a_{\alpha}-1\ldots\alpha}}.$$  

and

$$\Delta K^{'\alpha}_{\alpha\ldots\alpha} = L^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} \quad (r = 2, 3, \ldots).$$  

**Proof.** (The theorem has already been proved for $r = 2$.) Let us assume that we have proved it for some $r \geq 2$. Then we have from (6,2)

$$K^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} = K^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} + \epsilon L^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} + \ldots$$  

(6,3)

$$= K^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} + \epsilon [L^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} + V^{'\alpha}_{\alpha}\partial_{\alpha}K^{'\alpha}_{\alpha\ldots\alpha}] + \ldots.$$  

On the other hand

$$D_{\alpha}K^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} = \partial_{\alpha}K^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} + \epsilon \left[ V^{'\alpha}_{\alpha}(\partial_{\alpha}K^{'\alpha}_{\alpha\ldots\alpha})T^{'\alpha}_{\alpha+1\ldots\alpha}K^{'\alpha}_{\alpha\ldots\alpha} \right.$$  

$$- \sum_{1}^{r} W_{a_{\alpha}}K^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha+1|a_{\alpha}-1\ldots\alpha} + \ldots$$  

(6,4)

and consequently by virtue of (6,3)

$$K^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} = \partial_{\alpha}K^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} + \epsilon \left\{ \partial_{\alpha}L^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha} + \epsilon \left[ V^{'\alpha}_{\alpha}\partial_{\alpha}K^{'\alpha}_{\alpha\ldots\alpha} \right.$$  

$$+ (\partial_{\alpha}V^{'\alpha}_{\alpha})\partial_{\alpha}K^{'\alpha}_{\alpha\ldots\alpha} + V^{'\alpha}_{\alpha}\partial_{\alpha}K^{'\alpha}_{\alpha\ldots\alpha} + \epsilon \left[ V^{'\alpha}_{\alpha}(\partial_{\alpha}K^{'\alpha}_{\alpha\ldots\alpha})T^{'\alpha}_{\alpha+1\ldots\alpha}K^{'\alpha}_{\alpha\ldots\alpha} \right.$$  

$$- \sum_{1}^{r} W_{a_{\alpha}}K^{'\alpha\ldots\alpha}_{\alpha\ldots\alpha+1|a_{\alpha}-1\ldots\alpha} + \ldots$$  

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\[ \pm D_{a_{r+1}}K_{a_r \ldots a_1}^r + \varepsilon \left\{ D_{a_{r+1}}L_{a_r \ldots a_1}^r \right\} + (\nabla_{a_{r+1}} V^\nu)\nabla_\nu K_{a_r \ldots a_1}^r \]

\[ - \frac{1}{r} W_{a_{r+1}+1} \left( \sum_{s=1}^r K_{a_r \ldots a_s+1 \ldots a_1}^s \right) + V^\nu \left[ (\partial_\nu \Gamma_{\mu \lambda}^r) T_{a_{r+1}}^\mu K_{a_r \ldots a_1}^\lambda + \partial_\nu \partial_\mu K_{a_r \ldots a_1}^r \right] + \ldots. \]  

The tensor \( K_{a_{r+1} \ldots a_1}^r \) satisfies the following relation

\[ K_{a_{r+1} \ldots a_1}^{r'} = K_{a_{r+1} \ldots a_1}^r + \varepsilon V^\nu \partial_\nu K_{a_{r+1} \ldots a_1}^r + \ldots \]

\[ = K_{a_{r+1} \ldots a_1}^r + \varepsilon V^\nu \partial_\nu D_{a_{r+1}} K_{a_r \ldots a_1}^r + \ldots \]

\[ \pm K_{a_{r+1} \ldots a_1}^r + \varepsilon V^\nu \left[ \partial_\nu \partial_\mu K_{a_r \ldots a_1}^r \right] + \ldots. \]

(6.5)

Subtracting (6.5) from (6.4) we have

\[ \star K_{a_{r+1} \ldots a_1}^{r'} - K_{a_{r+1} \ldots a_1}^{r''} = \varepsilon \left\{ D_{a_{r+1}} L_{a_r \ldots a_1}^r \right\} + (\nabla_{a_{r+1}} V^\nu)\nabla_\nu K_{a_r \ldots a_1}^r \]

\[ + (\nabla_{a_{r+1}} V^\nu)\nabla_\nu K_{a_r \ldots a_1}^r - \sum_{s=1}^r Q_{a_{r+1} a_{r+1}+1}^s K_{a_r \ldots a_s+1 \ldots a_1}^s \]

\[ + V^\nu (\partial_\nu \partial_\mu + \partial_\mu \partial_\nu) K_{a_r \ldots a_1}^r + \ldots. \]

(6.6)

Since

\[ (\partial_{a_{r+1}} \partial_\nu - \partial_\mu \partial_{a_{r+1}}) K_{a_r \ldots a_1}^r = [T_{a_{r+1}}^\mu \partial_\mu \partial_\nu - (\partial_\nu T_{a_{r+1}}^\mu) \partial_\mu - T_{a_{r+1}}^\mu \partial_\mu \partial_\nu] K_{a_r \ldots a_1}^r \]

\[ \pm - (\nabla_\nu T_{a_{r+1}}^\mu) \nabla_\mu K_{a_r \ldots a_1}^r, \]

the equation (6.6) reduces to
In this tensor equation we may obviously suppress the dot in \(\dot{=}\) and consequently if we put
\[
L_{a_{r+1}}^{\prime}\ldots a_1 = D_{a_{r+1}}L_{a_r}\ldots a_1 + L_{a_{r+1}}^{\prime}\nabla_{\rho}K_{a_r}\ldots a_1
- \sum_1^r \xi a_{a_{r+1}}^{\prime}K_{a_r}\ldots a_{r+1}a_{r+1}\ldots a_1,
\]
we have from (6.7)
\[
\Delta K_{a_{r+1}}^{\prime}\ldots a_r = L_{a_{r+1}}^{\prime}\ldots a_1.
\]
Hence if (6.2) holds for some \(r\) then it holds also for \(r+1\). But (6.2) holds for \(r=2\) and consequently it holds for \(r=2, 3, \ldots\). The remaining part of the theorem (equation (6.1)b) may be proved by means of an equation similar to (5.10)

7. The differential equation \(L_a^\prime=0\).

**Theorem** (7.1). The system of differential equations

\[
(7.1) \quad L_a^\prime = 0 \quad \text{or} \quad \nabla_a V^\prime = V^\lambda \nabla_a T_a^\prime
\]
is completely integrable and its general solution involves \(n\) arbitrary constants.

**Proof.** We have first
\[
(7.2) \quad \nabla_{\{b} V_{a\}} = \nabla_{\{b} T_{a\}}^{\mu} \nabla_\mu V^\prime = (\nabla_{\{b} T_{a\}}^{\mu} V^\prime + T_{\{b} T_{a\}}^{\mu \nu} \nabla_\mu \nabla_\nu V^\prime)
- \frac{1}{2} R_{\omega \lambda}^{\mu \nu} V^{\lambda} T_{ba}^{\omega \mu}
\]
and by virtue of (7.1)
\[
(7.3) \quad \nabla_{\{b} V^\lambda \nabla_{\mid a} T_{\mid a\}} = \nabla_{\{b} V^\lambda \nabla_{\mid a} T_{\mid a\}}^{\prime} + V^\mu T_{\{b} \nabla_{\mid a} \nabla_\mu T_{\mid a\}}^{\prime}
\]
\[
= V^\mu [(\nabla_{\mid a} T_{\{b\}} (\nabla_{\mid a} T_{\mid a\}) + T_{\{b} \nabla_{\mid a} \nabla_\mu T_{\mid a\})].
\]
On the other hand
\[
0 = V^\mu \nabla_{\mid a} T_{\{b} \nabla_{\mid a} T_{\mid a\}}^{\prime} = V^\mu (\nabla_{\mid a} T_{\{b\} \nabla_{\mid a} T_{\mid a\}} + V^\mu T_{\{b} \nabla_{\mid a} \nabla_\mu T_{\mid a\})
\]

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and consequently (7,3) reduces to

\( \nabla_{[b}V^\lambda_{\varphi]}T^r_{a]} = V^\mu T^\mu_{[b}(\nabla_{[\mu}V^{\varphi}_{\nu]\lambda} - \nabla_{[\varphi}V^\nu_{\mu]\lambda})T^r_{a]} = V^\mu T^\mu_{[b}R^r_{\varphi\lambda}. \)

The integrability conditions of (7,1) are obtained by substituting in

\( \nabla_{[b}\nabla_{\varphi]}V^r = \nabla_{[b}V^\lambda_{\varphi]}T^r_{a]} \)

from (7,2) and (7,4). In doing so we obtain

\[
\frac{1}{2} V^\lambda T^\mu_{ba}( - R^r_{\varphi\mu\lambda} - R^r_{\lambda\varphi\mu} + R^r_{\lambda\mu\varphi}) = \frac{1}{2} V^\lambda T^\mu_{ba}(R^r_{\varphi\mu\lambda} + R^r_{\lambda\mu\varphi} + R^r_{\lambda\varphi\mu}) = 0
\]

since

\( R^r_{\varphi\mu\lambda} + R^r_{\lambda\varphi\mu} + R^r_{\lambda\mu\varphi} = 0. \)

Hence the integrability conditions of (7,1) are identically satisfied and consequently (7,1) is completely integrable and its general solution involves \( n \) arbitrary constants.

\textbf{Theorem (7,2).} Let \( V_1, \ldots, V_n \) be \( n \) linear independent solutions of (7,1). Then

\( W^\varphi = V_{[a}^\varphi \partial^\mu_{a}\partial^\nu_{a} V^\nu_{a}] \quad (a, \epsilon = 1, \ldots, n; \ a \neq \epsilon) \)

is a vector solution of (7,1).

\textbf{Proof.} \( W^\varphi \) is obviously a vector. Hence to prove our theorem we have to show that it satisfies the equation

\( \partial_\epsilon W^\varphi = W^\lambda \partial_\lambda T^r_{a}. \)

From (7,1) we have for

\( \partial_\epsilon W^\varphi = \partial_\epsilon V_{[a}^\varphi \partial_\mu_{a} V^\nu_{a}] = V_{[a}^\varphi \partial_\mu_{a} \partial_\nu_{a} V^\nu_{a}], \)

On the other hand \( (\partial_\epsilon \partial_\mu - \partial_\mu \partial_\epsilon) V^\varphi = -(\partial_\mu T^\lambda_{a}) \partial_\lambda V^\varphi \)

and consequently (7,7) reduces to

\( \partial_\epsilon W^\varphi = (\partial_\mu T^\lambda_{a})(V_{[a}^\varphi \partial_\nu_{a} V^\nu_{a}] - V_{[a}^\varphi \partial_\mu_{a} V^\mu_{a}]) + V_{[a}^\varphi \partial_\mu \partial_\nu_{a} V^\nu_{a}] = V_{[a}^\varphi \partial_\mu \partial_\nu_{a} V^\nu_{a}] \partial_\lambda T^\lambda_{a} = W^\lambda \partial_\lambda T^r_{a}. \)

\( 10 \) It is assumed that in the equations (7,5)-(7,8) the brackets [ ] do not affect the Greek and Latin indices.
Remark. The theorem (7,2) shows that the set of all generators $V^\lambda \partial_\lambda$ of infinitesimal transformations (2,1) is closed under the operations of taking linear combination and forming commutators. Hence these generators form a Lie algebra.

8. Special cases.

Definition (8,1). We say that a transformation (2,1) reproduces the family $(V_m)$ if

$$\Delta K^a_{\alpha_1 \ldots \alpha_i} = 0 \quad (r = 1, 2, \ldots; K^a = T^a).$$

Theorem (8,1). A necessary and sufficient condition that $(V_m)$ be reproduced under (2,1) is: $V^\rho$ is a solution of

$$L^\rho_a = 0.$$  

Proof. Let $(V_m)$ be reproduced by (2,1). Then from the first of (8,1) we have (8,2) by virtue of (3,1)c. Conversely, if (8,2) is satisfied, the following equations hold (cf. (3,4)c and (4,8))

$$\star g_{\alpha \beta} = g'_{\alpha \beta} + \ldots,$$

$$\star \Gamma^\rho_{\alpha \beta} = \Gamma^\rho'_{\alpha \beta} + \ldots.$$  

Consequently we have from (4,1)c and (8,3)b

$$\star \Gamma^\rho_{\alpha \beta} = \star \Gamma^\rho_{\alpha \beta} + \star Q^\rho_{\alpha \beta} + \ldots$$  

On the other hand we get by virtue of (8,3)a and (8,3)b

$$\star \Gamma^\rho_{\alpha \beta} = \Gamma^\rho_{\alpha \beta} + \ldots.$$  

Comparing (8,3)(c) and (d) we have

$$Q^\rho_{\alpha \beta} = 0.$$  

The equations (8,2), (8,4) together with (6,2) and (6,1)a lead to (8,1) for $r = 2, 3, \ldots$.

In the next theorem we shall consider a family $(V_m)$ of totally geodesic subspaces, that is, a family for which

$$K^\rho_{\alpha \beta} = 0$$  

and we prove first the following lemma.

Lemma (8,1). If (8,5) holds, then $W_{\alpha \beta \gamma \delta}$ (cf. the equation (4,2)) satisfies

\[11\] This equation imposes some conditions on the structure of the large space $V_n$ (cf. the equation (8,12)).
the following equation

\[ W_{bad} = W_{a bd} = P_{\beta \delta \alpha \mu}^c W_{\alpha} + D_a D_b W_d \]

where \( P_{\beta \delta \alpha \mu}^c \) is the curvature tensor of \( V_m \) and \( W_{\alpha} = \Gamma_{\alpha \beta}^c V_{\beta} \).

**Proof.** Let \( W'(N') \) be the tangential (normal) component of \( V' \)

\[ V' = W' + N', \quad W' = V^\lambda T_\lambda T_c, \quad N_c T' = 0. \]

From (3,5) we have

\[ W_{ab} = T_{(a b)}^\mu V_\mu = T_{(a b)}^\mu V_\mu + T_{(a b)}^\mu V_\mu N_\mu \]

\[ = D_{(a b)} W_b - N_\mu V_{(a T_b^\mu)}. \]

If (8,5) holds then

\[ 0 = N_\mu D_a T_b^\mu = N_\mu (\nabla_a T_b^\mu - \Gamma_{ba}^c T_c^\mu) = N_\mu \nabla_a T_b^\mu \]

and (8,8)a reduces to

\[ (8,8)b \quad W_{ab} = D_{(a W_b)}. \]

Consequently, we get from (4,2)a

\[ (8,9)a \quad W_{a b d} = D_{[a D_d]} W_b + D_{[b D_d]} W_a + D_{[b D_a]} W_d + D_a D_b W_d. \]

Because for any vector \( P_b \) whatsoever

\[ D_{[a D_d]} P_b + D_{[a D_b]} P_a + D_{[b D_a]} P_d = 0 \]

\[ D_{[b D_d]} P_a = 1/2 P_{b a d} P_c \]

where \( P_{b a d}^c \) is the curvature tensor of \( V_m \), (8,9)a reduces to (8,6).

**Definition (8,2).** If \( A = 0 \) is an analytic expression for a property of \( (V_m) \), then we say that (2,1) preserves this property if \( \Delta A = 0 \).

**Theorem (8,2).** Let \( (V_m) \) be a family of totally geodesic subspaces. A necessary and sufficient condition that this property be preserved by (2,1) is: The vector \( V' \) is a solution of any one of the following equations

(a) \( L_{ab} = 0 \)

(b) \( D_b D_a V' - V^\mu T_{ba}^{\mu \lambda} R_{\mu \lambda} - W_{ba} T_c' = 0 \)

(c) \( D_b D_a N' - N^\mu T_{ba}^{\mu \lambda} R_{\mu \lambda} = 0 \)

where in the last equation

\[ N' = V^\lambda (\delta_\lambda^c - T_c T_\lambda^c). \]
A sufficient but not necessary condition is:

The vector $V'$ is a solution of (8,2). If (8,10) but not (8,2) is satisfied by $V'$, then $(V_m)$ is not reproduced. If (8,2) holds, then (8,10) is satisfied and $(V_m)$ is reproduced.

**Proof.** According to our assumption the transformation (2,1) carries a family $(V_m)$ in a family $(V'_m)$. Hence in order to prove our theorem we have only to look for conditions that the equation $K_{ab}^* = 0$ be satisfied. The necessity and sufficiency of (8,10)a follows at once from (5,8). If (8,5) is satisfied, then we have

\[(8,11)a \quad K_{ab}^* = K_{ab} + \epsilon V^\lambda \partial_\lambda K_{ab} + \cdots = 0 + \cdots\]

and consequently

\[(8,11)b \quad \Delta K_{ab}^* = (\tau - \omega)K_{ab}^* = \tau K_{ab}^*.
\]

Hence, we have from (8,11)b, (5,8) and (5,2) that in our case a necessary and sufficient condition for $\Delta K_{ab}^* = 0$ is that $V'$ be a solution of (8,10)b and this equation is (in our case) equivalent to (8,10)a. On the other hand, the integrability conditions of (8,5) are

\[(8,12) \quad P_{cba}^d T_d^r = T_{cba}^{\mu\lambda} K_{ab}^r
\]

and consequently $T_{cba}^{\mu\lambda} R_{ab}^r$ is in $(V_m)$. If we decompose $V'$ according to (8,7), then we have from (8,12)

\[(8,13)a \quad - V^r T_{ba}^{\mu\lambda} R_{ab}^r = - W T_{cba}^{\mu\lambda} R_{ab}^r - N T_{ba}^{\mu\lambda} R_{ab}^r
\]

On the other hand, if (8,5) holds then

\[(8,13)b \quad D_b D_a V' = D_b D_a W^c T_c^r + D_b D_a N^c = (D_b D_a W^c) T_c^r + D_b D_a N^c.
\]

Furthermore, if (8,5) holds we have according to (8,6)

\[(8,13)c \quad - W^c T_c^r = - P_{bda}^e W^d T_c^r - (D_b D_a W^c) T_c^r
\]

Comparing (8,10)b with (8,13) we have

\[(8,14)a \quad D_b D_a V' - V^r T_{ba}^{\mu\lambda} R_{ab}^r - W_{ba}^c T_c^r
\]

\[= T_e [D_b D_a W^d - W^e P_{cba}^d + P_{cba}^d W^e - D_b D_a W^d]
\]

\[+ D_b D_a N^c - N^c T_{ba}^{\mu\lambda} R_{ab}^r = 0.\]

\[\text{\textsuperscript{11} We use here the well known identities } P_{basc} = P_{asbc} = - P_{a先把.}\]
Since
\[
(D_b D_a - D_a D_b) W^d = - P_{bac}^d W^e = P_{abc}^d W^e = -(P_{bae}^d + P_{cae}^d) W^e
\]
we obtain from (8,14)a
\[
(8,14)b \quad D_b D_a V^e - V^e T_{ba} R_{a\mu\lambda} - W^e T_c = D_b D_a N^e - N^e T_{ba} R_{a\mu\lambda}
\]
and this equation proves the statement about (8,10)c. The last part of the theorem follows easily from Theorem (8,1).

**Remark I.** The statement about the equation (8,10)b is a generalization of Levi-Civita's result for \( m = 1 \).

**Remark II.** Let \( m = 1, n = 2 \). Then
\[
R_{a\mu\lambda}^* = 2 K g_{\lambda\mu},
\]
(8,15)
\[
N^{\mu} T^\nu_\lambda R^\lambda_{\nu\mu\lambda} = - K N^\nu T_{\mu}, \quad (T = g_{\mu\lambda} T^\lambda_{\nu\mu\lambda}).
\]
If \( \eta^1 = s \), \( s \) being the arc of \( V_{11} \), then \( D_1 = T^1_{\lambda} T_{\lambda} \) and \( T = 1 \). Moreover, if \( j^\nu \) is the unit vector in the direction of \( D_1 T^\nu_{1} \), then
\[
(8,16)
D_1 T^\nu_{1} = j^\nu k, \quad D_1 j^\nu = -k T^\nu_{1},
\]
where \( k \) is the curvature of \( V_{11} \). Because we may put \( N^\nu = y j^\nu \), \( y \) being a factor of proportionality, we have on account of (8,16)
\[
(8,17)
D_1 N^\nu = y j^\nu - yk T^\nu_{1}.
\]
Consequently
\[
D_1^2 N^\nu = y^\nu j^\nu - 2 y' k T^\nu_{1} - yk^2 T^\nu_{1} - yk^2 j^\nu.
\]
If \( (V_1) \) consists of geodesic lines, we have \( k = 0 \) and
\[
(8,18)a \quad j^\nu D_1^2 N^\nu = y^\nu.
\]
On the other hand, we obtain from (8,15) for \( \eta^1 = s \)
\[
(8,18)b \quad j^\nu N^{\nu} T^\lambda_{1} T_{1} R^\lambda_{1 \mu\lambda} = - K y.
\]
Comparing (8,10)c and (8,18) we have
\[
y'' + K y = 0,
\]
---

which is the well known equation found by Jacobi.

**Bibliography**


**Indiana University**