ON THE MATRIC EQUATION $X^2 + AX + XB + C = 0$

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The equation

$$(1) \quad X^2 + AX + XB + C = 0,$$

where $A$, $B$, and $C$ are $n \times n$ matrices with elements in $F$, the field of complex numbers or a subfield thereof will be solved for $n \times n$ matrices $X$ with elements in $F$. Throughout the discussion which follows, the elements of all matrices and the coefficients of all polynomials which arise will be in the field $F$ even when this is not specifically stated. Moreover the similarity of matrices and the reducibility of polynomials will be valid under the rational operations of $F$.

It should be remarked that equation (1) becomes unilateral if $X$ be replaced by $Y - A$ or by $Z - B$ and in that form has been studied heretofore.¹ This fact was noted by the writer only after the essentials of the present note had been discovered.

1. Necessary conditions. Let

$$R = \begin{pmatrix} -B & I \\ -C & A \end{pmatrix}$$

and let $X$ be any solution of (1) with elements in $F$;² then

$$(X, I) \begin{pmatrix} 0 & I \\ I & -X \end{pmatrix} R \begin{pmatrix} 0 & I \\ I & -X \end{pmatrix} = \begin{pmatrix} X + A & -X^2 - AX - XB - C \\ I & -X - B \end{pmatrix}$$

(2)

$$= \begin{pmatrix} X + A & 0 \\ I & -X - B \end{pmatrix}.$$

Thus the matrices

$$R \quad \text{and} \quad \begin{pmatrix} X + A & 0 \\ I & -X - B \end{pmatrix}$$

are similar, and $|R - \lambda I|$ is reducible to polynomials $f(\lambda)$ and $g(\lambda)$,


² The letter $I$ will represent unit matrices of order $n$ or $2n$ to agree with that of other matrices in the same expression. Thus in $R$ above $I$ is of order $n$. 586
namely the characteristic polynomials of $X + A$ and $-X - B$, respectively, with coefficients in $F$. The following theorem, which is implied by the above, will be a guide in the quest of solutions of the given equation.

**Theorem I.** If equation (1) has a solution with elements in $F$, then there exists at least one pair of polynomials $f_\alpha(\lambda)$ of degree $\alpha \leq n$ and $g_\beta(\lambda)$ of degree $\beta \leq n$ with coefficients in $F$ such that $f_\alpha(R)g_\beta(R) = 0$, where $f_\alpha(\lambda)g_\beta(\lambda)$ is not necessarily the minimum polynomial satisfied by $R$ but is a divisor of $|R - \lambda I|$ and such that $f_\alpha(X + A) = 0$ and $g_\beta(-X - B) = 0$.

Polynomials $f_\alpha(\lambda)$ and $g_\beta(\lambda)$, of degree $\alpha \leq n$ and $\beta \leq n$ respectively, such that $f_\alpha(\lambda)g_\beta(\lambda)$ is a divisor of $|R - \lambda I|$ and a multiple of the minimum polynomial satisfied by $R$ will be designated as admissible polynomials.

Let $f_\alpha(\lambda)$ be an admissible polynomial such that $f_\alpha(X + A) = 0$, where $X$ is a solution of (1) and let

$$f_\alpha(R) = \begin{pmatrix} U & M \\ V & N \end{pmatrix},$$

where $U$, $V$, $M$, and $N$ are polynomials in the matrices $A$, $B$, and $C$; hence according to (2) we have

$$\begin{pmatrix} X, I \\ I, 0 \end{pmatrix} f_\alpha(R) \begin{pmatrix} 0, I \\ I, -X \end{pmatrix} = \begin{pmatrix} f_\alpha(X + A), & 0 \\ f_\alpha(-X - B), \ast \end{pmatrix} = \begin{pmatrix} XM + N, XU + V - (XM + N)X \\ M, U - MX \end{pmatrix};$$

and consequently the equations,

$$(3) \quad XM + N = 0, \quad XU + V = 0,$$

are simultaneously satisfied whether $M$ be singular or not, and

$$(4) \quad (X, I)f_\alpha(R) = (X, I)\begin{pmatrix} U, M \\ V, N \end{pmatrix} = 0.$$

Now let

$$f_\alpha(R)\begin{pmatrix} I \\ -X_1 \end{pmatrix} = \begin{pmatrix} U, M \\ V, N \end{pmatrix}\begin{pmatrix} I \\ -X_1 \end{pmatrix} = 0,$$

and $M$ be nonsingular. Then $f_\alpha(R)$ and $M$ have the same rank $n$.

*It can be shown that $M$ is singular only if, but not necessarily if, $f_\alpha(\lambda)$ and $g_\beta(\lambda)$ have a common factor.*
and a unique solution of this equation exists such that
\[(5) \quad U - MX_1 = 0, \quad V - NX_1 = 0.\]
We shall show that \(X_1\) so obtained is also a solution of (1) and that
\[(6) \quad X_1 = -M^{-1}(X + B)M - A,\]
where \(X\) is a solution of (1) and as such satisfies (4).

Because \(R\) and \(f_a(R)\) are commutative, we readily obtain the relations:
\[(7) \quad N = U + MA + BM,\]
\[V = BU - UB - MC = AN - NA - CM,\]
and by (5)
\[V - NX_1 = BU - UB - MC - (U + MA + BM)X_1\]
\[= BMX_1 - MX_1B - MC - (MX_1 + MA + BM)X_1\]
\[= -M(X_1^2 + AX_1 + X_1B + C) = 0.\]
Now \(M\) is nonsingular and \(X_1\) is therefore a solution of (1). Equation (6) follows from (3), (5), and the first equation of (7).

The solutions \(X\) and \(X_1\) of equation (1) are such that \(f_a(X + A) = 0\) and \(f_a(-X_1 - B) = 0\) and arise in case \(f_a(R)\) and
\[
\begin{pmatrix}
0, 0 \\
M, 0
\end{pmatrix}
\]
are equivalent under elementary transformations since
\[
\begin{pmatrix}
X, I \\
I, 0
\end{pmatrix} f_a(R)\begin{pmatrix}
0, I \\
I, -X_1
\end{pmatrix} = \begin{pmatrix}
X, I \\
I, 0
\end{pmatrix} \begin{pmatrix}
U, M \\
V, N
\end{pmatrix} \begin{pmatrix}
0, I \\
I, -X_1
\end{pmatrix} = \begin{pmatrix}
0, 0 \\
M, 0
\end{pmatrix}.
\]

The following theorem is proved.

**Theorem II.** If \(X\) is a solution of equation (1) with elements in \(F\), then there exists a polynomial \(f_a(\lambda)\) of degree \(\alpha \leq n\) with coefficients in \(F\) such that equation (4) is satisfied. If in addition \(M\) is nonsingular, \(X_1 = -M^{-1}(X + B)M - A\) is also a solution of (1) with elements in \(F\).

2. **The converse problem.** We shall state the theorem:

**Theorem III.** If \(f_a(\lambda)\) is a polynomial of degree \(\alpha \leq n\) with coefficients in \(F\) such that
is of rank $n$ and if $M$ is nonsingular, then $X$ such that $(X, I) f_{\alpha}(R) = 0$

is a solution of equation (1) with elements in $F$ and $X_1 = -M^{-1}(X + B)M$

$- A$ is likewise such a solution.

The proof of this theorem is entirely analogous to that above by which $X_1$ was shown to be a solution of (1) and will not be given here. Though Theorem III holds for any polynomial $f_{\alpha}(\lambda)$ of whatever degree in $\lambda$; those of lowest degree which may satisfy the hypotheses will be found only among the admissible polynomials defined above.

**Addendum.** The referee's suggestion that the results above might be extended to the equation

$$XDX + AX + XB + C = 0,$$

where $D$ is nonsingular, was anticipated by Professor James H. Bell in a letter to the writer dated March 3, 1949. This generalization appeared of slight significance because the product of this equation on the right (on the left) by $D$ can readily be transformed to (1) in case $D$ is nonsingular. However, the writer has subsequently noted that if $I$ be replaced by $D$ in $R$ and consequently if $X + A$ be replaced by $XD + A$ and $X + B$ by $DX + B$, as suggested by the referee, the content of the above paper after appropriate minor changes is still valid even when $D$ is singular. Moreover, in case $D$ is singular, this equation is not transformable to the unilateral form and consequently is of particular interest. The writer wishes to express his indebtedness to Professor Bell and to the referee.

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