A NOTE ON THE CHARACTERISTIC POLYNOMIALS OF CERTAIN MATRICES

WILLIAM T. REID

The purpose of this note is to establish the following theorem on matrices with elements in an arbitrary field \( \mathbb{F} \). In particular, \( I \) denotes the \( n \times n \) identity matrix; \( 0 \) is used for the zero matrix, square or rectangular, its dimensions being clear in each case from the context. The determinant of an \( n \times n \) matrix \( M(\lambda) \) with elements in \( \mathbb{F}(\lambda) \), the ring of polynomials in \( \lambda \) with coefficients in \( \mathbb{F} \), is denoted by \( |M(\lambda)| \); in particular, if \( M \) is an \( n \times n \) matrix with elements in \( \mathbb{F} \), then \( |\lambda I - M| \) is the characteristic polynomial of \( M \).

**Theorem.** If \( A \) is an \( n \times m \) matrix, and \( D \) is an \( n \times n \) matrix, then in order that \( |\lambda I - AB| = |\lambda I - AB - D| \) for arbitrary \( m \times n \) matrices \( B \) it is necessary and sufficient that \( D \) be nilpotent and \( DA = 0 \). In particular, if \( C \) is an \( m \times n \) matrix, then in order that \( |\lambda I - AB| = |\lambda I - A(B + C)| \) for arbitrary \( m \times n \) matrices \( B \) it is necessary and sufficient that \( ACA = 0 \).

Presented to the Society, September 2, 1949; received by the editors April 5, 1949 and, in revised form, July 23, 1949.

1 In the original manuscript of this paper attention was restricted to \( \mathbb{F} \) the field of complex numbers. The proof there given of the fact that the nilpotency of \( D \) and \( DA = 0 \) imply \( |\lambda I - AB| = |\lambda I - AB - D| \), for arbitrary \( B \), was valid for arbitrary fields \( \mathbb{F} \). The proof here given of the necessity of \( DA = 0 \) for \( \mathbb{F} \) an arbitrary field was suggested by the referee.
An alternate proof of the sufficiency part of the statement in the last sentence of the theorem has been obtained recently by Parker,\(^2\) in generalizing an earlier result of A. Brauer.\(^3\)

The principal tool used in the proof of the above theorem is the fact that a matrix \(D\) is nilpotent if and only if \(\lambda^n = |\lambda I - D|\). That \(\lambda^n = |\lambda I - D|\) implies the nilpotency of \(D\) follows from the Cayley-Hamilton theorem. On the other hand, if \(D\) is nilpotent of index \(s\), then \(|\lambda I - D|^t = \lambda^n\) for \(r \geq s\), while if \(\lambda^n = |\lambda I - D|^t\) for \(r \geq t+1 > 1\), then \(|\lambda I - D|^t = |\lambda I - D|^{t+1} = |\lambda^2 I| = \lambda^{2n}\), so that \(\lambda^n = |\lambda I - D|^t = \lambda^n\), and by induction \(|\lambda I - D| = \lambda^n\).

The sufficiency part of the statement in the first sentence of the above theorem is established by noting that if \(D\) is nilpotent and \(DA = 0\), then \(\lambda^n |\lambda I - AB| = (\lambda I - D)(\lambda I - AB)\) implies \(\lambda^n (\lambda I - AB - D)\) for arbitrary \(m \times n\) matrices \(B\).

On the other hand, suppose that \(|\lambda I - AB| = |\lambda I - AB - D|\) for all \(m \times n\) matrices \(B\). The choice \(B = 0\) gives \(|\lambda I - D| = \lambda^n\), and hence the nilpotency of \(D\). Then \(|\lambda^2 I - \lambda(AB + D)| = \lambda^n |\lambda I - (AB + D)| = |\lambda I - D| \cdot |\lambda I - AB| = |\lambda^2 I - \lambda(AB + D) + DAB|\) for arbitrary \(m \times n\) matrices \(B\). Now if \(DA = K = \|K_{ij}\| (i = 1, \ldots, n; \beta = 1, \ldots, m)\) has a nonzero element \(K_{pq}\), then the choice \(B_0 = \| \delta_{\alpha q} \delta_{p j} K^{-1}_{pq} \| (\alpha = 1, \ldots, m; j = 1, \ldots, n)\) gives \(DAB_0 = G = \| G_{ij}\| (i, j = 1, \ldots, n)\) with \(G_{ij} = 0\) if \(j \neq p, G_{pp} = 1\). If \(AB_0 + D = H = \| H_{ij}\| (i, j = 1, \ldots, n)\), then from the above, \(|\lambda^2 I - \lambda H| = |\lambda^2 I - \lambda H + G|\). This is impossible, however, since \(|\lambda^2 I - \lambda H + G| = |\lambda^2 I - \lambda H| + |R(\lambda)|\), where \(R_{ij}(\lambda) = \lambda^2 \delta_{ij} - \lambda H_{ij}\) if \(j \neq p, R_{pp}(\lambda) = G_{pp}\), and consequently \(|R(\lambda)|\) is a non-zero polynomial \(\lambda^{2n-2}\) + terms of lower degree. Hence \(DA = 0\), and the statement of the first sentence of the theorem is proved.

The second sentence of the theorem follows from the first part, together with the remark that if \(D = AC\), then \(0 = DA = ACA\) implies \(0 = DAC = D^4\) so that \(D\) is nilpotent.

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