the equation \( \sum_{m=0}^{s} A_m \times K_m X^m = 0 \), if the matrices involved are square.

Results similar to those in this paper may be obtained, in some cases more readily, in the consideration of \( \sum_{m=0}^{s} (X^m K_m) \times A_m = 0 \), \( \sum_{m=0}^{s} A_m \times (K_m X^m) = 0 \), and \( \sum_{m=0}^{s} (X^m K_m) \times A_m = 0 \), and so on, where \( A \times B = (a_{ij} B) \).

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**AVERAGES OF CHARACTER SUMS**

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Suppose that \( \chi \) is a primitive residue character\(^1\) modulo \( k \), \( k > 1 \), and that for \( y \) non-negative, \( S(y) = \sum_{0 \leq l \leq y} \chi(l) \). It is important [see, for example, 11] in the analytic theory of numbers to have as much information as possible about the sums \( S(y) \), in particular about their maximum order of magnitude; it is known (cf. [13; 14; 8]), for example, that \( S(y) \leq k^{1/2} \log k \), but unknown whether or not \( M(\chi) = o(k^{1/2} \log k) \) as \( k \) tends to infinity, where \( M(\chi) \) is the maximum of \( |S(1)|, \cdots, |S(k-1)| \). Hua [4; 5; 6] has shown that it is often helpful to consider the averages \( n^{-1} \sum_{m=0}^{n} S(m) \). In this paper we consider some further developments of this idea.

1. **Preliminaries.** We recall [7, pp. 483–486, 492–494] that if \( \chi \) is a primitive residue character mod \( k \) and if \( \tau(\chi) = \sum_{n=1}^{k} \chi(n) e^{2\pi i n/k} \), then \( |\tau(\chi)| = k^{1/2} \) and

\[
\sum_{n=1}^{k} \chi(n) e^{2\pi i mn/k} = \bar{\chi}(m) \tau(\chi)
\]

for any integer \( m \), \( \bar{\chi} \) being the complex conjugate of \( \chi \).

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\(^1\) For the basic facts about residue characters see [7, pp. 401–414, 478–494]. Numbers in brackets refer to the bibliography.
The function \( S(kx) \) is of bounded variation and period unity, and as such has an everywhere convergent Fourier series. It may be deduced from (1) (cf. \([13, \text{pp. 23}-24]\) and \([8, \text{pp. 81}-82]\)) that this Fourier series is

\[
A(x) = \frac{\tau(x)}{2\pi i} \sum_{m=0}^\infty \frac{\overline{\chi}(m)}{m} e^{-2\pi i mx},
\]

where \( m \) runs over the positive and negative integers and

\[
A(x) = \int_0^1 S(kx) dx = \frac{1}{k} \sum_{m=0}^{k-1} S(m).
\]

Thus if we define a function \( S^*(y) \) as \( S(y) \) if \( y \) is not an integer and as \( S(y) - \chi(y)/2 \) if \( y \) is an integer, we have

\[
S^*(kx) = \left\{
\begin{array}{ll}
A(x) + \frac{\tau(x)}{\pi} \sum_{n=1}^\infty \frac{\overline{\chi}(n)}{n} \sin 2\pi nx & \text{if } \chi(-1) = 1 \\
A(x) - \frac{\tau(x)}{\pi i} \sum_{n=1}^\infty \frac{\overline{\chi}(n)}{n} \cos 2\pi nx & \text{if } \chi(-1) = -1
\end{array}
\right.
\]

for all non-negative \( x \).

2. Another proof of Paley's \( \Omega \)-result for \( S(n) \). If we put \( x = 0 \) in (4) we get a formula for the arithmetic mean \( A(x) \) of \( S(0), \ldots, S(k-1) \), namely

\[
A(x) = \begin{cases} 
0 & \text{if } \chi(-1) = 1 \\
\frac{\tau(x)}{\pi} L(1, \overline{\chi}) & \text{if } \chi(-1) = -1
\end{cases}
\]

where \( L(s, \overline{\chi}) = \sum_{n=1}^\infty \overline{\chi}(n)n^{-s} \) for \( R(s) > 0 \). The second half of formula (5) can of course be proved by the method used in \([9]\) to prove the second part of Satz 217 (of which the second half of (5) is a generalization), but some use of Fourier series is essential.\(^2\)

The second part of (5) enables us, when \( \chi(-1) = -1 \), to use information about the order of magnitude of \( L(1, \overline{\chi}) \) to get information about the order of magnitude of \( A(x) \). For example, a slight change in the argument of \([2]\) shows that for real primitive \( \chi \) with \( \chi(-1) = -1 \) we have

\[
\limsup_{k \to \infty} \frac{L(1, \chi)}{\log \log k} \geq \varepsilon^\gamma,
\]

\(\footnote{Naturally (5) can be proved in a roundabout way by showing that both sides are equal to \( L(0, \chi) \). The argument indicated here essentially stems from Minkowski (cf. \([13, \text{p. 26}]\)).} \)
\( \gamma \) being Euler's constant. (The lim sup is unambiguous, since for each \( k \) there is at most one real primitive \( \chi \) with \( \chi(-1) = -1 \).) Now \( \tau(\chi) = tk^{1/2} \) for real primitive \( \chi \) with \( \chi(-1) = -1 \) [9, Satz 215] and thus (6) gives the following result.

**Theorem 1.** If \( \chi \) runs through all real primitive characters with \( \chi(-1) = -1 \) in order of increasing size of the modulus \( k \) of \( \chi \) and if \( A(\chi) \) is as defined in (3), then

\[
\limsup_{k \to \infty} \frac{A(\chi)}{k^{1/2} \log \log k} \leq \frac{e^\gamma}{\pi}.
\]

This theorem is a slightly stronger form of Paley's result (see [12; 10; 1]) that if \( \chi \) is a primitive character modulo \( k \), then \( M(\chi) = \Omega(k^{1/2} \log \log k) \), where \( M(\chi) \) is the maximum of \( |S(0)|, \ldots, |S(k-1)| \); for our theorem shows that this \( \Omega \)-result is true even for the arithmetic mean of \( S(0), \ldots, S(k-1) \).

We remark in passing that Parseval's formula applied to (2) or (4) gives

\[
\frac{1}{k} \sum_{m=0}^{k-1} |S(m) - A(\chi)|^2 = \int_0^1 |S(kx) - A(\chi)|^2 dx = \frac{k}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{k}{12} \prod_{p | k} \left( 1 - \frac{1}{p^2} \right),
\]

where the ' indicates summation over the positive integers relatively prime to \( k \). Alternatively we have, in view of (5),

\[
\frac{1}{k} \sum_{m=0}^{k-1} |S(m)|^2 = \begin{cases} 
\frac{k}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} & \text{if } \chi(-1) = 1 \\
3 \left( \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} + |L(1, \chi)|^2 \right) & \text{if } \chi(-1) = -1.
\end{cases}
\]

3. **Estimation of certain partial averages of \( S(n) \).** We saw above that if \( \chi \) is a primitive character with \( \chi(-1) = 1 \), then the arithmetic mean of the numbers \( S(0), \ldots, S(k-1) \) is zero. Hence in this case we should expect a fairly good estimate of the arithmetic mean of \( S(0), \ldots, S(n-1) \) for \( n < k \). And in fact Hua [4; 5; 6] has proved that we have

\[
\left| \frac{1}{n} \sum_{m=0}^{n-1} S(m) \right| \leq \frac{1}{2} \left( k^{1/2} - \frac{n}{k^{1/2}} \right) \quad \text{for } 0 \leq n \leq k, \chi(-1) = 1.
\]

If \( \chi \) is a primitive character with \( \chi(-1) = -1 \), such a neat result
cannot be expected, since the arithmetic mean of \(S(0), \cdots, S(k-1)\) is not zero. However we prove the following result, which is valuable for \(n\) not too small relative to \(k\), especially if \(|L(1, \chi)|\) is large.

**Theorem 2.** If \(\chi\) is a primitive character modulo \(k\) with \(\chi(-1) = -1\) and if \(\alpha k \leq n \leq k\), where \(\alpha\) is a number between 0 and 1, then

\[
\left| \frac{1}{n} \sum_{m=0}^{n-1} S(m) - \frac{\tau(x)}{\pi i} L(1, \chi) \right| < \left( 2.1 + \frac{\pi}{4} \log \frac{1}{\alpha} \right) k^{1/2}.
\]

**Proof.** Put \(q = [k/2]\), \(\rho = e^{2\pi i/k}\). Then by (1) we have

\[
\tau(x) S(m) = \sum_{l=0}^{m} \chi(l) \tau(\chi) = \sum_{l=0}^{m} \sum_{h=-q}^{q} \chi(h) \rho^{hl}
\]

\[
= \sum_{h=1}^{q} \chi(h) \sum_{l=0}^{m} (\rho^{hl} - \rho^{-hl})
\]

\[
= \sum_{h=1}^{q} \chi(h) \frac{\rho^{h(m+1)} - 1 - \rho^h + \rho^{-hm}}{\rho^h - 1}
\]

\[
= \sum_{h=1}^{q} \chi(h) \frac{\rho^{h(m+1)} + \rho^{-hm}}{\rho^h - 1}
\]

\[
+ i \sum_{h=1}^{q} \chi(h) \cot \frac{\pi h}{k}.
\]

Hence

\[
\tau(\chi) S(m) = \frac{1}{n} \sum_{h=1}^{q} \chi(h) \frac{\rho^{h(n+1)} - \rho^{-h(n-1)}}{(\rho^h - 1)^2}
\]

\[
+ i \sum_{h=1}^{q} \chi(h) \cot \frac{\pi h}{k}
\]

\[
= \frac{1}{2in} \sum_{h=1}^{q} \chi(h) \frac{\sin (2\pi hn/k)}{\sin^2 (\pi h/k)}
\]

\[
+ i \sum_{h=1}^{q} \chi(h) \cot \frac{\pi h}{k}.
\]

Now \(\tau(x) \tau(\chi) = -\tau(\chi) \tau(\chi) = -k\) for \(\chi(-1) = -1\) and thus

\[
\tau(\chi) \left\{ \frac{1}{n} \sum_{m=0}^{n-1} S(m) - \frac{\tau(x)}{\pi i} L(1, \chi) \right\}
\]

\[
= \frac{\tau(\chi)}{n} \sum_{m=0}^{n-1} S(m) - \frac{i k}{\pi} L(1, \chi)
\]
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\[ (8) \]
\[
\frac{1}{2\pi n} \sum_{h=1}^{q} \overline{\chi}(h) \frac{\sin(2\pi hn/k)}{\sin^2(\pi h/k)} + i \sum_{h=1}^{q} \overline{\chi}(h) \left( \cot \frac{\pi h}{k} - \frac{1}{\pi h/k} \right) - \frac{i k}{\pi} \sum_{h=q+1}^{\infty} \overline{\chi}(h) \] 

say. If we put \( S_1(h) = \sum_{n=q+1}^{h} \overline{\chi}(n) \) and use the fact that \( |S_1(h)| < 2^{-k} \), we have

\[ (9) \]
\[
| R_3 | = \left| \frac{k}{\pi} \sum_{h=q+1}^{\infty} \frac{S_1(h)}{h(h+1)} \right| \leq \frac{2^{-1} k^2}{\pi} \sum_{h=q+1}^{\infty} \frac{1}{h(h+1)} 
\]

Using the fact that \( 0 < x^{-1} - \cot x \leq 2/\pi \) for \( 0 < x \leq \pi/2 \), we find

\[ (10) \]
\[
| R_2 | \leq \frac{2}{\pi} \frac{q}{k} \leq \frac{1}{\pi} k. 
\]

We divide the summation in \( R_1 \) into two parts, namely \( 1 \leq h \leq [\alpha^{-1}+1] \) and \( [\alpha^{-1}+2] \leq h \leq q \). In the first part we use \( \sin(2\pi hn/k) \leq 2\pi hn/k \), in the second part we use \( \sin(2\pi hn/k) \leq 1 \), and in both parts we use \( \sin(\pi h/k) \geq 2h/k \). Thus

\[ (11) \]
\[
| R_1 | \leq \frac{1}{2n} \sum_{h=1}^{[\alpha^{-1}+1]} \frac{2\pi hn/k}{4h^2/k^2} + \frac{1}{2n} \sum_{h=[\alpha^{-1}+2]}^{q} \frac{1}{4h^2/k^2} \leq \frac{\pi}{4} k \left( \frac{3}{2} + \log \frac{1}{\alpha} \right) + \frac{k^2}{8n} \alpha 
\]

Combining (8), (9), (10), and (11), we get our theorem.

4. A theorem of Davenport. Davenport \cite{x} has proved that if \( s \) is a fixed complex number with \( 0 < \sigma = \Re(s) < 1 \), then for any primitive \( \chi \) we have \( |L(s, \chi)| \leq C k^{(1-\sigma)/2} \) where \( C \) is a constant depending on \( s \). In this section we show how Hua's inequality (7) gives a very simple proof of this result in the case \( \chi(-1) = 1 \), with a specific value of the
constant. Unfortunately our Theorem 2 is too weak to give Davenport's result for the case \( \chi(-1) = -1 \).

**Theorem 3.** If \( \chi \) is a primitive character with \( \chi(-1) = 1 \) and if \( 0 < \sigma = R(s) < 1 \), then

\[
| L(s, \chi) | \leq \frac{|s(s + 1)|}{\sigma(1 - \sigma)} k^{(1-\sigma)/2}.
\]

**Proof.** Put \( T(n) = \sum_{m=0}^{n} S(m) \). Then we get by two partial summations

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} S(n)s \int_{0}^{1} \frac{dy}{(n + y)^{s+1}}
\]

\[
= \sum_{n=1}^{\infty} T(n)s(s + 1) \int_{0}^{1} \int_{0}^{1} \frac{dxdy}{(n + x + y)^{s+2}}.
\]

For \( n \leq [k^{1/2}] \) clearly \( |T(n)| \leq \sum_{m=0}^{n} m \leq n^2 \), while for \( n \geq [k^{1/2} + 1] \) we have by (7)

\[
|T(n)| \leq 2^{-1} k^{1/2} \{ n + 1 - (n + 1)^2/k \} \leq 2^{-1} k^{1/2} n.
\]

Thus

\[
| L(s, \chi) | \leq |s(s + 1)| \sum_{n=1}^{\infty} \frac{|T(n)|}{n^{s+2}}
\]

\[
= |s(s + 1)| \left\{ \sum_{n=1}^{[k^{1/2}]} \frac{1}{n^s} + \sum_{n=([k^{1/2}]+1)}^{\infty} \frac{2^{-1} k^{1/2}}{n^{s+1}} \right\}
\]

\[
\leq |s(s + 1)| \left\{ 1 + \int_{1}^{[k^{1/2}]} \frac{dx}{x^s} + 2^{-1} k^{1/2} \int_{[k^{1/2}]}^{\infty} \frac{dx}{x^{s+1}} \right\}
\]

\[
\leq |s(s + 1)| \left\{ \frac{k^{(1-\sigma)/2}}{1 - \sigma} + \frac{2^{-1} k^{1/2}}{\sigma [k^{1/2}]^s} \right\}
\]

\[
\leq |s(s + 1)| \left\{ \frac{k^{(1-\sigma)/2}}{1 - \sigma} + \frac{k^{(1-\sigma)/2}}{\sigma} \right\} = \frac{|s(s + 1)|}{\sigma(1 - \sigma)} k^{(1-\sigma)/2},
\]

which proves our theorem.

**Bibliography**


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