A FACTORIZATION THEOREM FOR HOMOMORPHISMS

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1. The general factoring theorem. The purpose of this note is to prove a general factoring theorem (Theorem 2) for homomorphisms of general algebras, and to show its usefulness by proving the Chevalley-Jacobson density theorem for irreducible rings of endomorphisms (Theorem 3). Theorem 2 is probably true for a very general class of algebraic structures, in the sense of Bourbaki [2, §8], Birkhoff [1], Jónsson-Tarski [6], or as outlined in [3]. However, we prove it here only for a class of algebras $A$ which have a binary operator $+$ and a neutral element $0$, under which $A$ is a group, and which share a class $\Omega$ of operators $\alpha$, applicable to finite or transfinite sequences of elements of $A$, and such that $\alpha(0, 0, \cdots) = 0$. These restrictions imply that $\{0\}$ is a subalgebra of $A$, and allow the definition of "inner" direct sums of subalgebras. In what follows, "algebra" will always refer to such an algebraic structure.

We begin by stating the following well known elementary factoring theorem.

Theorem 1. Let $A$, $B$, and $C$ be algebras. Let $\phi$ and $\psi$ be homomorphisms of $A$ onto $B$, and $A$ into $C$, respectively. Let $\ker(\phi) \subseteq \ker(\psi)$. Then, there is a (unique) homomorphism $\theta$ of $B$ into $C$ such that $\psi = \theta \circ \phi$.

One need only choose $\theta$ as $\{(\phi(x), \psi(x)) | x \in A\}$, a subset of $B \times C$. If $\phi(x) = \phi(x')$, then $x - x' \in \ker(\phi)$ so that $x - x' \in \ker(\psi)$, and $\psi(x) = \psi(x')$; thus, $\theta$ is a function mapping $B$ into $C$ and such that $\psi(x) = \theta(\phi(x))$ for all $x \in A$. The verification that $\theta$ is a homomorphism is easily made.

Our principal theorem is a generalization of this, with one additional restriction. We recall that an algebra $B$ is said to be simple (or irreducible) if $B$ contains no proper subalgebras.

Theorem 2. Let $A$, $B_1$, $B_2$, $\cdots$, $B_n$, and $C$ be algebras, with the $B_i$ simple. Let $\phi_i$ be a homomorphism of $A$ onto $B_i$, and let $\psi$ be a homomorphism of $A$ into $C$; let $K_i = \ker(\phi_i)$, and $K = \ker(\psi)$. Then, the condition $\bigcap K_i \subseteq K$ is necessary and sufficient for the existence of homomorphisms $\theta_i$ of $B_i$ into $C$ such that $\psi = \sum \theta_i \phi_i$.

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1 Numbers in brackets refer to the references at the end of the paper.
We shall say that a finite collection \( \{S_1, S_2, \ldots, S_r\} \) of sets is "free" if no \( S_i \) contains the intersection of the remaining sets. Given any finite collection \( \{S_1, S_2, \ldots, S_r\} \), it is possible to relabel these in such a manner that for some \( r \geq 1 \), the subset \( \{S_1, S_2, \ldots, S_r\} \) is free, while if \( i > r \), \( S_i \cap \cdots \cap S_r \subset S_i \). Doing this to the collection \( \{K_1, \ldots, K_n\} \), we may assume that \( \{K_1, \ldots, K_r\} \) is free, and that \( K_1 \cap \cdots \cap K_r \subset K \). Let \( B = PB_i \), the "outer" direct product of the algebras \( B_1, \ldots, B_r \). \( B \) is again such an algebra, and may be regarded as the "inner" direct sum of subalgebras \( B_i \), isomorphic to the factors \( B_i \). Define a function \( \Phi \) by: \( \Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_r(x)) \); this is a homomorphism of \( A \) into \( B \), and its kernel is exactly \( \cap_i K_i \). Thus, we have two homomorphisms, \( \Phi \) and \( \psi \), of \( A \) into \( B \) and into \( C \), with \( \ker(\Phi) \subset \ker(\psi) \). In order to apply Theorem 1, we must show that \( \Phi \) is onto. Let \( K_j = \cap'_{i \neq j} K_i \), the prime indicating that \( K_j \) is omitted; this is a subalgebra of \( A \), which is mapped by \( \Phi \) into the \( j \)th summand \( B_j \). Since \( K_j \) is not contained in \( K_j \), \( \Phi(K_j) \) is nonzero, and since \( B_j \) is simple, must be \( B_j \) itself. Thus, \( \Phi(A) \) contains each of the summands \( B_j \), and must therefore be \( B \). By Theorem 1, we conclude that there is a homomorphism \( \theta \) of \( B \) into \( C \) such that \( \psi = \theta \Phi \). Splitting \( \theta \) into its components \( \theta_i \), acting on the factors \( B_i \) of \( B \), we have \( \psi = \theta \Phi = \sum \theta_i \phi_i \), where \( \theta_i \) is a homomorphism of \( B_i \) into \( C \). This proves the sufficiency; the necessity is obvious.

We observe that a corresponding theorem may be proved in which the requirement that the algebras \( B_i \) be simple is relaxed to the condition that each is a finite direct sum of simple algebras. The proof is the same, since \( B = PB_i \) is again a direct sum of simple algebras. The same can be said if \( B \) is such that any homomorphism of a subalgebra of \( B \) into \( C \) may be extended to a homomorphism of \( B \) itself into \( C \).

2. The Chevalley-Jacobson density theorem. As an application of Theorem 2, we shall obtain a proof of the fundamental density theorem for irreducible rings of endomorphisms \([4, 5, 7]\).

Theorem 3. Let \( R \) be a simple subring of the ring \( E(V) \) of all endomorphisms of the abelian group \( V \). Let \( D = L_R(V) \), the ring of all \( R \)-linear endomorphisms of \( V \). Then \( R \) is dense in \( L_D(V) \).

What has to be shown is that if \( a_1, \ldots, a_{n+1} \) is any finite ordered set of \( D \)-independent points of \( V \), then there is an element \( x \) of \( R \) such that \( x(a_1) = \cdots = x(a_n) = 0 \), while \( x(a_{n+1}) \neq 0 \). We suppose that no such \( x \) exists. Define functions \( \phi_i \) and \( \psi \) by: \( \phi_i(x) = x(a_i) \), \( \psi(x) = x(a_{n+1}) \). Then, \( \phi_i \) and \( \psi \) are homomorphisms of \( R^+ \), the additive
group of $R$, onto $V$, with the ring $R$ as a ring of operators. Since $R$ is irreducible, $V$ is simple. By the assumption above, $\phi_i(x) = 0$ for $i \leq n$ implies that $\psi(x) = 0$, so that $\cap K_i \subseteq K$. If Theorem 2 is applied, there exist homomorphisms $\theta_i$ of $V$ into $V$—whence $\theta_i \subseteq D$—such that $\psi = \sum \theta_i \phi_i$. This in turn implies that $a_{n+1} = \sum \theta_i a_i$, which contradicts the $D$-independence of $a_1, \ldots, a_{n+1}$.

References