A CHARACTERIZATION OF MINIMAL SURFACES IN ISOTHERMIC REPRESENTATION

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1. **Introduction.** The real functions

$$x_j \equiv x_j(u, v) \equiv x_j(z), \qquad j = 1, 2, 3, \tag{1}$$

defined and continuous in the unit disc $\mathcal{D}$: $|z| < 1$ will be said to define a surface S. In this note we shall study the equation

$$\sum_{j=1}^{3}\left[\iint_{D(z,r)} (\zeta - z)x_j(\zeta)d\xi d\eta\right]^2 = 0, \qquad \zeta = \xi + i\eta, \tag{2}$$

where $D(z, r)$ denotes the closed disc in $\mathcal{D}$ with center z and radius r. The equation (2) may be considered to be a generalization of a familiar equation of Féderoff [3, p. 512].[2]

If the first partial derivatives of the functions (1) are continuous and satisfy

$$E(u, v) = G(u, v), \qquad F(u, v) = 0 \tag{3}$$

in $\mathcal{D}$, where

$$\begin{gathered} E(u, v) \equiv \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)^2, \qquad F(u, v) \equiv \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\,\frac{\partial x_j}{\partial v}\right), \\ G(u, v) \equiv \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial v}\right)^2 \end{gathered} \tag{4}$$

are the coefficients of the first fundamental differential quadratic form of S, then the parameters u, v are said to be *isothermic parameters*, and S is said to be given in *isothermic representation*. If (3) holds, then the map of $\mathcal{D}$ on S is conformal except where $E=G=0$. If $EG-F^2\neq 0$ in $\mathcal{D}$, then we say S is *regular*. From (4) it follows that (3) may be written in the form

$$\sum_{j=1}^{3} (\lambda x_j)^2 = 0, \tag{5}$$

where $\lambda \equiv \partial/\partial u + i\partial/\partial v$ is a differential operator.

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

If the functions (1) are harmonic and satisfy (5) in $\mathcal{D}$, then they have been called a *triple of conjugate harmonic functions* [1]. In terms of these triples, a familiar theorem of Weierstrass may be stated as follows.

THEOREM 1. *A necessary and sufficient condition that the functions* (1), *defined in* $\mathcal{D}$, *be the coordinate functions of a minimal surface in isothermic representation is that they constitute a triple of conjugate harmonic functions.*

2. **Principal results.** We shall make use of the following result.

LEMMA. *If the functions* (1) *are not identically constant in* $\mathcal{D}$, *if they have continuous partial derivatives of the third order in* $\mathcal{D}$, *and if they are the coordinate functions of a regular surface* S, *then a necessary and sufficient condition that they map* $\mathcal{D}$ *isothermically on a surface that lies on a sphere* $\mathcal{S}$ *of finite non-null radius is that*

$$\sum_{j=1}^{3}\left[\iint_{D(z,r)}(\zeta - z)x_j(\zeta)d\xi d\eta\right]^2 = o(r^6) \tag{6}$$

hold at each point z *in* $\mathcal{D}$, *and that*

$$\sum_{j=1}^{3}\left[\iint_{D(z_1,r)}(\zeta - z_1)x_j(\zeta)d\xi d\eta\right]^2 \neq o(r^8) \tag{7}$$

hold at some point z_1 *of* $\mathcal{D}$.

Necessity. If the functions (1) map $\mathcal{D}$ isothermically on a spherical surface S, then it follows from the formulas of Gauss [5, p. 359] that the functions (1) have continuous partial derivatives of all orders in $\mathcal{D}$. Hence, if we use Taylor expansions, we find

$$\begin{aligned}\sum_{j=1}^{3}&\left[\iint_{D(z,r)}(\zeta - z)x_j(\zeta)d\xi d\eta\right]^2 \\ &= \frac{\pi^2 r^4}{16}\sum_{j=1}^{3}(\lambda x_j)^2 + \frac{\pi^2 r^6}{96}\sum_{j=1}^{3}\lambda x_j\cdot\Delta\lambda x_j \\ &\quad + \frac{\pi^2 r^8}{9216}\sum_{j=1}^{3}(3\lambda x_j\cdot\Delta^2\lambda x_j + 4(\Delta\lambda x_j)^2) + o(r^8)\end{aligned} \tag{8}$$

where $\Delta \equiv \partial^2/\partial u^2 + \partial^2/\partial v^2$ is a differential operator. Moreover it is also known that for this representation of S,

$$\sum_{j=1}^{3}\lambda x_j\cdot\Delta\lambda x_j = 0 \tag{9}$$

holds throughout $\mathcal{D}$ [5, p. 361]. From (5), (8), and (9) it follows that (6) holds throughout $\mathcal{D}$.

To show that there exist points at which (7) holds, we shall use a counterpositive proof. Suppose that

$$\sum_{j=0}^{3}\left[\iint_{D(z,r)}(\zeta - z)x_j(\zeta)d\xi d\eta\right]^2 = o(r^8) \tag{10}$$

holds at each point of $\mathcal{D}$. Then it follows from (8) that

$$\sum_{j=1}^{3}(3\lambda x_j\cdot\Delta^2\lambda x_j + 4(\Delta\lambda x_j)^2) = 0 \tag{11}$$

holds throughout $\mathcal{D}$. But for S it is known [5, p. 359] that

$$\sum_{j=1}^{3}\lambda x_j\Delta^2\lambda x_j = -4\alpha^2E\lambda^2E, \quad \sum_{j=1}^{3}(\Delta\lambda x_j)^2 = 4\alpha^2(\lambda E)^2, \quad \alpha \neq 0, \tag{12}$$

where α is a real constant. From (11) and (12) we obtain

$$3E\lambda^2E - 4(\lambda E)^2 = 0. \tag{13}$$

Now we want to show the existence of an open subset $\mathcal{D}^*$ of $\mathcal{D}$ where $\lambda E\neq 0$ and $E\neq 0$. First, if E were identically constant in $\mathcal{D}$, then S would be a plane surface, contrary to hypothesis. Hence there is an open subset $\mathcal{D}_1$ or $\mathcal{D}$ where $\lambda E\neq 0$. Second, if $E=0$ throughout $\mathcal{D}_1$, then $\lambda E=0$ there. Hence there is an open subset of $\mathcal{D}_1$ where $\lambda E\neq 0$ and $E\neq 0$, and this subset is denoted by $\mathcal{D}^*$.

For z in $\mathcal{D}^*$, (13) yields $3\lambda \log(\lambda E) = 4\lambda(\log E)$, and hence

$$\lambda E = E^{4/3}\Phi(z), \tag{14}$$

where

$$\lambda\Phi(z) = 0. \tag{15}$$

From the imaginary part of (13) we obtain

$$3\frac{E_{uv}}{E_u}dv = 4\frac{E_v}{E}dv, \qquad 3\frac{E_{uv}}{E_v} = 4\frac{E_u}{E}du,$$

and hence

$$E_u = E^{4/3}\Phi_1(u), \qquad E_v = E^{4/3}\Phi_2(v), \tag{16}$$

where $\Phi_1(u)$ and $\Phi_2(v)$ are real functions of u and v, respectively. From (14), (15), and (16) we obtain

$$\Phi(z) \equiv \Phi_1(u) + i\Phi_2(v), \tag{17}$$

which, by (15), is an analytic function for z in $\mathcal{D}^*$. If we use the Cauchy-Riemann equations for (17), then we find

$$\Phi_1(u) \equiv 2a_0u + a_1, \qquad \Phi_2(v) \equiv 2a_0v + a_2,$$

where a_0, a_1, a_2 are real constants. Therefore (14) yields

$$\frac{E_u}{E^{4/3}}du = (2a_0u + a_1)du, \qquad \frac{E_v}{E^{4/3}}dv = (2a_0v + a_2)dv,$$

from which we obtain

$$E = -\frac{27}{[a_0(u^2 + v^2) + a_1u + a_2v + a_3]^3}, \tag{18}$$

where a_3 is another real constant. But it is well known that the Gaussian curvature of S is given by [2]

$$K = -\frac{1}{2E}\Delta(\log E). \tag{19}$$

From (18) and (19), we find

$$K = \frac{a_1^2 + a_2^2 - 4a_0a_3}{18}[a_0(u^2 + v^2) + a_1u + a_2v + a_3], \tag{20}$$

which holds throughout $\mathcal{D}^*$. But S is on the sphere $\mathcal{S}$, so that (20) must be identically constant in $\mathcal{D}^*$. Hence $a_0 = a_1 = a_2 = 0$ and therefore $K = 0$; therefore $\mathcal{D}^*$ is mapped on a spherical surface with vanishing curvature. But $\mathcal{S}$ has finite radius, so that we have been led to a contradiction by assuming (10) to hold throughout $\mathcal{D}$. Hence (7) holds for at least one point of, and hence in an open subset of, $\mathcal{D}$.

Sufficiency. By the use of finite Taylor expansions we obtain

$$\begin{aligned}\sum_{j=1}^{3}\left[\iint_{D(z,r)}(\zeta - z)x_j(\zeta)d\xi d\eta\right]^2 \\ = \frac{\pi^2r^4}{16}\sum_{j=1}^{3}(\lambda x_j)^2 + \frac{\pi^2r^6}{96}\sum_{j=1}^{3}\lambda x_j\Delta\lambda x_j + o(r^6).\end{aligned} \tag{21}$$

From (6) and (21) we obtain (5) and (9). If we operate on (5) with the operator $\bar{\lambda}$, we obtain

$$\sum_{j=1}^{3}\lambda x_j\Delta x_j = 0. \tag{22}$$

Operating on (22) with λ, and applying (9) to the result, we obtain

$$\sum_{j=1}^{3} \lambda^2 x_j \Delta x_j = 0. \tag{23}$$

There are four real, linear, homogeneous equations in Δx_j, $j=1, 2, 3$, implied by (22) and (23):

$$\begin{aligned} &\sum_{j=1}^{3} x_{j,u}\Delta x_j = 0, && \sum_{j=1}^{3} x_{j,v}\Delta x_j = 0, \\ &\sum_{j=1}^{3} (x_{j,uu} - x_{j,vv})\Delta x_j = 0, && \sum_{j=1}^{3} x_{j,uv}\Delta x_j = 0. \end{aligned} \tag{24}$$

One solution to the system (24) is

$$\Delta x_j \equiv 0, \qquad j = 1, 2, 3. \tag{25}$$

In this case the functions (1) are harmonic in $\mathcal{D}$, and hence (8) yields (10) for each $D(z, r)$ in $\mathcal{D}$. This contradicts (7). Hence the set $\mathcal{D}^{**}$,

$$\mathcal{D}^{**} \equiv \left[z \,\middle|\, z \in \mathcal{D};\ \sum_{j=1}^{3} (\Delta x_j)^2 \neq 0;\ (24) \text{ holds} \right],$$

is a non-null open subset of $\mathcal{D}$.

For z in $\mathcal{D}^{**}$, the system (24) has a nontrivial solution. Therefore the rank of the matrix

$$\begin{Vmatrix} x_{1,u} & x_{2,u} & x_{3,u} \\ x_{1,v} & x_{2,v} & x_{3,v} \\ x_{1,uu} - x_{1,vv} & x_{2,uu} - x_{2,vv} & x_{3,uu} - x_{3,vv} \\ x_{1,uv} & x_{2,uv} & x_{3,uv} \end{Vmatrix}$$

is less than three. Hence it follows from the definitions of the coefficients e, f, g, of the second fundamental differential quadratic form of S, that

$$e = g, \qquad f = 0, \tag{26}$$

holds in $\mathcal{D}^{**}$. Hence it follows from (5) and (26) that the functions (1) map $\mathcal{D}^{**}$ isothermically on a spherical surface S^{**}. If S^{**} were either a plane or a point, then (25) would hold in $\mathcal{D}^{**}$; hence S^{**} lies on a sphere of finite non-null radius.

Now consider the subset $\mathcal{D}-\mathcal{D}^{**}$ of $\mathcal{D}$, and let $z_0 = u_0 + iv_0$ be a point of $\mathcal{D}-\mathcal{D}^{**}$. If z_0 is a frontier point of $\mathcal{D}-\mathcal{D}^{**}$, then a simple continuity argument shows that the functions (1) map z_0 on the boundary of S^{**}, lying on the same sphere containing S^{**}; moreover, since S is regular, we know that $E(u_0, v_0) \neq 0$, so that the con-

tinuity of the Gaussian curvature shows that $K(u_0, v_0) > 0$. Now if $\mathcal{D} - \mathcal{D}^{**}$ has a component $\mathcal{C}$ with non-null *interior*, then the functions (1) map this interior on a minimal surface $\mathcal{M}$, such that the frontier of $\mathcal{C}$ is mapped on the boundary of $\mathcal{M}$. Since a minimal surface is a surface of nonpositive Gaussian curvature, it follows by a continuity argument again that the Gaussian curvature at points of the boundary of $\mathcal{M}$ satisfy $K(u_0, v_0) \leqq 0$. But at the beginning of this paragraph it was pointed out that $K(u_0, v_0) > 0$ at the points corrresponding to frontier points of $\mathcal{D} - \mathcal{D}^{**}$. Since S^{**} and $\mathcal{M}$ are merely parts of the regular surface S, it follows from this contradiction that the set $\mathcal{D} - \mathcal{D}^{**}$ has no component with a non-null interior, and hence all points $\mathcal{D} - \mathcal{D}^{**}$ are mapped on the boundary of S^{**}. Therefore the functions (1) map all of $\mathcal{D}$ isothermically on a spherical surface, lying on a sphere with finite non-null radius.

The following result does not appear to demand the regularity of the surfaces involved.

THEOREM 2. *If the functions* (1) *have continuous partial derivatives of the third order in* $\mathcal{D}$, *then a necessary and sufficient condition that they be the coordinate functions of a minimal surface in isothermic representation is that*

$$\sum_{j=1}^{3} \left[\iint_{D(z,r)} (\zeta - z) x_j(\zeta) d\xi d\eta \right]^2 = 0 \tag{27}$$

hold for each $D(z, r)$ *in* $\mathcal{D}$.

Necessity. If the functions (1) map $\mathcal{D}$ isothermically on a minimal surface, then it follows from Theorem 1 that (5) and (25) hold, and hence, by the use of Fourier expansions, it follows that (27) holds for each $D(z, r)$ in $\mathcal{D}$.

Sufficiency. If (27) holds, then (5), (22), and (23) follow from (21). Hence we obtain the system (24). Now consider the set $\mathcal{D}^{**}$ defined in the proof of the lemma. As in the proof, the functions (1) map $\mathcal{D}^{**}$ isothermically on a surface S^{**} that lies on a sphere with finite non-null radius. Hence the functions (1) are analytic in the real variables u, v [**5**, p. 358]. Therefore we may use (8) to obtain (11). Hence we find that (18) holds in $\mathcal{D}^{**}$, and consequently, as in the proof of the necessity part of the lemma, the Gaussian curvature of S^{**} is identically zero. Therefore the spherical surface S^{**} is a plane surface, so that (25) holds in $\mathcal{D}^{**}$. From this contradiction we conclude that the open set $\mathcal{D}^{**}$ contains no interior points, hence is a null set. Therefore (5) and (25) hold throughout $\mathcal{D}$, and hence S is a minimal surface given in isothermic representation by the functions (1).

COROLLARY. *If the functions* (1) *have continuous partial derivatives of the third order in* $\mathcal{D}$, *then a necessary and sufficient condition that they be the coordinate functions of a minimal surface in isothermic representation is that* (10) *hold at each point of* $\mathcal{D}$.

3. **Characterization of plane isothermic maps.** We take this occasion to offer a simplification of a proof of a characterization of plane isothermic maps, a result which may be considered to be a generalization to space of the Cauchy and Morera theorems. The simplification consists in deriving some fundamental identities without the use of conformal mapping and *schlicht* functions [**5**, pp. 368–370].

THEOREM 3. *If the functions* (1) *have continuous partial derivatives of the third order in* $\mathcal{D}$, *then a necessary and sufficient condition that they be the coordinate functions of a plane surface in isothermic representation is that*

$$\sum_{j=1}^{3}\left[\int_{\gamma} x_j(\zeta)d\zeta\right]^2 = 0 \tag{28}$$

hold for each closed rectifiable Jordan curve γ *in* $\mathcal{D}$.

PROOF. In the earlier proof it was shown that (28) is a necessary condition that the functions (1) map $\mathcal{D}$ isothermically on a plane surface. It was also shown that if (28) holds for all closed rectifiable Jordan curves γ in $\mathcal{D}$, then the functions (1) constitute a triple of conjugate harmonic functions in $\mathcal{D}$ and hence may be written in the form

$$x_j(z) \equiv F_j(z) + \overline{F_j(z)}, \qquad j = 1, 2, 3,$$

where $F_j(z)$ is analytic in $\mathcal{D}$ and where $\overline{F_j(z)}$ is its complex conjugate.

Now let $z = u + iv$ be an arbitrary point in $\mathcal{D}$, and write

$$F_j(\zeta) = \sum_{n=0}^{\infty} a_{j,n}(\zeta - z)^n, \qquad \overline{F_j(z)} = \sum_{n=0}^{\infty} \bar{a}_{j,n}(\overline{\zeta - z})^n. \tag{29}$$

Now consider the family of triangles $\gamma(r, \alpha)$, with vertices z, $z + re^{i\alpha}$, $z + re^{-i\alpha}$. Then a direct computation, using (28) and (29), yields

$$\begin{aligned} &\sum_{j=1}^{3}\left[\int_{\gamma(r,\alpha)} x_j(\zeta)d\zeta\right]^2 \\ &= -4\cos^2\alpha \sum_{n=0}^{\infty} r^{n+1} \sum_{k=0}^{n} \sin k\alpha \sin (n-k)\alpha \left[\sum_{j=1}^{3} \bar{a}_{j,k}\bar{a}_{j,n-k}\right] = 0, \end{aligned} \tag{30}$$

which must hold for $\gamma(r, \alpha)$ in $\mathcal{D}$. Since (30) holds for all sufficiently

small r, it follows that

$$\sum_{k=0}^{n} \sin k\alpha \sin (n-k)\alpha \left[\sum_{j=1}^{3} \bar{a}_{j,k}\bar{a}_{j,n-k} \right] = 0, \qquad n \geqq 0, \tag{31}$$

holds for all α. From (31) we obtain

$$\sum_{k=1}^{n-1} [\cos (n-2k)\alpha - \cos n\alpha] \left[\sum_{j=1}^{3} \bar{a}_{j,k}\bar{a}_{j,n-k} \right] = 0, \qquad n \geqq 2, \tag{32}$$

which must hold for all α. Now let p be a fixed integer, $1 \leqq p \leqq n-1$. If we multiply both members of (32) by $\cos(n-2p)\alpha$ and then integrate the result over the interval $(0, 2\pi)$, we obtain

$$\bar{a}_{n-p}, \bar{a}_{n+p} = 0, \qquad 1 \leqq p \leqq n-1, n \geqq 2. \tag{33}$$

Now (33) are the identities that had to be proved in order to prove that the functions (1) map $\mathcal{D}$ isothermically on a plane surface.

4. **Conclusion.** The results contained in this note are similar to those obtained before [5]. The one outstanding point of difference is the lack of a characterization of isothermic plane maps by means of an equation similar to (5). It would be interesting to obtain such a characterization.

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