ON THE WALSH-KACZMARZ SERIES
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1. Let \( \{\psi_n(t)\} \) be the Walsh-Kaczmarz functions, which form a normalized set of orthogonal functions. Every function \( f(t) \) integrable in the interval \((0, 1)\) can be expanded by means of the functions \( \psi_n(t) \) in the form

\[
f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t),
\]

where \( c_n \) are defined by the equation \( c_n = \int_0^1 f(t) \psi_n(t) \, dt \). We term (1) the Walsh-Kaczmarz series of \( f(t) \). By \( s_n(t) \) and \( \sigma_n(t) \) we denote the \( n \)th partial sum and the \( n \)th arithmetic mean of the series (1), respectively. We prove the following theorems.

**Theorem 1.** If \( s_{n,r}(t) \) denotes the \( v \)th partial sum of the Walsh-Kaczmarz series of \( f_n(t) \), then

\[
\int_0^1 \left( \sum_{n=0}^{\infty} |s_{n,r}(t)|^2 \right)^{r/2} \, dt \leq A_r \int_0^1 \left( \sum_{n=0}^{\infty} |f_n(t)|^2 \right)^{r/2} \, dt, \quad r > 1.
\]

**Theorem 2.** If \( f(t) \) belongs to \( L^r (r > 1) \), then

\[
B_r \int_0^1 \left( \frac{1}{n} \sum_{n=1}^{\infty} \left| \frac{s_n(t) - \sigma_n(t)}{n} \right|^2 \right)^{r/2} \, dt \leq \int_0^1 |f(t)|^r \, dt \leq C_r \int_0^1 \left( \sum_{n=1}^{\infty} \left| \frac{s_n(t) - \sigma_n(t)}{n} \right|^2 \right)^{r/2} \, dt.
\]

**Theorem 3.** If \( f(t) \) belongs to \( L^r (r > 1) \), then

\[
D_r \int_0^1 \left( \frac{1}{n} \sum_{n=1}^{\infty} \left| s_n(t) - \sigma_n(t) \right|^2 \right)^{r/2} \, dt \leq \int_0^1 |f(t)|^r \, dt \leq E_r \int_0^1 \left( \sum_{n=1}^{\infty} \left| s_n(t) - \sigma_n(t) \right|^2 \right)^{r/2} \, dt.
\]

**Theorem 4.** If \( f(t) \) belongs to \( L^r (r > 1) \), then

\[
\int_0^1 \left( \sup_{0 \leq s_n \leq t} |\sigma_n(t)| \right)^r \, dt \leq F_r \int_0^1 |f(t)|^r \, dt.
\]

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\(^1\) \( A_r, B_r, \cdots \) denote constants depending only on \( r \).
THEOREM 5. If \( f(t) \in L^r \ (r > 1) \) and \( 2 \geq k > 1 \), then

\[
\int_0^1 \left( \sup_{0 \leq n < \infty} \frac{1}{n+1} \sum_{i=0}^n | s_k(t) - f(t) |^k \right)^{r/k} dt \leq C \int_0^1 | f(t) |^r dt.
\]

In particular

\[
\frac{1}{n+1} \sum_{i=0}^n | s_n(t) - f(t) |^2
\]

converges to zero p.p. in \((0, 1)\).

THEOREM 6. If \( f(t) \) belongs to \( L^r \ (r > 1) \), then for almost all \( t \) the sequence of natural numbers can be divided into two complementary subsequences \( \{v_k\} \) and \( \{\mu_k\} \) such that \( s_k(t) \) tends to \( f(t) \) and \( \sum 1/\mu_k \) converges.

Theorem 4 is a special case of a theorem due to Paley [3] which is the maximal theorem for \( \sigma_n^k(t) \ (\delta > 0) \), \( \sigma_n^k(t) \) being the \( n \)th Cesàro mean of the \( \delta \)th order of the series (1). But the complete proof of the latter theorem seems not yet to be published.

Theorem 5 was also stated without proof by Paley [3].

2. Lemma 1. If \( f(t) \) belongs to \( L^r \ (r > 1) \), then

\[
A_r \int_0^1 \left( \sum_{n=0}^\infty | \Delta_n(t) |^2 \right)^{r/2} dt 
\]

\[
\leq \int_0^1 | f(t) |^r dt \leq B_r \int_0^1 \left( \sum_{n=0}^\infty | \Delta_n(t) |^2 \right)^{r/2} dt,
\]

where

\[
\Delta_n(t) = \sum_{\nu=2^\nu}^{2^{\nu+1} - 1} c_{n,\nu} \psi_\nu(t).
\]

This is due to Paley [2].

LEMMA 2. Let

\[
f_n(t) = \sum_{\nu=0}^\infty c_{n,\nu} \psi_\nu(t),
\]

then

\[
\Delta \nu(f_n) = \sum_{\nu=2^\nu}^{2^{\nu+1} - 1} c_{n,\nu} \psi_\nu(t),
\]

Numbers in brackets refer to the bibliography at the end of the paper.
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\[ C_r \int_0^1 \left( \sum_{n,p} |\Delta_s(f_n)|^2 \right)^{r/2} dt \]
\[ \leq \int_0^1 \left( \sum_n f_n^2(t) \right)^{r/2} dt \leq D_r \int_0^1 \left( \sum_{n,p} |\Delta_s(f_n)|^2 \right)^{r/2} dt, \]

where \( r > 1 \).

This is a generalization of Lemma 1. Marcinkiewicz [1] proved this in the trigonometrical case, where the proof depends on the trigonometrical analogue of Lemma 1. Following his argument and using Lemma 1, we can prove Lemma 2.

We shall now prove Theorem 1.\(^2\) Let us put

\[ k = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_h}, \quad g(t) = f(t) \cdot \psi_k(t) = \sum_{m=0}^\infty c_m \psi_m(t), \]

and

\[ g_k(t) = \sum_{m=2^k}^{2^{k+1}-1} c_m \psi_m(t). \]

For

\[ m = 2^{m_1} + 2^{m_2} + \cdots, \quad 0 \leq m_1 < m_2 < \cdots, \]
\[ k = 2^{k_1} + 2^{k_2} + \cdots, \quad 0 \leq k_1 < k_2 < \cdots, \]

define

\[ m + k = \sum_{m_i \neq k_j} 2^{m_i} + \sum_{k_j = m_i} 2^{k_j}, \]
\[ 0 + k = k + 0 = k. \]

Then \( \psi_m(t) \psi_k(t) = \psi_{m+k}(t) \), so

\[ \psi_k(t)f(t) \sim \sum_n c_n \psi_n(t) \psi_k(t) = \sum_n c_n \psi_{n+k}(t) = \sum_m c_{m+k} \psi_m(t), \]

for \( m = n + k \) implies \( n = m + k \). Hence \( c'_m = c_{m+k} \).

Now

\[ \sum_{r=1}^\lambda g_k(t) = \sum_{r=1}^\lambda \sum_{m=2^k}^{2^{k+1}-1} c_m \psi_m(t) \]
\[ = \sum_{r=1}^\lambda \sum_{m=2^k}^{2^{k+1}-1} \psi_k(t)c_m \psi_{m+k}(t) \]
\[ = \psi_k(t) \sum_{n \in A} c_n \psi_n(t) \]

\(^2\) The author has to thank the referee for his valuable suggestions for this proof.
where \( A \) is the set of indices, \( k \) in number,
\[
m + k = (2^{2r} \leq m < 2^{2r+1}; r = 1, 2, \ldots, \lambda).
\]
No two indices are equal, since the non-negative integers form a group under \( \dagger \). Also \( m+k < k \), since
\[
m = 2^{2r} + \mu \ (0 \leq \mu < 2^{2r}), \quad k = 2^{2r} + (k - 2^{2r}),
\]
so
\[
m + k = \mu + (k - 2^{2r}) \leq \mu + k - 2^{2r} < k.
\]
Hence the \( k \) indices \( m+k \) satisfy \( 0 \leq m+k < k \) and are distinct;
\[
\sum_{n \in A} c_n \psi_n(t) = \sum_{n=0}^{k-1} c_n \psi_n(t) = s_k(t).
\]
Thus we get
\[
s_k(t) \psi_k(t) = g_{k_1}(t) + g_{k_2}(t) + \cdots + g_{k_\lambda}(t).
\]
Put
\[
p_n = 2^k + 2^{k_2} + \cdots + 2^{k_\lambda}, \quad k_1 = k_1(n), \ldots, k_\lambda = k_\lambda(n);
\]
\[
\lambda = \lambda(n), \quad 0 \leq k_1 < k_2 < \cdots.
\]
Then we would have, by Lemma 2,
\[
\int_0^1 \left( \sum_{n=1}^{\infty} \left| s_{n,p_n}(t) \right|^2 \right)^{r/2} dt
\]
\[
= \int_0^1 \left( \sum_{n=1}^{\infty} \left| g_{n,k_1} + \cdots + g_{n,k_\lambda} \right|^2 \right)^{r/2} dt
\]
\[
\leq B_r \int_0^1 \left( \sum_{n=1}^{\infty} \sum_{\lambda=1}^{\lambda(n)} \left| g_{n,k_\lambda}(t) \right|^2 \right)^{r/2} dt
\]
\[
\leq B_r \int_0^1 \left( \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left| g_{n,k}(t) \right|^2 \right)^{r/2} dt
\]
\[
= B_r \int_0^1 \left( \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \Delta_k(\psi_{p_n}(t) f_n(t)) \right)^{r/2} dt
\]
\[
\leq B_r D_r \int_0^1 \left( \sum_{n=1}^{\infty} \psi_{p_n}(t) f_n(t) \right)^{r/2} dt
\]
\[
= B_r D_r \int_0^1 \left( \sum_{n=1}^{\infty} \left| f_n(t) \right|^2 \right)^{r/2} dt,
\]
which is the required inequality.
Lemma 3. If \( f(t) \in L^r (r > 1) \), then we have
\[
\int_0^1 \left( \sum_{n=1}^\infty \left| s_n(t) - \sigma_n(t) \right|^2 \right)^{r/2} dt
\]
\[
\leq E_r \int_0^1 \left( \sum_{n=1}^\infty | \Delta_n(t) |^2 \right)^{r/2} dt
\]
\[
\leq F_r \int_0^1 \left( \sum_{n=1}^\infty \left| s_n(t) - \sigma_n(t) \right|^2 \right)^{r/2} dt
\]
\[
\leq G_r \int_0^1 \left( \sum_{n=1}^\infty \left( \sum_{m=0}^{n-1} \left| \delta_{m} \right|^2 \right)^{r/2} \right) dt.
\]

Except for the last inequality, the lemma is proved by Zygmund [4] for the case of Fourier series. He proved it by using the Fourier series analogue of Theorem 1. His method is also applicable for our case. The proof of the last inequality runs as follows. Since
\[
s_n(t) = \sum_{r=0}^n c_r \psi_r(t),
\]
\[
\sigma_n(t) = \sum_{r=0}^n s_r(t)/(n + 1) = \sum_{r=0}^n (1 - r/(n + 1)) c_r \psi_r(t),
\]
we have
\[
s_n(t) = (r + 1) \sigma_n(t) - n \sigma_{n-1}(t), \quad s_n(t) - \sigma_n(t) = n(\sigma_n(t) - \sigma_{n-1}(t))
\]
and
\[
s_n(t) - \sigma_n(t) = \left\{ 1c_1 \psi_1(t) + 2c_2 \psi_2(t) + \cdots + nc_n \psi_n(t) \right\} / (n + 1).
\]
Hence
\[
| \sigma_{n+1}(t) - \sigma_n(t) | \leq \sum_{i=2^{n+1}}^{2^{n+1}+1} | \sigma_i(t) - \sigma_{i-1}(t) |
\]
\[
\leq \sum_{i=2^{n+1}}^{2^{n+1}+1} i(\sigma_i(t) - \sigma_{i-1}(t))^2 \sum_{i=2^{n+1}}^{2^{n+1}+1} 1/i
\]
\[
\leq 2 \sum_{i=2^{n+1}}^{2^{n+1}+1} i(\sigma_i(t) - \sigma_{i-1}(t))^2
\]
\[
\leq 2 \sum_{i=2^{n+1}}^{2^{n+1}+1} | s_i(t) - \sigma_i(t) |^2 / i.
\]
On the other hand
\[ \int_0^1 \left( \sum_{k=1}^\infty \left| s^k(t) - \sigma^k(t) \right|^2 \right)^{r/2} dt \]

\[ \leq \int_0^1 \left\{ \sum_{k=1}^\infty \frac{1}{(2^k + 1)^2} \left( \sum_{i=1}^{2^k} i c\psi_i(t) \right)^2 \right\}^{r/2} dt \]

\[ \leq 2 \int_0^1 \left\{ \sum_{k=1}^\infty \left( \sum_{i=1}^{2^k} i c\psi_i(t) \right)^2 \sum_{\nu=2^k}^{2^{k+1} - 1} 1/\nu^{1/2} \right\}^{r/2} dt \]

\[ \leq A_r \int_0^1 \left\{ \sum_{k=1}^\infty \sum_{\nu=2^k}^{2^{k+1} - 1} \frac{1}{\nu^{1/2}} \left( \sum_{i=1}^{r} i c\psi_i(t) \right)^2 \right\}^{r/2} dt \]

\[ \leq B_r \int_0^1 \left( \sum_{n=1}^\infty \frac{|s_n(t) - \sigma_n(t)|^2}{n} \right)^{r/2} dt \]

by Theorem 1. Since

\[ |s_{n+1}(t) - s_n(t)| \leq |s_{n+1}(t) - s_{n+1}(t)| + |s_n(t) - s_n(t)| \]

we get the required inequality.

By Lemmas 2 and 3, Theorems 2 and 3 are now immediate.

**Lemma 4.** If \( r > 1 \), \( \|A\| \)

\[ A \int_0^1 \left\{ \sum_{n=1}^\infty n(\sigma_n(t) - \sigma_{n-1}(t))^2 \right\}^{r/2} dt \leq \int_0^1 |f(t)|^r dt. \]

Since

\[ s_n(t) = (n + 1)\sigma_n(t) - n\sigma_{n-1}(t), \]

and

\[ n(\sigma_n(t) - \sigma_{n-1}(t)) = s_n(t) - \sigma_n(t) \]

the lemma follows from Theorem 2.

**Lemma 5.** If \( r > 1 \), \( \sup_{0 \leq n < \infty} |s_n(t)| \)

\[ \int_0^1 \left( \sup_{0 \leq n < \infty} |s_n(t)| \right)^r dt \leq A \int_0^1 |f(t)|^r dt. \]

This is due to Paley [2].

With these preparations we will now prove Theorem 5. For \( 2^n < k < 2^{n+1} \), we have

\[ (\sigma_k - \sigma_{k-1})^2 \leq \left\{ \sum_{i=2^n+1}^{2^{n+1}} |\sigma_i(t) - \sigma_{i-1}(t)| \right\}^2 \]
\[ \leq \sum_{i=2^n+1}^{2^{n+1}} i(\sigma_i(t) - \sigma_{i-1}(t))^2 \sum_{i=2^n+1}^{2^{n+1}} 1/i \]
\[ = 2 \sum_{i=2^n+1}^{2^{n+1}} i(\sigma_i(t) - \sigma_{i-1}(t))^2. \]

Since
\[ \sigma_k(t) = \sigma_{2^n}(t) + (\sigma_k(t) - \sigma_{2^n}(t)), \]
we have
\[ \sup |\sigma_k(t)| \leq \sup \sigma_{2^n}(t) + 2 \sum_{i=1}^{\infty} i(\sigma_i(t) - \sigma_{i-1}(t))^2. \]

By Lemmas 4 and 5, we get Theorem 4.

Theorem 5 is easily deduced from Theorems 4 and 2.

Theorem 6 is proved by Zygmund’s argument [5].

**BIBLIOGRAPHY**


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