ON THE PRODUCTS OF FUNCTIONS REPRESENTED
AS CONVOLUTION TRANSFORMS

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The authors have in a number of papers, [1], [2], and [3],¹ considered the convolution transforms

\begin{equation}
(1) \quad f(x) = \int_{-\infty}^{\infty} G(x - t) d\alpha(t)
\end{equation}

with kernels of the form

\begin{equation}
G(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{E(s)^{-1} e^{st}}{s} ds,
\end{equation}

\begin{equation}
E(s) = e^{bs} \prod_{k=1}^{\infty} \left( 1 - \frac{s}{a_k} \right) e^{s/a_k}.
\end{equation}

Here \( b, \{a_k\}_1^{\infty} \) are real constants subject only to the restriction that \( \sum_{k=1}^{\infty} a_k^{-2} < \infty \). Let \( \lambda_1 \) and \( \lambda_2 \) be strictly positive real numbers such that \( \lambda_1 + \lambda_2 \leq 1 \). In the present paper we shall consider results concerning products of functions representable in the form (1) which are consequences of the identity

\begin{equation}
(1 - Da^{-1})f(\lambda_1 x)g(\lambda_2 x) = \lambda_2 f(\lambda_1 x) [g(\lambda_2 x) - a^{-1} g'(\lambda_2 x)]
+ \lambda_1 g(\lambda_2 x) [f(\lambda_1 x) - a^{-1} f'(\lambda_1 x)] + (1 - \lambda_1 - \lambda_2) f(\lambda_1 x) g(\lambda_2 x).
\end{equation}

We recall the definitions \( a_1 = \max_{a_k < 0} [a_k, -\infty] \), \( a_2 = \min_{a_k > 0} [a_k, \infty] \), see [2; 1].

**Theorem 1.** Let \( G(x) \subseteq \text{class I} \) and let

\begin{equation}
f(x) = \int_{-\infty}^{\infty} G(x - t) d\alpha(t), \quad g(x) = \int_{-\infty}^{\infty} G(x - t) d\beta(t)
\end{equation}

with \( \alpha(t) \in \uparrow, \beta(t) \in \uparrow \). If \( 0 < \lambda_1, 0 < \lambda_2, \lambda_1 + \lambda_2 \leq 1 \), then

\begin{equation}
f(\lambda_1 x)g(\lambda_2 x) = \int_{-\infty}^{\infty} G(x - t) d\gamma(t)
\end{equation}

with \( \gamma(t) \in \uparrow \).

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¹ The first number in brackets refers to the references cited at the end of the paper. The number following the semicolon gives the section number of the reference cited.

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Proof. By [3; 36] necessary and sufficient conditions for \( h(x) \) to have the representation

\[
(4) \quad h(x) = \int_{-\infty}^{\infty} G(x - t) d\delta(t)
\]

with \( \delta(t) \in \uparrow \) are:

a. \( h(x) \in C^\infty \quad (-\infty < x < \infty) \),

b. \( h(x) = o(e^{x^2}) \quad (x \to -\infty) \)

\[ = o(e^{x^2}) \quad (x \to +\infty), \]

c. \( \prod_{i=1}^{n} (1 - DA_i^{-1}) h(x) \geq 0 \quad (-\infty < x < \infty) \)

for every selection \( \{A_1, A_2, \cdots, A_n\} \) from \( \{a_1, a_2, \cdots\} \). Our assumptions imply that \( f(x) \) and \( g(x) \) fulfill these conditions; it follows that \( f(\lambda_1 x)g(\lambda_2 x) \) also satisfies these conditions (a and b trivially, and c because of (3)). Our desired result follows.

The following theorem may be proved similarly. See [3; 36, 37].

Theorem 2. Let \( G(x) \in \text{class II or III} \) and let

\[
f(x) = \int_{-\infty}^{\infty} G(x - t) d\alpha(t) \quad (x > r_1),
\]

\[
g(x) = \int_{-\infty}^{\infty} G(x - t) d\beta(t) \quad (x > r_2)
\]

with \( \alpha(t) \in \uparrow \), \( \beta(t) \in \uparrow \). If \( 0 < \lambda_1, 0 < \lambda_2, \lambda_1 + \lambda_2 \leq 1 \), then

\[
f(\lambda_1 x)g(\lambda_2 x) = \int_{-\infty}^{\infty} G(x - t) d\gamma(t) \quad (x > \text{Max} [r_1/\lambda_1, r_2/\lambda_2])
\]

where \( \gamma(t) \in \uparrow \).

Using a further representation theorem [3; 32] one may establish

Theorem 3. Let \( p, q, \) and \( r \) be numbers greater than one such that \( r^{-1} = p^{-1} + q^{-1} \), and let

\[
f(x) = \int_{-\infty}^{\infty} G(x - t)e^{-c_1 t} \phi(t) dt,
\]

\[
g(x) = \int_{-\infty}^{\infty} G(x - t)e^{-c_2 t} \psi(t) dt,
\]

where \( \alpha_1 < c_1 < \alpha_2, \alpha_1 < c_2 < \alpha_2, \) and where \( \phi(t) \in L_p(-\infty, \infty), \psi(t) \)
If $0 < \lambda_1$, $0 < \lambda_2$, $\lambda_1 + \lambda_2 \leq 1$, then

$$
f(\lambda_1 x)g(\lambda_2 x) = \int_{-\infty}^{\infty} G(x - t)e^{-c t} \chi(t) dt
$$

where $c = \lambda_1 c_1 + \lambda_2 c_2$, and where $\chi(t) \in L_r(-\infty, \infty)$. More precisely we have

$$
\|\chi(t)\|_r \leq E(c_1)^{-1}E(c_2)^{-1/\rho} \lambda_1^{-1/\rho} \lambda_2^{-1/\rho} \|\phi(t)\|_p \|\psi(t)\|_q.
$$

As an application of Theorem 1 we have the following result concerning the Stieltjes transform. If

$$
A(x) = \int_{0+}^{\infty} (x + t)^{-1} d\alpha(t), \quad B(x) = \int_{0+}^{\infty} (x + t)^{-1} d\beta(t)
$$

where $\alpha(t) \in \uparrow$, $\beta(t) \in \uparrow$, and if $0 < \lambda_1$, $0 < \lambda_2$, $\lambda_1 + \lambda_2 \leq 1$, then

$$
A(x^{\lambda_1})B(x^{\lambda_2}) = \int_{0+}^{\infty} (x + t)^{-1} d\gamma(t)
$$

with $\gamma(t) \in \uparrow$. Choosing $A(x) = B(x) = x^{-1+\epsilon}$, $0 < \epsilon < 1$, we see that the condition $\lambda_1 + \lambda_2 \leq 1$ is essential.

REFERENCES


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